Some Results on Chromatic and Potts/Tutte Polynomials Including Zeros and Asymptotic Limits for Families of Graphs

Robert Shrock

C. N. Yang Institute for Theoretical Physics, Stony Brook University

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Potts/Tutte Polynomials

Recall some relevant definitions and results from Lecture 1:

Consider a graph $G = G(V, E)$, where V is the set of vertices (sites) and E is the set of edges (bonds). Denote $n = n(G) = |V|$ as the number of vertices, $e(G) = |E|$ as the number of edges, $k(G)$ as the number of connected components, and $c(G)$ as the number of linearly independent circuits in G.

Let $T =$ temperature, $\beta = 1/(k_B T)$, $J =$ spin-spin coupling. The Hamiltonian of the Potts model is

$$\mathcal{H} = -J \sum_{e_{ij}} \delta_{\sigma_i \sigma_j}$$

The couplings $J > 0$ and $J < 0$ favor FM and AFM spin ordering, respectively.

The Potts model partition function is (with $K = \beta J$)

$$Z = \sum_{\{\sigma_i\}} e^{-\beta \mathcal{H}} = \sum_{\{\sigma_i\}} e^{K \sum_{e_{ij}} \delta_{\sigma_i \sigma_j}} = \sum_{\{\sigma_i\}} \prod_{e_{ij}} e^{K \delta_{\sigma_i \sigma_j}}$$

A spanning subgraph G' of G is $G' = (V, E')$, where $E' \subseteq E$. Define $y = e^K$ and $v = y - 1$. An equivalent, graph-theoretic expression for Z is (don't confuse v, V)

$$Z(G, q, v) = \sum_{G' \subseteq G} q^{k(G')} v^{e(G')}$$

This shows $Z(G, q, v)$ is a polynomial in q and v (with positive coeffs).

At $T = 0$ in the AFM case, $K = -\infty$, so $y = 0$ and $v = -1$. The $T = 0$ Potts antiferromagnet (AFM) partition function $Z(G, q, -1) = P(G, q)$, the chromatic polynomial $P(G, q)$, which enumerates the number of ways of assigning q colors to the vertices of G such that no two adjacent vertices have the same color (proper q-coloring of G). Minimum number of colors needed for proper q-coloring of G is the chromatic number, $\chi(G)$.

Def. The degree of a vertex is the number of edges connecting to it. For a vertex-transitive lattice graph, this is the coordination number.

Def. A planar graph is one that can be drawn in the plane without any edges crossing.

Def. If $G(V, E)$ is a planar graph, then the (planar) dual graph G^* is the graph obtained by associating each vertex of G with a face of G^* and each face of G with a vertex of G^*.

The Tutte polynomial of a graph G is

$$T(G,x,y) = \sum_{G' \subseteq G} (x-1)^{k(G')-k(G)}(y-1)^{c(G')}$$

This is equivalent to the Potts model partition function:

$$Z(G,q,v) = q^{k(G)} v^{n(G)-1} T(G,x,y)$$

with

$$x = 1 + \frac{q}{v}\,, \quad y = v+1\,, \text{so} \;\; q = (x-1)(y-1)$$

Recall some specific types of graphs:

- empty graph (n vertices, no edges): E_n
- tree graph T_n: connected graph with n vertices, no circuits; e.g., line graph
- cyclic graph C_n
- strip graphs of lattices with fixed transverse width L_y vertices and variable length L_x vertices, with various boundary conditions
- complete graph, K_n: each vertex connected to every other vertex by an edge
- hierarchical graphs, e.g., Sierpinski graphs, Diamond hierarchical graphs

Method for Calculations of Potts/Tutte Polynomials on Recursive Families of Graphs of Arbitrary Size

Recall the deletion-contraction relation (DCR) from Lecture 1. Let $G - e$ denote the graph G with the edge e deleted and let G/e denote the graph G with the edge e deleted and the two vertices which it connected identified, i.e., G contracted on the edge e. $Z(G, q, v)$ satisfies the DCR

$$Z(G, q, v) = Z(G - e, q, v) + vZ(G/e, q, v)$$

Def. A recursive family of graphs G_m is a family for which the $(m + 1)$'th member is obtained from the m'th member graph either by simply gluing on some fixed subgraph or by cutting through the m'th member, inserting the subgraph, and gluing the graph together again. Denote formal limit $m \to \infty$ as $\{G\}$.

Examples: (i) a line graph L_m with m vertices and $m - 1$ edges; to construct L_{m+1}, one just adds an edge and vertex to the end of the line; (ii) a circuit graph C_m with m vertices and edges; to construct C_{m+1}, one cuts the graph anywhere, inserts another vertex and edge, and glues it together again; (iii) strip graph of a regular lattice, e.g., ladder graph.

For a variety of recursive families of graphs we have calculated Z and T via iterative application of the DCR in a manner which does not lead to an exponential increase in the number of terms. One way to proceed is to calculate a generating function for the recursive family of graphs G_m which is a rational function in q, v, and an expansion variable ξ, of the form

$$\Gamma_Z(G, q, v; \xi) = \frac{\mathcal{N}(q, v; \xi)}{\mathcal{D}(q, v, \xi)}$$

such that the Taylor series expansion in ξ about $\xi = 0$ yields $Z(G_m, q, v)$:

$$\Gamma_Z(G, q, v; \xi) = \sum_{m=0}^{\infty} Z(G_{m+1}, q, v)\, \xi^m$$

For the denominator, write

$$\mathcal{D}(q, v, \xi) = 1 + \sum_{j=1}^{N_{Z,\lambda}} b_j \xi^j = \prod_{j=1}^{N_{Z,\lambda}} (1 - \lambda_{Z,j}\xi)$$

This also means that the $Z(G_m, q, v)$ satisfy a recursion relation with $b_0 \equiv 1$:

$$\sum_{j=0}^{N_{Z,\lambda}} b_j Z(G_{m+N_{Z,\lambda}-j}, q, v) = 0$$

For example, for the family of circuit graphs $\{C\}$, one has

$$\Gamma_Z(C,q,v,\xi) = \frac{q\Big[(v+1) - v(q+v)\xi]}{[1-(q+v)\xi][1-v\xi]} = \sum_{m=0}^{\infty} Z(C_{m+1},q,v)\,\xi^m$$

Here $\mathcal{D} = 1-(q+2v)\xi + v(q+v)\xi^2$, i.e., $b_1 = -(q+2v)$, $b_2 = v(q+v)$, so $N_{Z,\lambda} = 2$ and $Z(C_m,q,v)$ satisfies the recursion relation

$$Z(C_{m+2},q,v) + b_1 Z(C_{m+1},q,v) + b_2 Z(C_m,q,v) = 0$$

Corresponding generating function and recursion relation for $T(G_m,x,y)$ with $b_{T1} = -(1+x)$, $b_{T2} = x$,

$$\Gamma_T(C,q,v,\xi) = \frac{y - x(y-1)\xi}{[1-x\xi][1-\xi]} = \sum_{m=0}^{\infty} T(C_{m+1},x,y)\,\xi^m$$

$$T(C_{m+2},x,y) + b_{T1}T(C_{m+1},x,y) + b_{T2}T(C_m,x,y) = 0$$

We consider strip graphs of various lattices Λ, of length $L_x = m$ and width L_y vertices, with various boundary conditions (BC's). We take the width to be fixed and the length to be variable and arbitrarily great. Lattice types include square, triangular, honeycomb, kagomé, etc. BC types are denoted F = free, P = periodic, T = twisted.

- free: FBCx, FBCy
- cyclic: PBCx, FBCy
- Möbius: TPBCx, FBCy
- cylindrical: FBCx, PBCy
- toroidal: PBCx, PBCy
- Klein-bottle: TPBCx, PBCy

We have also done calculations for other types of recursive graphs, such as

- self-dual strips of the square lattice with free and periodic longitudinal BC; e.g., for the periodic case, one adjoins a single external vertex to all of the vertices on one side of the strip.
- hammock graphs, $H_{k,r}$ with two endpoints, r "ropes" each of length $k-1$ edges, joining these two endpoints.
- augmentations of strip graphs, in particular, necklace graphs containing polygons joined by line graphs.

Some Calculational Results

For a cyclic strip graph of length $L_x = m$, denoted G_m, $Z(G_m, q, v)$ has a structure consisting of a sum of m'th powers of certain functions $\lambda_{Z,L_y,d,j}$ depending on q and v multiplied by coefficients $c^{(d)}(q)$ that are polynomials of degree d in q:

$$Z(G_m, q, v) = \sum_{d=0}^{L_y} c^{(d)} \sum_{j=1}^{n_Z(L_y,d)} (\lambda_{Z,L_y,d,j})^m$$

where

$$c^{(d)} = \sum_{j=0}^{d} (-1)^j \binom{2k-j}{j} q^{d-j}$$

so $c^{(0)} = 1$, $c^{(1)} = q - 1$, $c^{(2)} = q^2 - 3q + 1$, etc.

Essentially the same structural formula holds for the Tutte polynomial $T(G_m, x, y)$:

$$T(G_m, x, y) = \frac{1}{x-1} \sum_{d=0}^{L_y} c^{(d)} \sum_{j=1}^{n_T(L_y,d)} (\lambda_{T,L_y,d,j})^m$$

where $n_T(L_y, d) = n_Z(L_y, d)$ and the $\lambda_{T,L_y,d,j}$'s are obtained from the $\lambda_{Z,L_y,d,j}$'s by the transformation of variables $(q, v) \to (x, y)$.

The λ's are the roots of $\mathcal{D}(q, v, \xi)$ in the generating function as a polynomial in ξ:

$$\mathcal{D}(q, v, \xi) = \prod_{j=1}^{N_{Z,G,\lambda}} (1 - \lambda_{Z,G,j}\xi)$$

The total number of λ's that enter is

$$N_{Z,L_y,\lambda} = \sum_{d=0}^{L_y} n_Z(L_y, d)$$

For a given strip and boundary conditions, the number of λ's is independent of the length $L_x = m$, but it grows rapidly with the width L_y.

For example, for the cyclic strips, the total number of λ's, $N_{Z,\Lambda,L_y,\lambda}$, which we have shown is the same for strips of the square, triangular, and honeycomb lattices, is

$$N_{Z,\Lambda,L_y,\lambda} = \binom{2L_y}{L_y}$$

Thus, for $L_y = 1,\ 2,\ 3,\ 4,\ 5$, this total number is 2, 6, 20, 70, 252 and as $L_y \to \infty$, it grows asymptotically like

$$N_{Z,\Lambda,L_y,\lambda} \sim L_y^{-1/2}\, 4^{L_y} \quad \text{as} \ \ L_y \to \infty$$

Table of numbers $n_Z(L_y, d)$ and their sums, $N_{Z,G,\lambda}$, for strips of width L_y and arbitrary length of the sq, tri, and hc lattices.

$L_y \downarrow \quad d \rightarrow$	0	1	2	3	4	5	6	$N_{Z,L_y,\lambda}$
1	1	1						2
2	2	3	1					6
3	5	9	5	1				20
4	14	28	20	7	1			70
5	42	90	75	35	9	1		252
6	132	297	275	154	54	11	1	924

We have obtained exact results for strips for which Z or P involve up to 70 λ's. See references in bibliography.

