Fractal Structure of Zeros in Hierarchical Models
(after Derrida, De Seze, and Itzykson)

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I. Preliminaries.
Let $\Gamma$ be the diamond graph. The **diamond hierarchical lattice** is the sequence of graphs $\{ \Gamma_n \}_{n \in \mathbb{N}}$ such that

- $\Gamma_1 := \Gamma$.
- $\Gamma_{n+1}$ has two marked vertices $a, b$ and is obtained from $\Gamma_n$ by replacing each edge of $\Gamma_n$ by $\Gamma_1$.

Let $\Gamma_n = (V_n, E_n)$
Potts model on the DHL

A configuration of spins is a mapping

\[ \sigma : V_n \longrightarrow \{1, 2, \ldots, q\} \]

The Ising model is the case \( q = 2 \). The Energy of \( \sigma \) is

\[ \mathcal{H}_n(\sigma) := -J \sum_{(i,j) \in E_n} \delta_{\sigma_i, \sigma_j} \]

Figure: Here, \( q = 3 \) and \( \mathcal{H} = -10J \).
A configuration $\sigma$ occurs with probability proportional to the *Gibbs weight*

$$W_n(\sigma) := e^{-H_n(\sigma)/T}$$

Note that

- When $T$ is close to zero, then minimal energy configurations have much higher probabilities.

- When $T$ is close to $\infty$, all configurations have more or less the same probability.

Hence,

$$P_n(\sigma) = \frac{W_n(\sigma)}{Z_n}$$

where $Z_n := \sum_\sigma W(\sigma)$ is the *partition function*. 
we introduce the change of variables

\[ y := e^{J/T} \]

so that \( Z_n \) becomes a polynomial in \( y \) of degree \(|E_n|\):

\[ Z_n(y) = \sum_{\sigma} y^{l(\sigma)} \]

where \( l(\sigma) := \sum_{(i,j) \in E_n} \delta_{\sigma(i),\sigma(j)} \) is the interaction of \( \sigma \). There are exactly \( q \) configurations such that the spins are aligned, so:

\[ Z_n(y) = q \prod_{i=1}^{\left| E_n \right|} (y - y_i) \]

The zeros of \( Z_n \), \( \{y_i\}_{1 \leq i \leq \left| E_n \right|} \) are called the Fisher zeros.
II. Computing the Fisher zeros: Migdal - Kadanoff renormalization equations.
two conditional partition functions

Let

\[ U_n := \sum_{\sigma \text{ s.t. } \sigma(a) = \sigma(b) = 1} W_n(\sigma) \]

\[ V_n := \sum_{\sigma \text{ s.t. } \sigma(a) = 1, \sigma(b) = 2} W_n(\sigma) \]

(U\textsubscript{n} and V\textsubscript{n} are functions of y). Clearly,

\[ Z_n = qU_n + q(q - 1)V_n \]

Finding an expression for U\textsubscript{n} and V\textsubscript{n} in terms of U\textsubscript{n-1} and V\textsubscript{n-1} is not hard (see blackboard) and since

\[ U_0 = y, \quad V_0 = 1 \]

we can compute Z\textsubscript{n} via an iterative procedure.
We have obtained:

\[ Z_n(y) = L \circ R^n(y, 1) \]

where \( R : \mathbb{C}^2 \rightarrow \mathbb{C}^2 \) is given by

\[
R(U, V) := \left( \left( U^2 + (q - 1)V^2 \right)^2, \ V^2(2U + (q - 2)V)^2 \right)
\]

and \( L : \mathbb{C}^2 \rightarrow \mathbb{C} \) is

\[
L(U, V) := qU + q(q - 1)V
\]
In the paper of **Derrida, De Seze and Itzykson (1983)** a different iterative procedure is used: Define $T : \mathbb{C} \to \mathbb{C}$ as

$$T(y) := \left( \frac{y^2 + q - 1}{2y + q - 2} \right)^2$$

Then $Z_n(y)$ are the $4^{n-1}$ preimages of $1 - q$ by the $(n - 1)$-th iterate of $T$.

**Derrida, De Seze and Itzykson (1983)** studies numerically what happens in the *thermodynamic limit* $n \to \infty$. 
Recall that

- The *Julia set* of $T$, $\mathcal{J}(T)$, is the closure of the set of repelling periodic points of $T$.

- Mikhail Lyubich and Alexandre Freires, Artur Lopes, and Ricardo Mañé have shown (1983) that if a point $y_0$ is not *exceptional* for $T$ (see below) then the probability measures $\mu_n(y_0)$ supported on $\{ T^{-n}(y_0) \}$ converge, as $n \to \infty$ to the measure of maximal entropy, which is supported on $\mathcal{J}(T)$.

- $y_0 = 1 - q$ is not exceptional, since it is not a critical value of $T$.

Hence, in the thermodynamic limit $n \to \infty$, the Fisher zeros converge, in the sense explained above, to $\mathcal{J}(T)$. 
Figure: Here, $q = 2$. No bias in the Monte Carlo procedure.
results of the numerical simulations (pictures)

Figure: Here, $q = 2$, as before. Biased Monte Carlo procedure.
Figure: Here, $q = 1.5$. Biased Monte Carlo procedure.
results of the numerical simulations (pictures)

Figure: Here, $q = 2.5$. Biased Monte Carlo procedure.
results of the numerical simulations (pictures)

Figure: Here, $q = 3$. Biased Monte Carlo procedure.
results of the numerical simulations (pictures)

Figure: Here, $q = 4$. Biased Monte Carlo procedure.