# Lectures on Finsler Geometry

Zhongmin Shen
Department of Mathematical Sciences
Indiana University-Purdue University Indianapolis
Indiana 46202-3216, USA
zshen@math.iupui.edu

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# Preface

In 1854, B. Riemann introduced the notion of curvature for spaces with a family of inner products (Riemannian metrics). There was no significant progress in the general case until 1918, when P. Finsler studied the variation problem for spaces with a family of norms (Finsler metrics). Meanwhile, A. Einstein used Riemannian geometry to present his general relativity. By that times, however, the geometry of Finsler spaces was still at its infant stage. Until 1926, L. Berwald extended Riemann's notion of curvature to Finsler spaces and discovered a new non-Riemannian quantity using his connection. In his Paris address in 1900, D. Hilbert formulated 23 problems, the 4th and 23rd problems being in Finsler's category. Finsler geometry has broader applications in many areas of natural science [AIM] and will continue to develop through the efforts of many geometers around the world [BCS3].

This book comes out of a series of lecture notes based on my work at IHES (*Curvature, distance and volume in Finsler geometry*, preprint, 1997). Viewing Finsler spaces as regular metric spaces, we discuss the problems from the modern geometric point of view. The book is intended to provide basic materials for Finsler geometry including Riemannian geometry. The presentation is aimed at a reader who has completed a one-year graduate course in differential geometry, and who knows the elements of homotopy theory in topology.

The first four chapters cover some basic theories on regular metrics (Finsler metrics) and regular measures (volume forms) on manifolds.

In Chapter 5, we introduce the notion of geodesics via the calculus of variation. Using the geodesics, we define the Chern connection. The Chern connection is an important tool to investigate the geometric structure of Finsler spaces.

In Chapter 6, we introduce the notion of Riemann curvature via geodesic variation. This is different from Riemann's and Berwald's approaches. We discuss the dependence of the Riemann curvature on geodesics. In Chapter 7, we introduce various non-Riemannian quantities (e.g., the Chern curvature, the Landsberg curvature and the S-curvature, etc) and their geometric meanings.

In Chapters 8 and 9, we use Cartan's exterior differential method to discuss various curvatures and relationships. In particular, we discuss Finsler spaces of constant curvature. Several important examples are studied. These examples make Finsler geometry more fruitful.

Starting with Chapter 10, we study the metric properties of Finsler spaces and metric-measure properties of Finsler m spaces. We first derive the second variation formula of length and show the geometric meaning of the T-curvature. Then in Chapters 11 and 12, we discuss the index form, Jacobi fields, exponential maps and their relationship. The discussion leads to some basic comparison theorems that are discussed in Chapter 13.

In Chapters 14, 15 and 16, we use a distance function to study the global geometric structure of Finsler spaces and Finsler m spaces. We obtain several estimates on the geometry of level sets of distance functions. The geometric meaning of the S-curvature lies in the volume comparison theorem.

In Chapters 17 and 18, we apply the Morse theory to the canonical energy functional on the loop space. In particular, we prove a vanishing theorem for the homotopy groups of Finsler spaces with pinched curvature.

In the last chapter, we view a Finsler space as a point of a space of Finsler spaces equipped with the Gromov-Hausdorff distance. We briefly discuss the precompactness and finiteness of certain classes of Finsler spaces.

I would like to take this opportunity to thank several people in my personal life and academic life. First, I would like to thank my thesis advisor Detlef Gromoll for his help and advice in Riemannian geometry during my graduate study in SUNY at Stony Brook, and thank another advisor after the thesis, S.S. Chern for bringing me to a wider field — Finsler geometry. I thank Bart Ng for providing me with a good research environment at my current institution. Finally, I thank my wife, Tianping, for her consistent support and understanding. Without them, I would not have written this book, for sure.

Zhongmin Shen
Indianapolis, USA

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