The $L_y = 1$ strip is just C_m, for which $n_Z(1,0) = n_Z(1,1) = 1$, so $N_{Z,C,\lambda} = 2$. The λ's are $\lambda_{Z,L_y=1,d=0,j=1} = q + v$ and $\lambda_{Z,L_y=1,d=1,j=1} = v$:

$$Z(C_m, q, v) = (q+v)^m + (q-1)v^m = c^{(0)}(\lambda_{Z,1,0,1})^m + c^{(1)}(\lambda_{Z,1,1,1})^m$$

For the cyclic square strip with $L_y = 2$, denoted $sq(2 \times m, cyc.)$, we calculated Z in Shrock, Physica A283, 388-446 (2000). It has $n_Z(2,0) = 2$, $n_Z(2,1) = 3$, and $n_Z(2,2) = 1$, for a total of $N_{Z,2,\lambda} = 6$ and is $Z(sq(2 \times m, cyc.), q, v) =$

$$\sum_{j=1}^{2} (\lambda_{Z,sq,2,0,j})^m + c^{(1)} \sum_{j=1}^{3} (\lambda_{Z,sq,2,1,j})^m + c^{(2)} (\lambda_{Z,sq,2,2,1})^m$$

where the λ's are

$$\lambda_{Z,sq,L_y=2,d=0,j=(1,2)} = \frac{1}{2}\left[q^2 + 3qv + 4v^2 + v^3 \pm \sqrt{R_{S12}}\ \right]$$

where $R_{S12} = q^4 + 6q^3v + 13q^2v^2 + 16qv^3 + 12v^4 - 2q^2v^3 - 2qv^4 + 4v^5 + v^6$,

$$\lambda_{Z,sq,L_y=2,d=1,j=1} = v(q + v)$$

$$\lambda_{Z,sq,L_y=2,d=1,j=(2,3)} = \frac{v}{2}\left[q + v(v + 4) \pm \sqrt{R_{C12}}\ \right]$$

where $R_{C12} = q^2 + 4qv - 2qv^2 + 12v^2 + 4v^3 + v^4$, and

$$\lambda_{Z,sq,L_y=2,d=2,j=1} = v^2$$

For the $L_y = 2$ Möbius strip of the square lattice, the λ's are the same, but there are some switches in the coefficients (and their signs), so that

$$Z(sq(2 \times m, cyc.), q, v) = \sum_{j=1}^{2} (\lambda_{Z,sq,2,0,j})^m$$
$$+c^{(1)}\Big[-(\lambda_{Z,sq,2,1,1})^m + (\lambda_{Z,sq,2,1,2})^m + (\lambda_{Z,sq,2,1,3})^m \Big]$$
$$-(\lambda_{Z,sq,2,2,1})^m$$

Although the λ's can be algebraic functions of q and v, they are the roots of algebraic equations with coefficients that are polynomials in q and v. Z involves symmetric polynomials of these λ's, and these are polynomials in the coefficients of the algebraic equations, and hence in q and v. This is guaranteed by the general cluster formula for any $Z(G, q, v)$.

We have obtained a similar structural formula for self-dual strips (Chang and Shrock, Physica A301, 301-329 (2001); Phys. Rev. E64, 066116 (2001). These involve slightly different coefficients.

Functions Defined in the Limit $n \to \infty$

Since our results apply for arbitrary strip length $L_x = m$, we can consider the limit $n \to \infty$ obtained by taking the length $L_x \to \infty$. Following our general notation, for a given type of strip graph we denote this as $\{G\}$. Recall the resultant definition of a dimensionless free energy (per site) of the Potts model,

$$f(\{G\}, q, v) = \lim_{n \to \infty} \frac{1}{n} \ln Z$$

with $G = -k_B T f$. Given that $Z(G_m, q, v) = \sum_j c_j \lambda_j^m$ for a recursive family of graphs, in the limit $m \to \infty$, the λ_j with the largest magnitude $|\lambda_{dom.}|$ (given that the corresponding $c_j \neq 0$), which we denote $\lambda_{dom.}$, will dominate this sum over j, and will hence determine f (early work by Beraha, Kahane, and Weiss). For a $L_x \times L_y$ strip with $n = L_x L_y$, $(\lambda_{dom.}^{L_x})^{1/n} = (\lambda_{dom.})^{1/L_y}$ and generically

$$f = \frac{1}{L_y} \ln \lambda_{dom.}$$

For example, for the tree graph, since there is only one λ, $f = q + v$. For the circuit graph, there are two λ's, and $f = q + v$ if $|q + v| > |v|$, while $f = v$ if $|v| > |q + v|$.

In general, for some special q values, q_s, there is noncommutativity of limits (Shrock and Tsai, Phys. Rev. $\mathrm{E}55$, 5165-5179 (1997); Shrock, Physica A283, 288 (2000)):

$$\lim_{q \to q_s} \lim_{n \to \infty} [Z(G_m, q, v)]^{1/n} \neq \lim_{n \to \infty} \lim_{q \to q_s} [Z(G_m, q, v)]^{1/n}$$

e.g., $Z(T_n, q, v) = q(q+v)^{n-1}$, so

$$\lim_{n \to \infty} [Z(T_n, q, v)]^{1/n} = q + v, \quad \Rightarrow \quad \lim_{q \to 0} \lim_{n \to \infty} [Z(T_n, q, v)]^{1/n} = v$$

while

$$\lim_{q \to 0} Z(T_n, q, v) = 0, \quad \Rightarrow \quad \lim_{n \to \infty} \lim_{q \to 0} [Z(T_n, q, v)]^{1/n} = 0$$

Our calculations give exact results for $Z(G,q,v)$ and $T(G,x,y)$ for recursive families of graphs of fixed width and arbitrary length. We have calculated Z and T for strips of various lattices, including square, triangular, honeycomb, etc., and also for wider strips. Much of this work was with thesis students S.-H. Tsai and S.-C. Chang.

Our results apply for arbitrary q and temperature variable $v = e^{J/(k_B T)} - 1$ and thus complement other types of exact calculations, such as Onsager's calculation of the partition function for a special value of q, namely $q = 2$ (the Ising value) for the square lattice, and subsequent calculations of the Ising partition function for other 2D lattices.

We have evaluated the various special cases of Z and T for lattice strips, including the chromatic polynomial and other special cases. See refs. at end. We next discuss further the special case of $P(G,q)$.

Chromatic Polynomials of Recursive Families of Graphs

Recall that $Z(G, q, v = -1) = P(G, q)$, the chromatic polynomial of G. Hence, for cyclic lattice strip graphs with members G_m, $P(G, q)$ has the form

$$P(G_m, q) = \sum_{d=0}^{L_y} c^{(d)} \sum_{j=1}^{n_P(L_y,d)} (\lambda_{P,L_y,d,j})^m$$

The total number of λ's is

$$N_{P,L_y,\lambda} = \sum_{d=0}^{L_y} n_P(L_y, d)$$

For $1 \le L_y \le 5$, $N_{P,L_y,\lambda} =$ 2, 4, 10, 26, 70, to be compared with 2, 6, 20, 70, 252 for $N_{Z,L_y,\lambda}$. For all but the lowest case $L_y = 1$, the numbers of λ's entering into the chromatic polynomial are less than those in the full Potts model partition function. Asymptotically,

$$N_{P,L_y,\lambda} \sim L_y^{-1/2}\, 3^{L_y} \quad \text{as} \quad L_y \to \infty$$

This is exponential growth, but $\sim 3^{L_y}$, hence slower then the growth $\sim 4^{L_y}$ for Z.

Table of numbers $n_P(L_y, d)$ and their sums, $N_{P,L_y,\lambda}$ for cyclic strips of width L_y and arbitrary length of the square (sq), triangular (tri), and honeycomb (hc) lattices.

$L_y \downarrow \quad d \rightarrow$	0	1	2	3	4	5	6	$N_{P,L_y,\lambda}$
1	1	1						2
2	1	2	1					4
3	2	4	3	1				10
4	4	9	8	4	1			26
5	9	21	21	13	5	1		70
6	21	51	55	39	19	6	1	192

Recall that from $P(G, q)$ one can calculate the ground state degeneracy, per site, of the Potts AFM:

$$W(\{G\}, q) = \lim_{n \to \infty} P(G, q)^{1/n}$$

On $\{G\}$, the associated g.s. entropy per site is $S_0 = k_B \ln W$. The q-state Potts AFM at $T = 0$ exhibits nonzero S_0, equiv. $W > 1$ for sufficiently large q

We have calculated W functions for a variety of limits of $\{G\}$'s and studied their behavior as a function of lattice type and q. Here we will generalize q from $\mathbb{N}$ to $\mathbb{C}$.

For the $n \to \infty$ limit of C_n, $W = q - 1$ if $|q-1| > 1$, i.e., for real q, if $q > 2$ or $q < 0$. If $|q-1| < 1$, i.e., for real q, if $0 < q < 2$, then P can be negative, so only the magnitude of W is determined. We obtain $|W| = 1$ if $|q-1| < 1$. There are thus nonanalytic changes in W at $q = 0$ and $q = 2$ in this case, with $q_c = 2$.

For the $L_x \to \infty$ limit of the cyclic or Möbius $L_y = 2$ strip of the square lattice,

$$W = \sqrt{q^2 - 3q + 3} \quad \text{if} \quad q > 2 \text{ or } q < 0$$

and

$$|W| = |q - 3| \quad \text{if} \quad 0 < q < 2$$

Thus, also $q_c = 2$ for the $L_x \to \infty$ limit of this $L_y = 2$ strip of the square lattice.

For the $L_y = 3$ cyclic or Möbius strip of square lattice, we obtain the following (from Shrock and Tsai, Phys. Rev. E60, 3512 (1999); Shrock, Phys. Lett. A261, 57 (1999)). Define intervals on the real axis R_1: $q > q_c$, where $q_c \simeq 2.34$ or $q < 0$; R_2: $2 < q < q_c$; R_3: $0 < q < 2$. Here, q_c is a solution of the degeneracy equation $|\lambda_{dom,R_1}| = |\lambda_{dom,R_2}|$, namely $2q^4 - 16q^3 + 51q^2 - 86q + 67 = 0$.

Then for $q \in R_1$,

$$W = 2^{-1/3}\Big[(q-2)(q^2-3q+5)+\Big[(q^2-5q+7)(q^4-5q^3+11q^2-12q+8)\Big]^{1/2}\Big]^{1/3}$$

For $q \in R_2$, $|W| = |q-4|^{1/3}$, and in R_3, W is the root of a certain cubic equation.

Since $P(G,q)$ is the $T=0$ Potts antiferromagnet (AF) partition function, the nonanalyticity at $q=q_c$ in $W(\{G\},q)$ is connected with a nonanalyticity of the $q=q_c$-state Potts AF free energy, as a function of temperature, at $T=0$.

From our calculations of W for the $L_x \to \infty$ limit of the $L_y = 4$ ($N_{P,\lambda} = 26$) and $L_y = 5$ ($N_{P,\lambda} = 70$) cyclic or Möbius strips of the square lattice, we obtain $q_c \simeq 2.49$ and 2.58, respectively. The q_c values that we have calculated for the infinite-length cyclic/Möbius strips exhibit a monotonic increase as a function of L_y, consistent with approaching 3 from below as $L_y \to \infty$.

We have calculated W for $L_x \to \infty$ limits of strips of the square lattice that are self-dual, and thus incorporate a property of the full 2D square lattice, and these exhibit $q_c = 3$, consistent with an inferred $T=0$ critical point in the $q=3$ Potts AF at $T=0$. For $L_y = 1$, we find $\mathcal{B}_q$: circle $|q-2| = 1$; also, see figures below.

From our exact calculations of $P(G_m, q)$ and hence $W(\{G\}, q)$, we find that, for a given L_y and q, W decreases as the vertex degree Δ increases. For example, restricting to real $q \geq \chi(G)$, for the $L_y = 2$ square-lattice strip,

$$W(sq(2 \times \infty), q) = \sqrt{q^2 - 3q + 3}$$

while for the $L_y = 2$ strip of the triangular (tri) lattice,

$$W(tri(2 \times \infty), q) = q - 2$$

So, e.g., $W(sq(2 \times \infty), q = 3) = \sqrt{3}$, larger than
$W(tri(2 \times \infty), q = 3) = 1$, etc.

We have made corresponding comparisons for wider lattice strips and have proved a number of rigorous upper and lower bounds on W for infinite limits of 2D lattices. Refs. Shrock and Tsai, Phys. Rev. E55, 6791-6794 (1997); Phys. Rev. E56, 2733-2737 (1997); Phys. Rev. E56, 4111-4124 (1997); Shrock and Xu, Phys. Rev. E81, 031134 (2010); Chang and Shrock, Phys. Rev. E91, 052142 (2015).

This ordering in the W values for a given q on different lattices can be understood as a consequence of the greater constraints on the proper q-coloring on a lattice with a greater vertex degree (coordination number).

Zeros of Chromatic and Potts/Tutte Polynomials and their Accumulation Sets as $n \to \infty$

Since $Z(G,q,v)$ is a polynomial in q and v, it is of interest to analyze its zeros. We have analyzed the zeros of $Z(G,q,v)$ in the complex q plane for fixed v, and in the complex v plane for fixed q for various families of graphs (as well as on submanifolds defined by $\phi(q,v) = 0$ such as $v^2 - q = 0$).

Since the coefficients of the polynomial terms of $Z(G,q,v)$ in q and v are real (actually positive integers), the sets of zeros (i) in q for fixed real v and (ii) in v for fixed real q are invariant under complex conjugation.

One can study the behavior of these zeros in the limit $n \to \infty$. One finds that in this limit, zeros accumulate to form certain curves and possible line segments, generically denoted as the loci $\mathcal{B}_q$ for fixed v and $\mathcal{B}_v$ for fixed q. These are determined by the condition that two dominant λ's are equal in magnitude, which defines algebraic curves for these recursive families of graphs.

For a long but finite-length strip, many zeros lie close to the asymptotic locus $\mathcal{B}_q$ or locus $\mathcal{B}_v$ on which they merge as $n \to \infty$.

As one crosses a boundary on $\mathcal{B}_q$ or $\mathcal{B}_v$, the dominant λ, and hence the analytic expression for W, changes. This was evident in the illustrative W functions given above.

We have calculated $\mathcal{B}_q$ and $\mathcal{B}_v$ for the $n \to \infty$ limits of many families of graphs. The study of these boundaries ties together graph theory, complex analysis, and algebraic geometry, and also relates these to statistical physics.

For the $L_x \to \infty$ limit of a family of strip graphs of a certain type, $\{G\}$, the maximal point at which $\mathcal{B}_q$ crosses the real q axis is denoted $q_c(\{G\})$.

One particularly interesting subcase is $\mathcal{B}_q$ for $v = -1$, the continuous accumulation set of the zeros of the chromatic polynomials.

The form of $\mathcal{B}_q$ depends on both the type of strip (e.g., type of lattice, sq, tri, hc...) and the boundary conditions.

We have shown that free b.c. yield loci $\mathcal{B}_q$ that consist of arcs and possible line segments.

We have found that a sufficient (not necessary) condition that $\mathcal{B}_q$ separates the q plane into regions is that the longitudinal b.c. are periodic or twisted periodic (here, $\mathcal{B}$

denotes the boundaries between these regions). This is also a sufficient (not necessary) condition that $\mathcal{B}_q$ crosses the real axis, defining a $q_c(\{G\})$.

We have characterized these regions for $v = -1$ and for other values of v; recall that $v \in [-1, 0]$ is Potts antiferromagnet $(J < 0)$, while $v \in [0, \infty]$ is Potts ferromagnet $(J > 0)$.

Correspondingly, one can analyze $\mathcal{B}_v$, or equivalently, $\mathcal{B}_y$, where $y = v + 1 = e^K$. In the thermodynamic limit, the curves on $\mathcal{B}_y$ separate complex-y extensions of physical phases of the q-state Potts model. These phases include the paramagnetic phase, where the S_q permutation symmetry is realized explicitly, the ferromagnetic phase, where the S_q symmetry is spontaneously broken by a nonzero uniform magnetization (preference for one value of σ_i), and can also include an antiferromagnetic phase.

Some illustrative figures for regular lattice strips follow. For others, see refs.

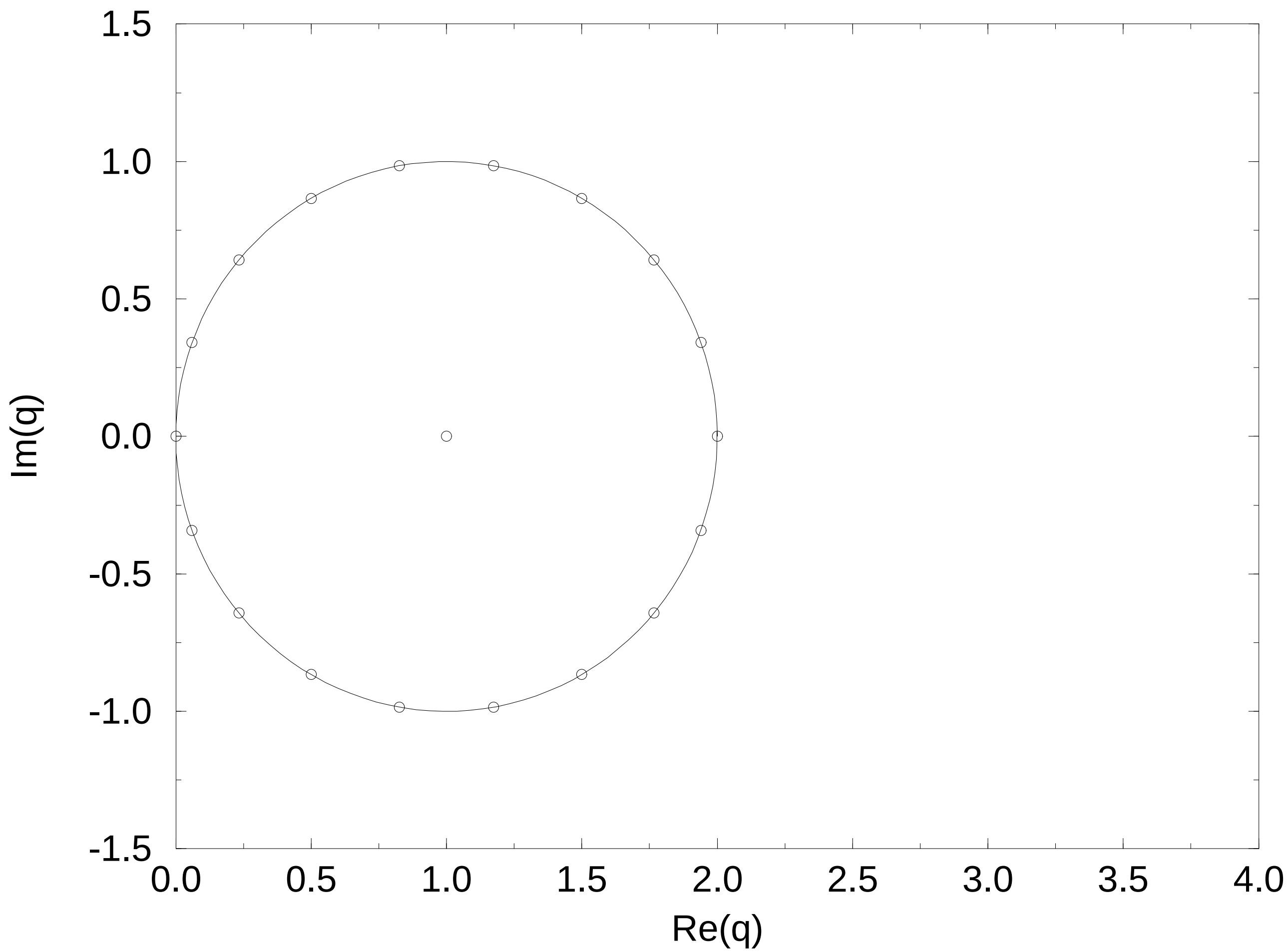


Figure 1: Locus $\mathcal{B}_q$ in the q plane for the $n \to \infty$ limit of the circuit graph C_n. Chromatic zeros for B_n with $n = 19$ are shown for comparison. From R. Shrock and S.-H. Tsai, Phys. Rev. E55, 5165 (1997).

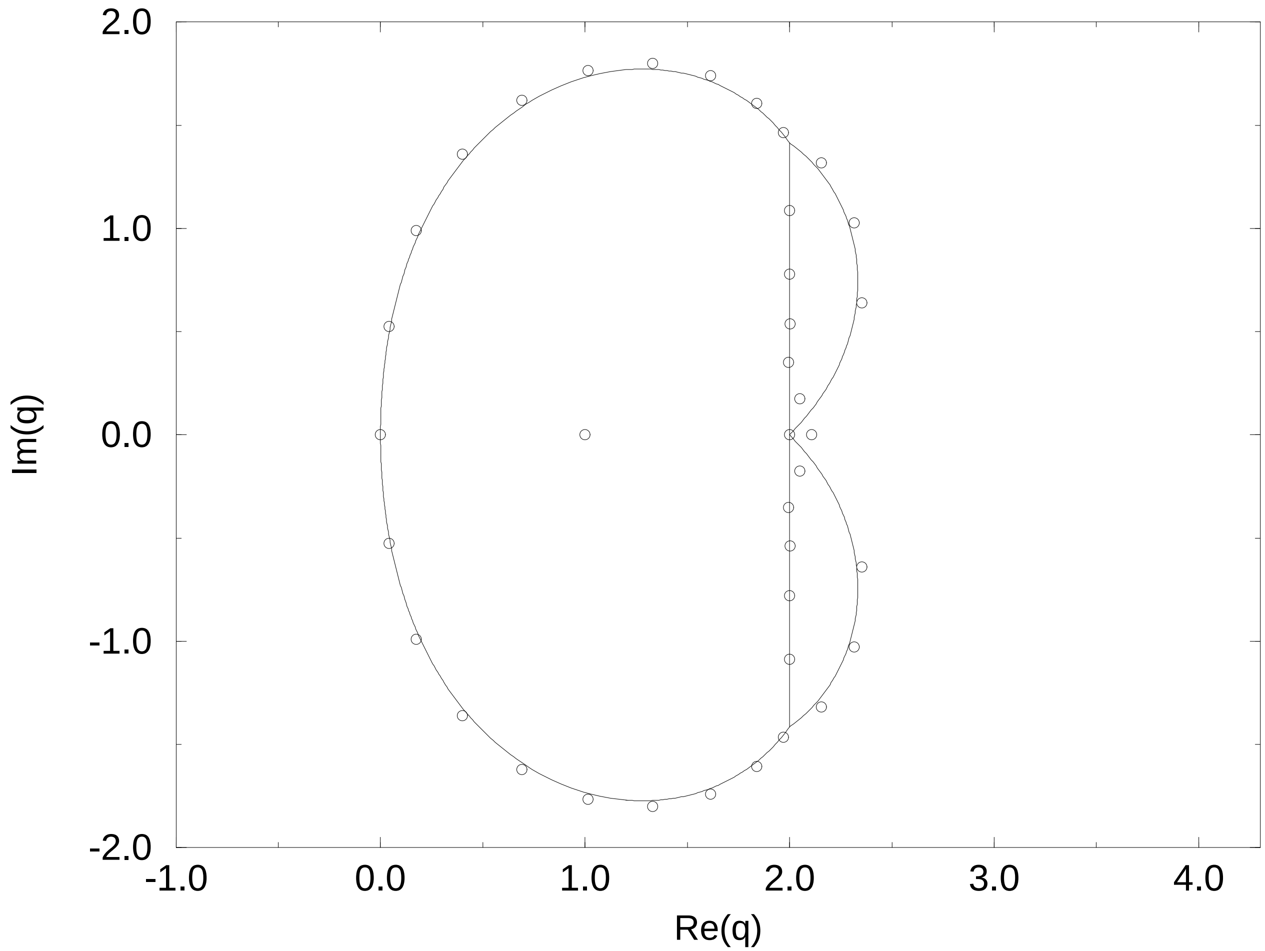


Figure 2: Locus $\mathcal{B}_q$ in the q plane for the $L_x \to \infty$ limit of the cyclic or Möbius strip of the square lattice with width $L_y = 2$. Chromatic zeros for the cyclic strip with $L_x = 19$ and thus $n = 38$ vertices are shown for comparison. From R. Shrock and S.-H. Tsai, Phys. Rev. E55, 5165 (1997).

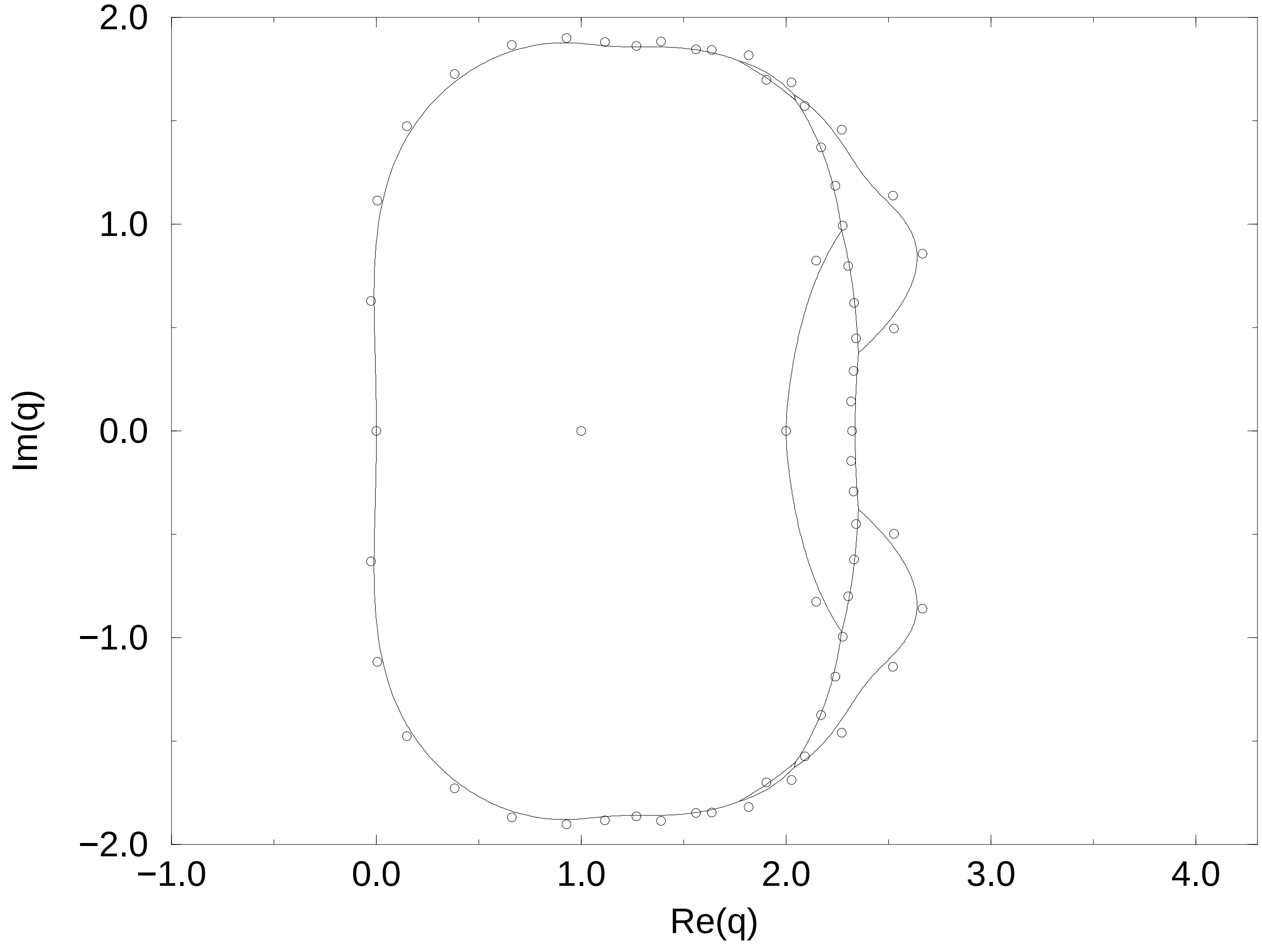


Figure 3: Locus $\boldsymbol{\mathcal{B}_q}$ for the $\boldsymbol{L_x \to \infty}$ limit of the cyclic or Möbius strip of the square lattice with width $\boldsymbol{L_y = 3}$. Chromatic zeros for the cyclic strip with $\boldsymbol{L_x = 20}$ and thus $\boldsymbol{n = 60}$ vertices are shown for comparison. From R. Shrock and S.-H. Tsai, Phys. Rev. E60, 3512 (1999).

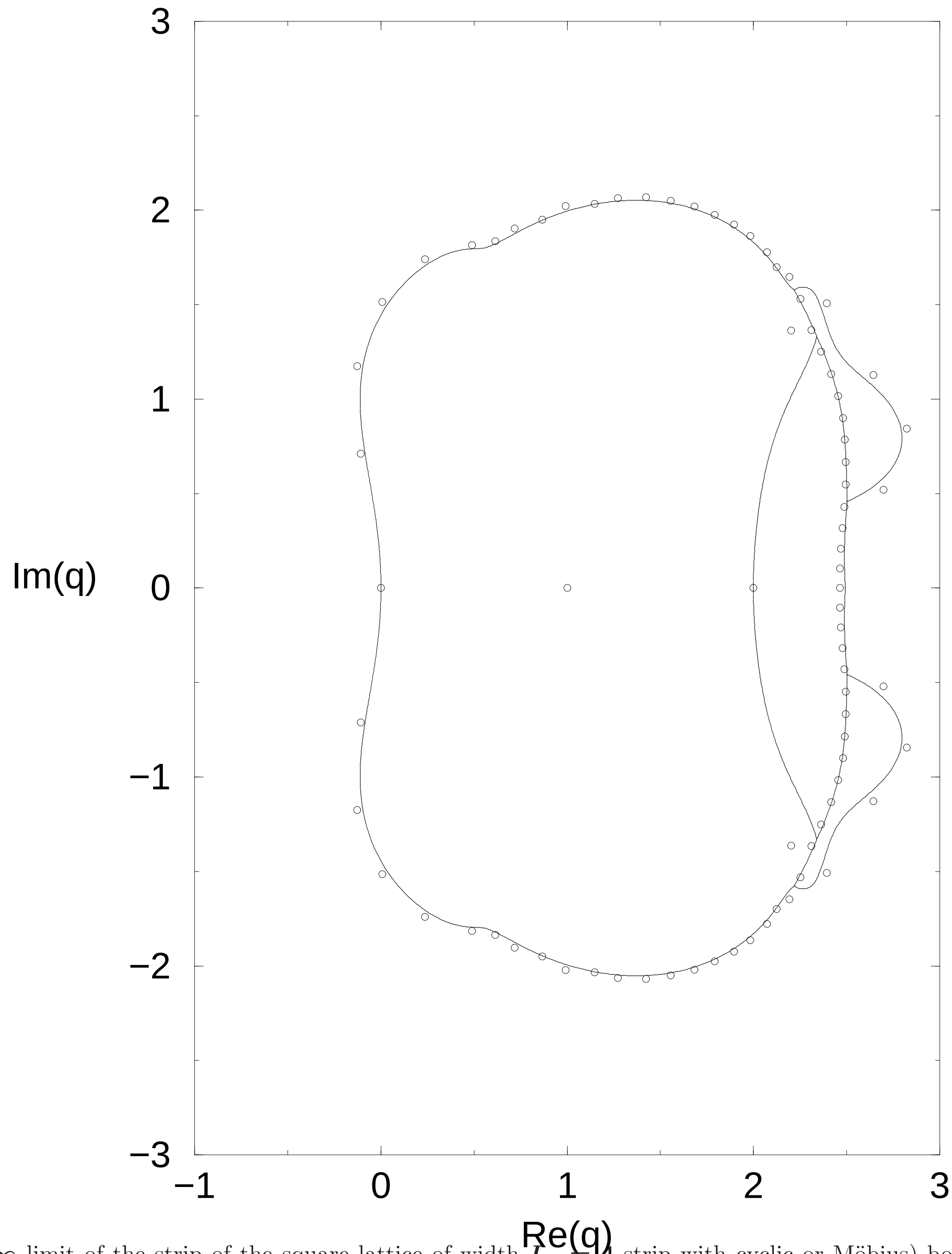


Figure 4: Locus $\mathcal{B}_q$ for the $L_x \to \infty$ limit of the strip of the square lattice of width $L_y = 4$ strip with cyclic or Möbius) boundary conditions. For comparison, chromatic zeros calculated for the cyclic strip with $L_x = 20$ and thus $n = 80$ vertices are shown. From S.-C. Chang and R. Shrock, Physics A290, 402 (2001), Physica A316, 335 (2002).

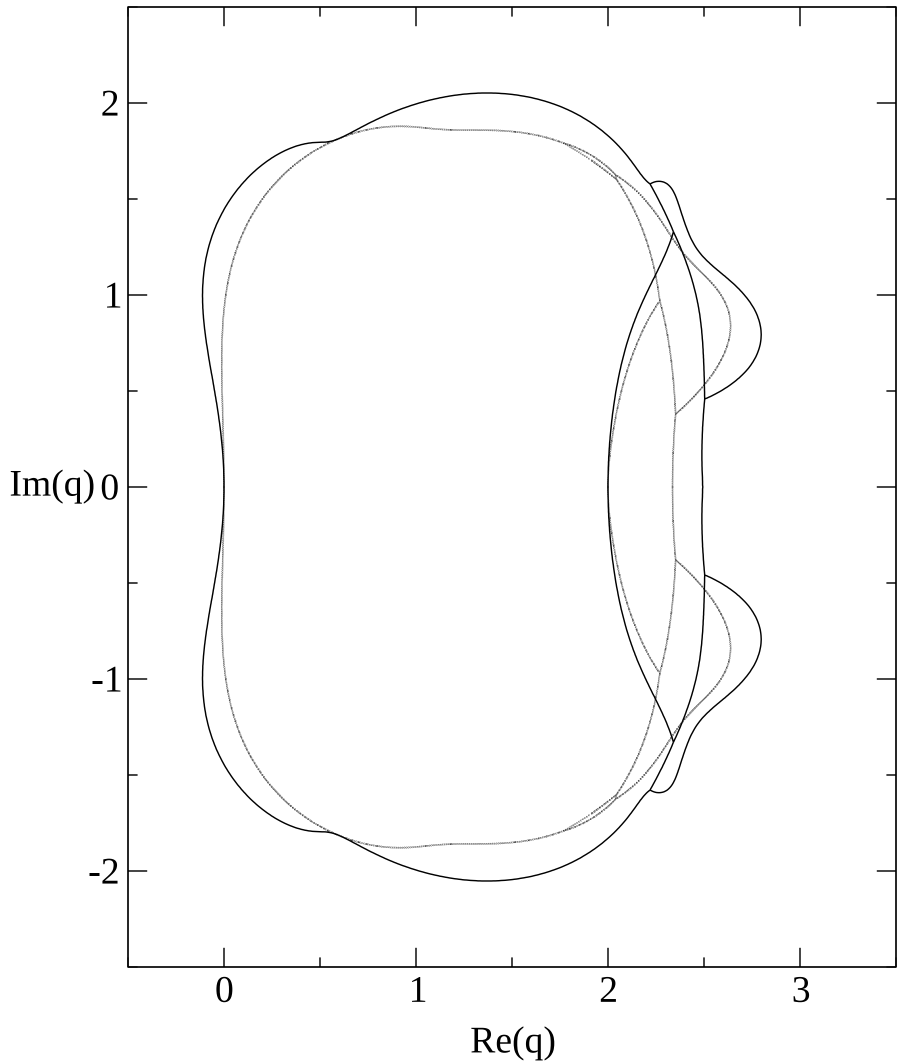


Figure 5: Comparison of loci $\mathcal{B}_q$ for the $L_x \to \infty$ limits of the strip of the square lattice with cyclic or Möbius boundary conditions with widths $L_y = 3$ and $L_y = 4$. From S.-C. Chang and R. Shrock, Physica A292, 307 (2001).

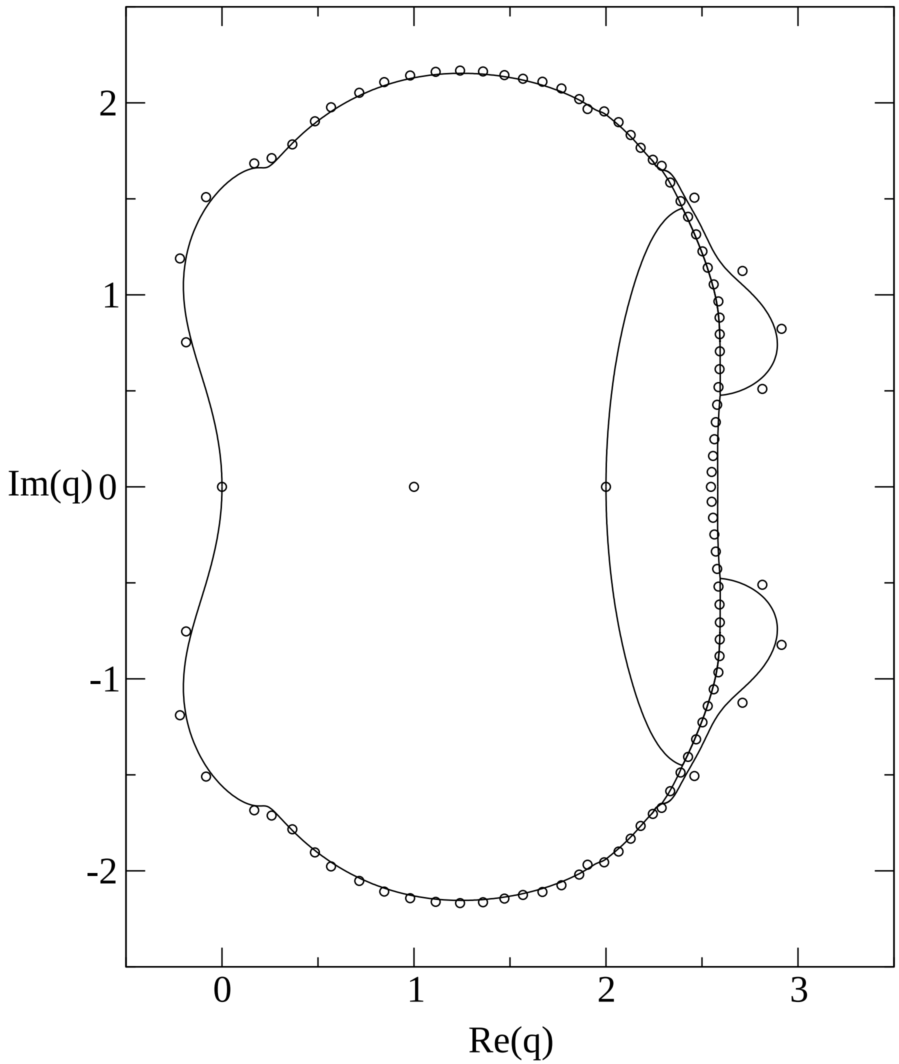


Figure 6: Locus $\mathcal{B}_q$ for the $L_x \to \infty$ limit of the cyclic or Möbius strip of the square lattice of width $L_y = 5$. For comparison, chromatic zeros calculated for the cyclic strip with $L_x = 20$ and thus $n = 100$ vertices are shown. From S.-C. Chang and R. Shrock, Physica A316, 335 (2002).

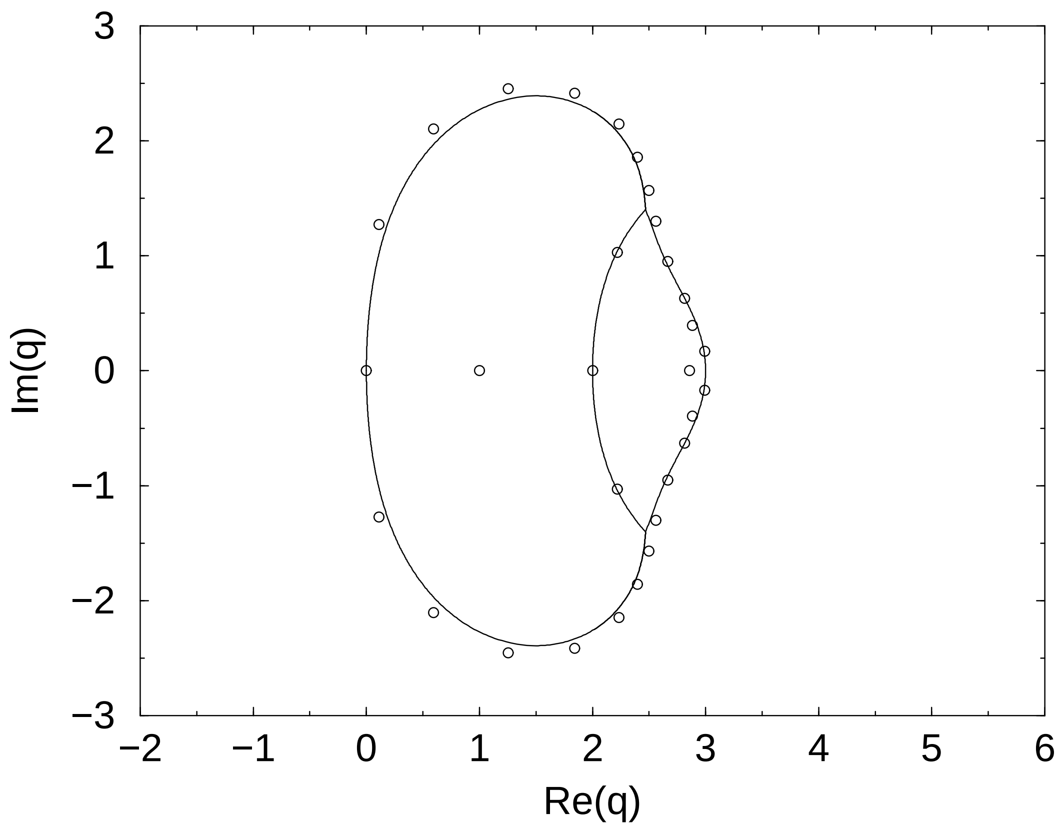


Figure 7: Locus $\mathcal{B}_q$ for the $L_x \to \infty$ limit of the strip of the square lattice with toroidal or Klein Bottle boundary conditions, of width $L_y = 3$, with chromatic zeros for a finite L_x shown for comparison; see N. Biggs and R. Shrock, J. Phys. A (Letts) 32, L489 (1999).

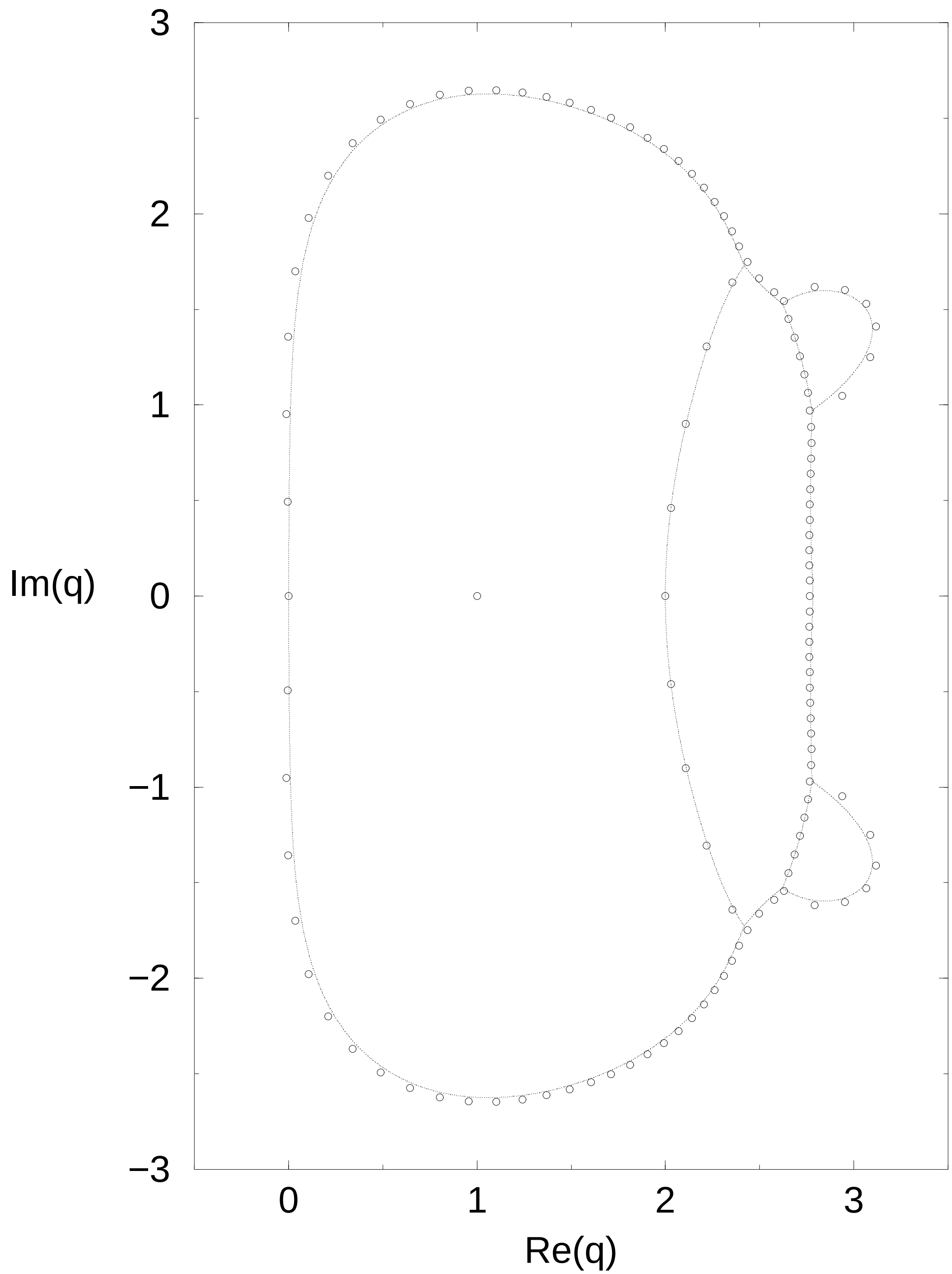


Figure 8: Locus $\mathcal{B}_q$ for the $L_x \to \infty$ limit of the strip of the square lattice with toroidal or Klein bottle boundary conditions, of width $L_y = 4$, with chromatic zeros for a finite L_x shown for comparison, from S.-C. Chang and R. Shrock, Physica A292, 307 (2001).

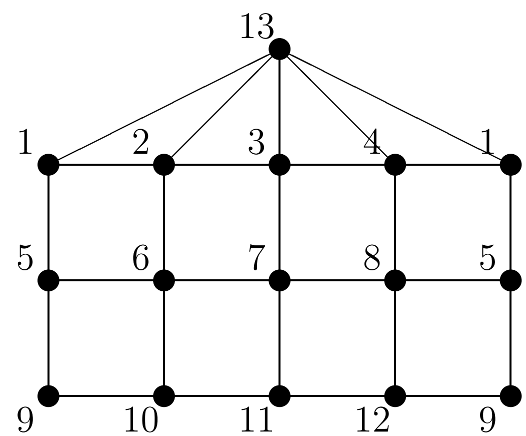


Figure 1: Example of a strip graph $G_D(L_y \times L_x)$ for the case $L_y = 3$, $L_x = 4$.

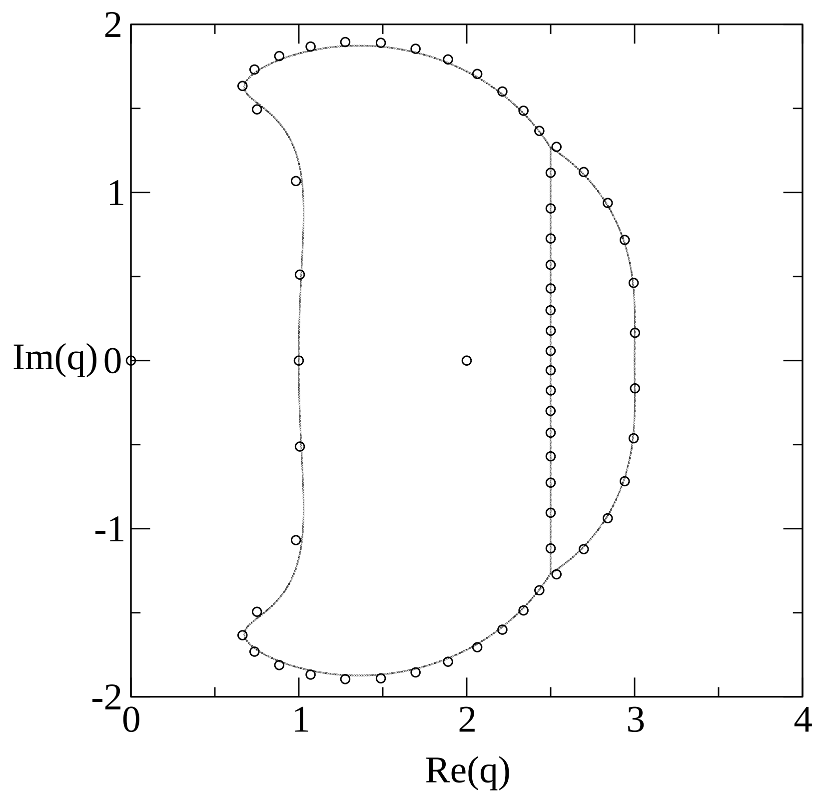


Figure 9: Locus $\mathcal{B}_q$ for the $L_x \to \infty$ limit of $G_D(2 \times L_x)$. For comparison, chromatic zeros are shown for $L_x = 30$, i.e., $n = 61$. For the $L_x \to \infty$ limit of the wheel graph $G_D(1 \times L_x)$, $\mathcal{B}$ is the circle $|q-2| = 1$. This figure and the others on G_D strips are from S.-C. Chang and R. Shrock, Physica A301, 301 (2001).

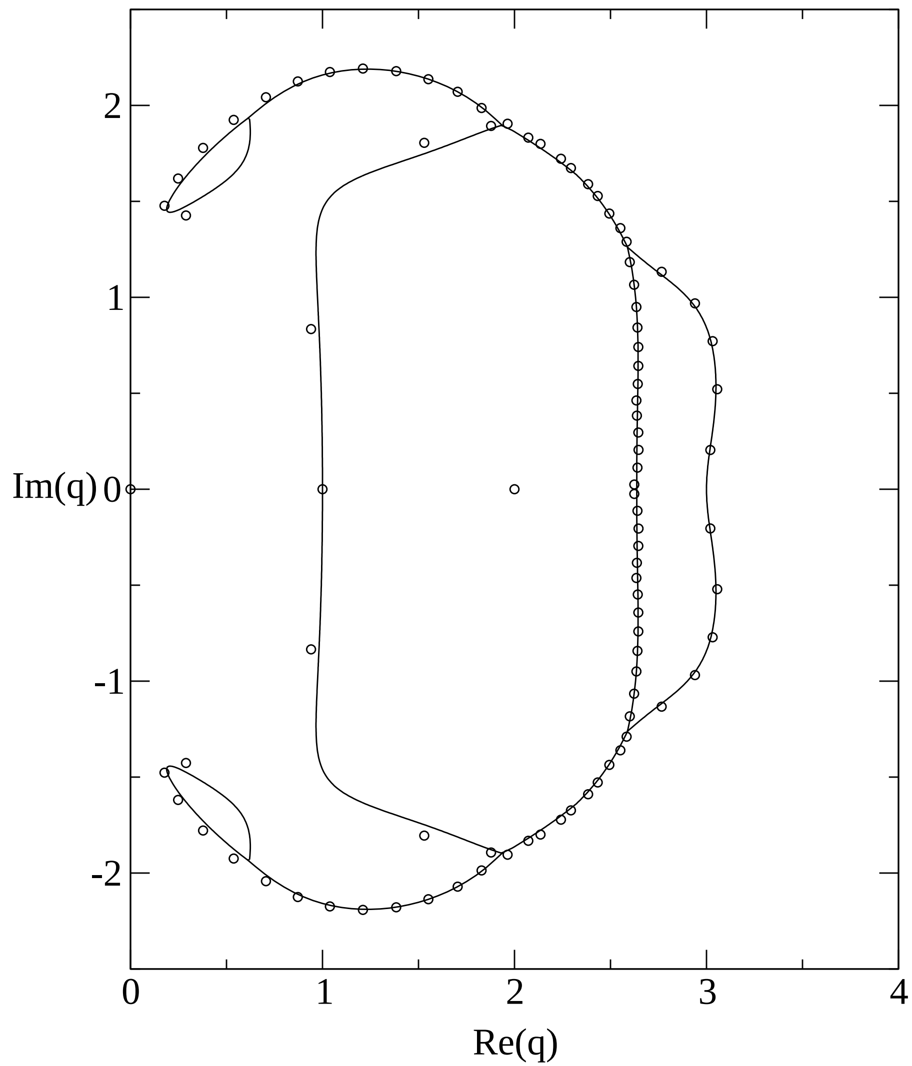


Figure 10: Locus $\mathcal{B}_q$ for the $L_x \to \infty$ limit of $G_D(3 \times L_x)$. For comparison, chromatic zeros are shown for $L_x = 30$, i.e., $n = 91$.

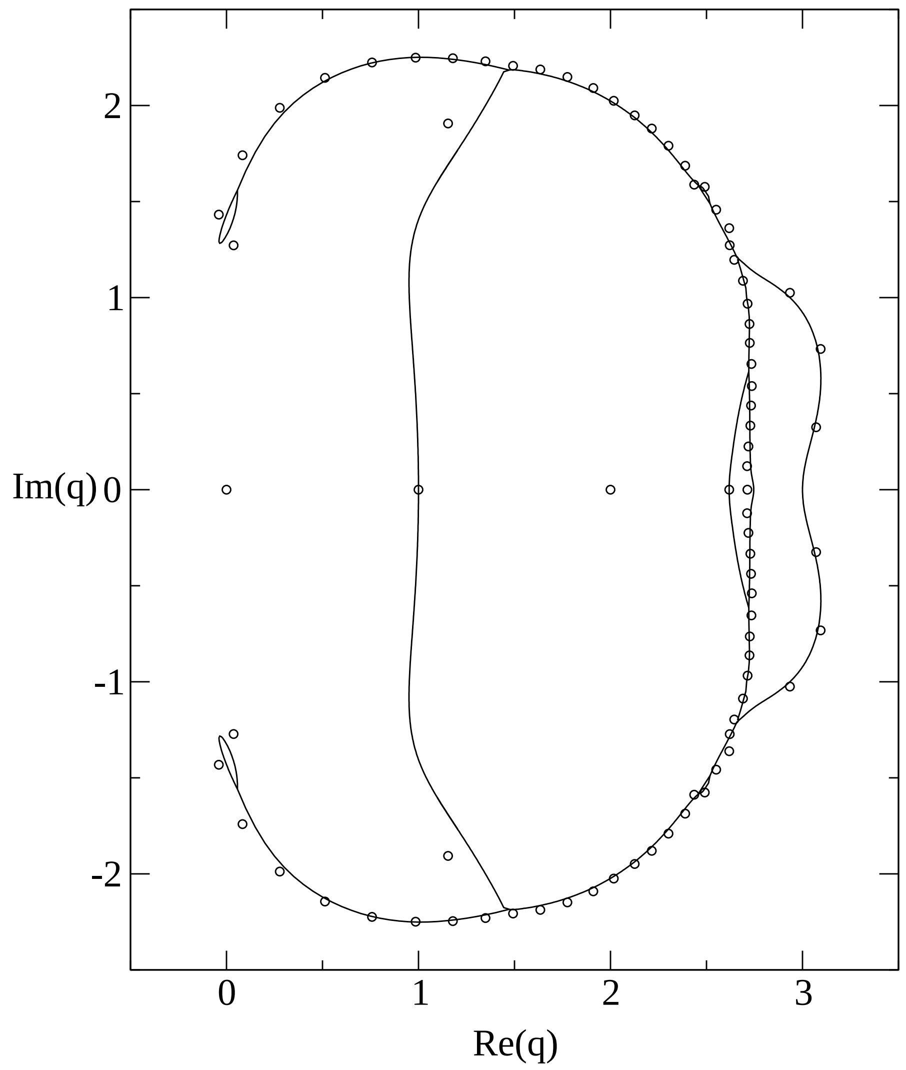


Figure 11: Locus $\mathcal{B}_q$ for the $L_x \to \infty$ limit of $G_D(4 \times L_x)$. For comparison, chromatic zeros are shown for $L_x = 20$, $n = 81$.

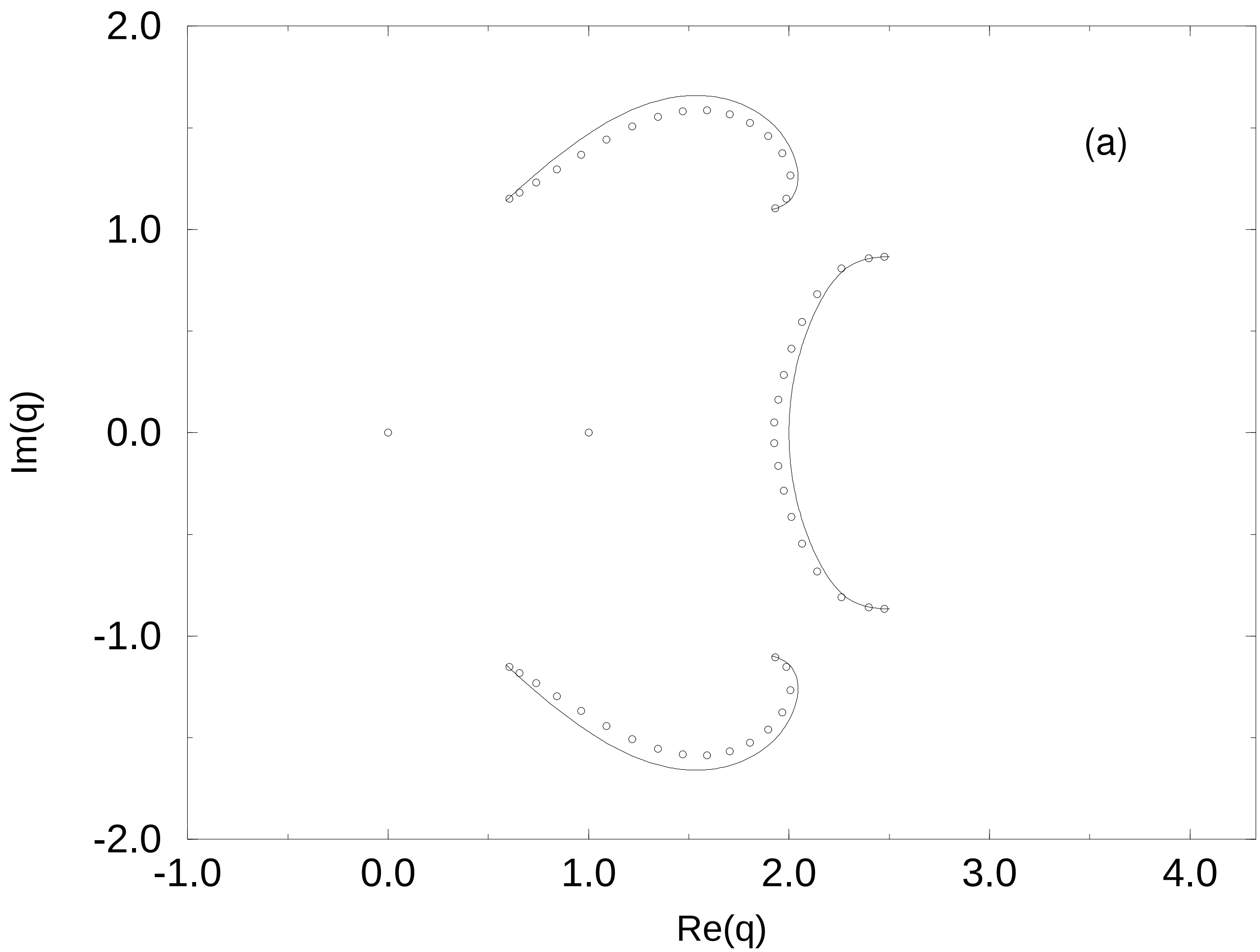


Figure 12: Locus $\mathcal{B}_q$ for the $L_x \to \infty$ limit of the square-lattice strip of width $L_y = 3$ with free boundary conditions, together with chromatic zeros on a finite-L_x strip shown for comparison. From M. Roček, R. Shrock, and S.-H. Tsai, Physica A252, 505 (1998).

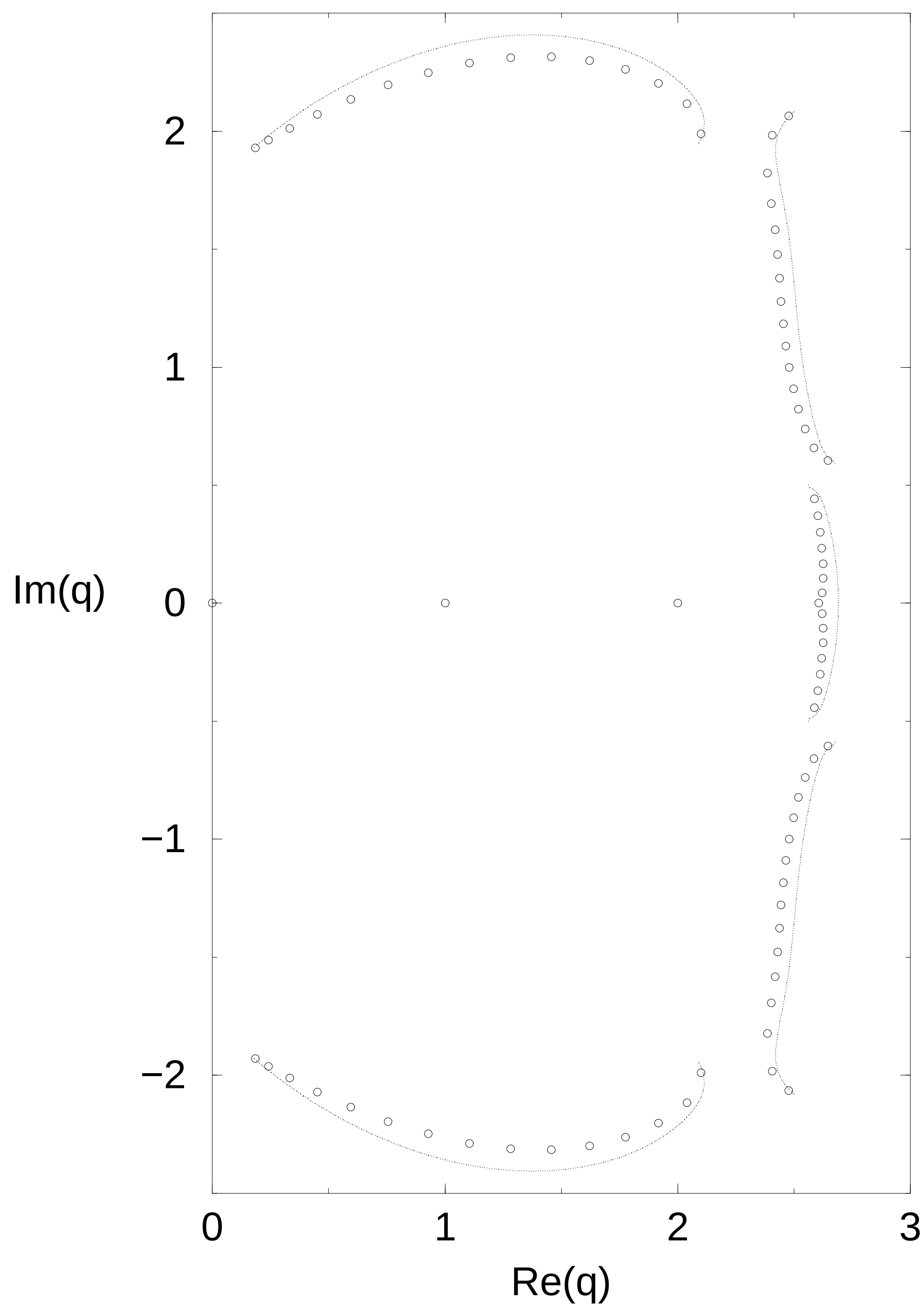


Figure 13: Locus $\mathcal{B}_q$ for the width $L_y = 5$ strip of the square lattice with cylindrical boundary conditions. For comparison, chromatic zeros calculated for the strip length $L_x = 16$, i.e., $n = 80$ vertices are shown. From S.-C. Chang and R. Shrock, Physica A290, 402 (2001).

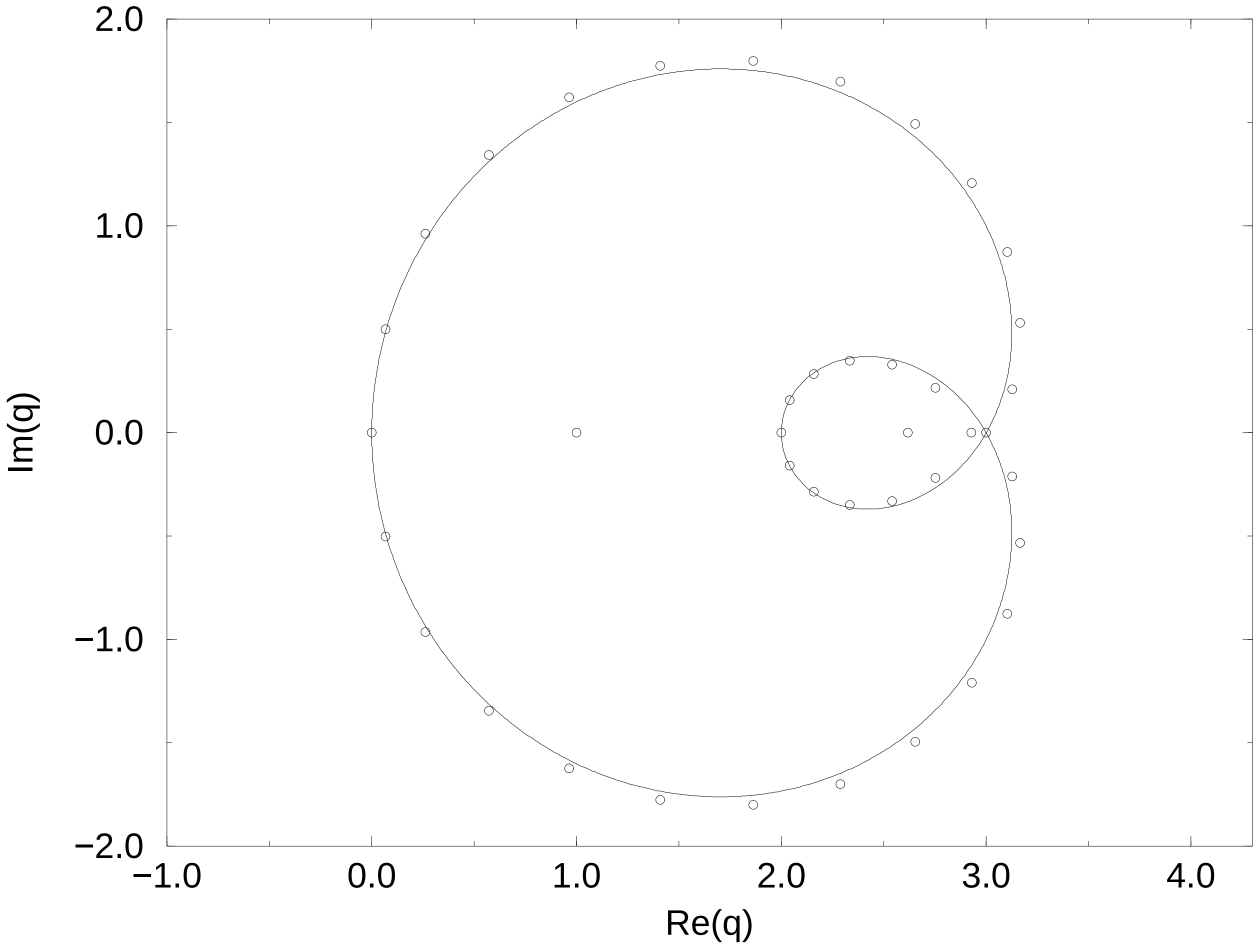


Figure 14: Boundary $\mathcal{B}_q$ for the $L_x \to \infty$ limit of the cyclic or Möbius strip of the triangular lattice with width $L_y = 2$. Chromatic zeros for the cyclic strip with $L_x = 20$ and thus $n = 40$ vertices are shown for comparison. From R. Shrock and S.-H. Tsai, Physica A275, 429 (2000).

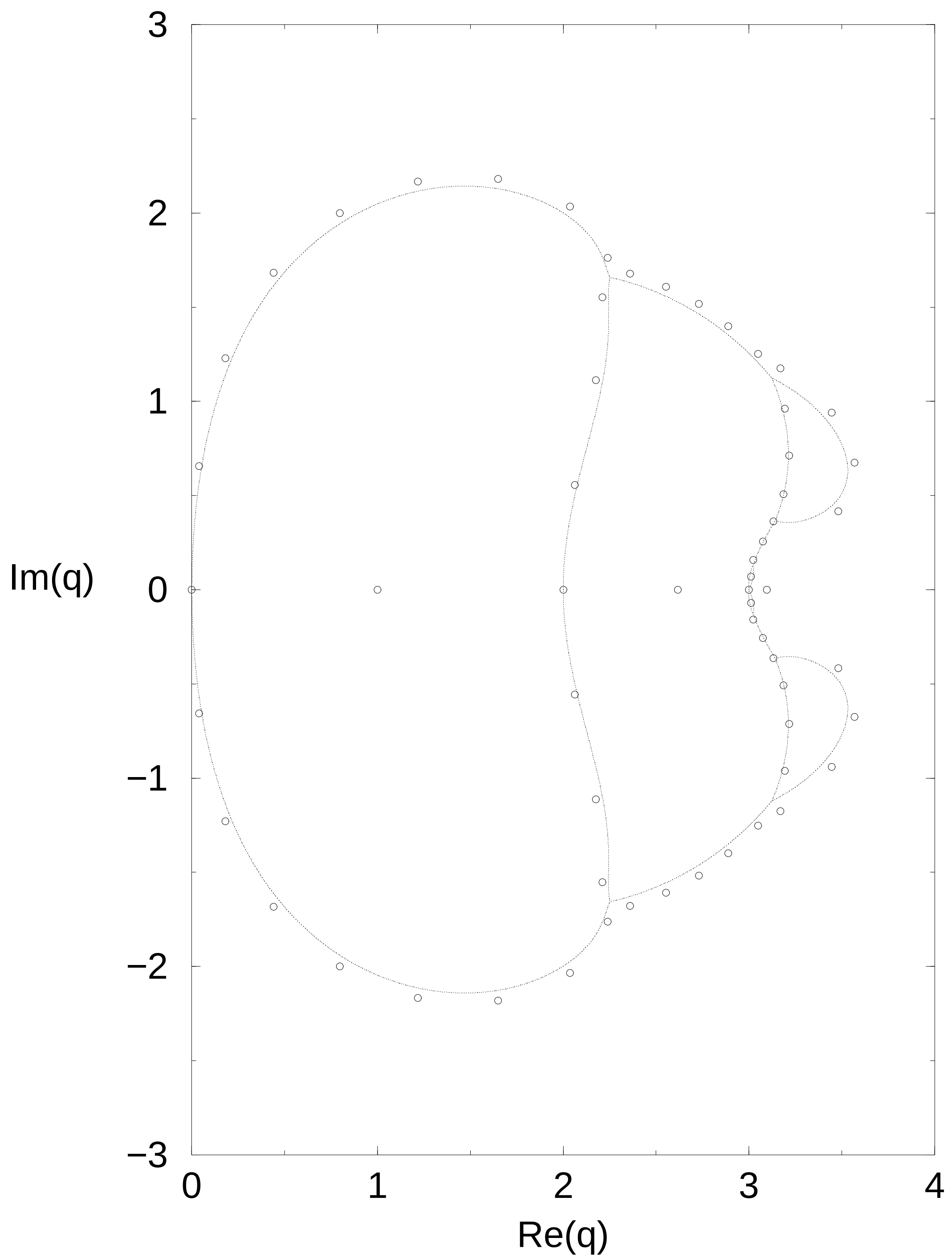


Figure 15: Locus $\mathcal{B}_q$ for the $L_x \to \infty$ limit of the cyclic or Möbius strip of the triangular lattice with width $L_y = 3$. Chromatic zeros for the cyclic strip with $L_x = 20$ and thus $n = 60$ vertices are shown for comparison. From S.-C. Chang and R. Shrock, Ann. Phys. 290, 124 (2001).

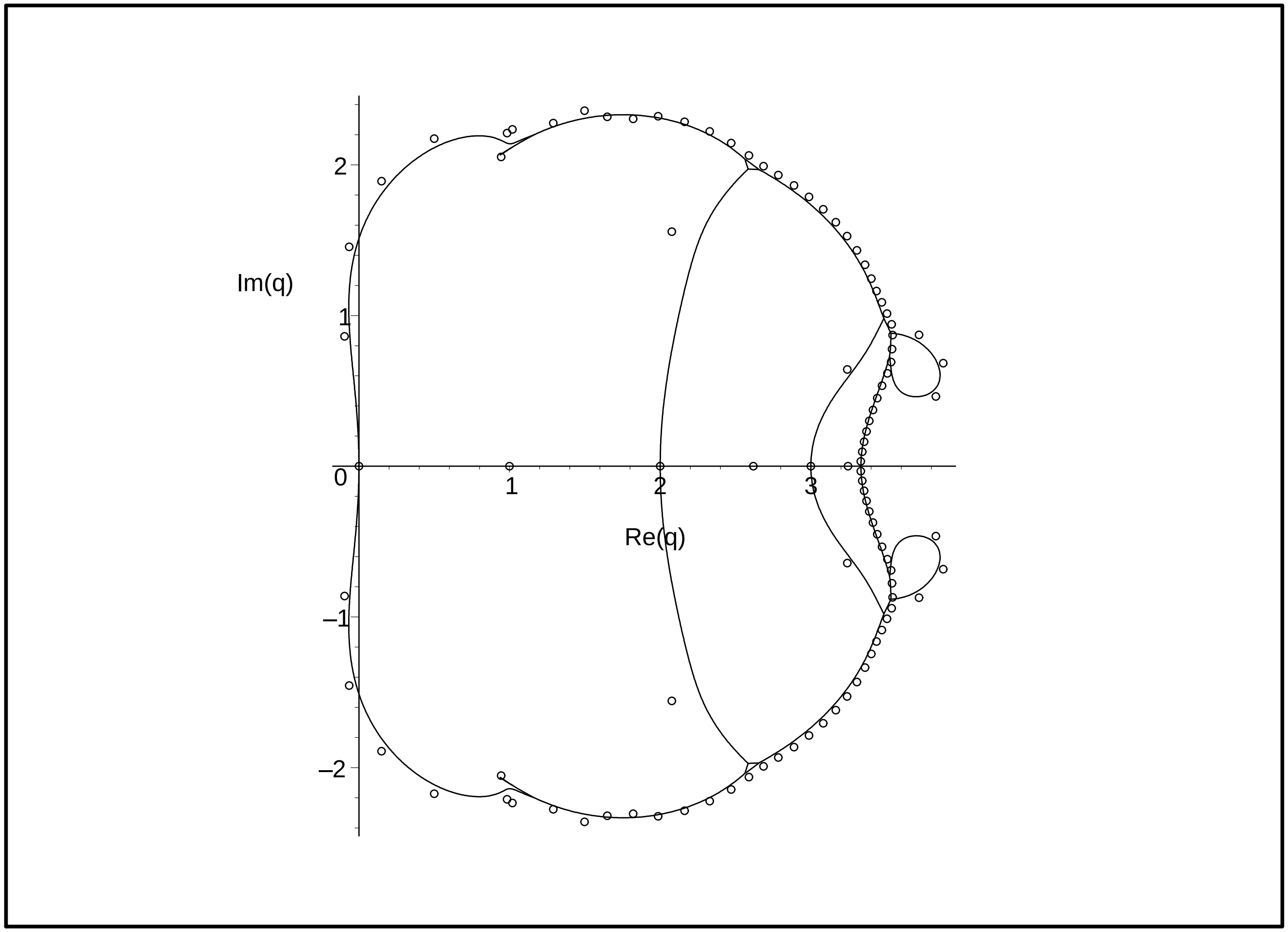


Figure 16: Locus $\mathcal{B}_q$ for the $L_x \to \infty$ limit of the cyclic or Möbius strip of the triangular lattice with width $L_y = 5$. Chromatic zeros for the cyclic strip with $L_x = 20$ and thus $n = 100$ vertices are shown for comparison. From S.-C. Chang and R. Shrock, Physica A 346, 400 (2005).

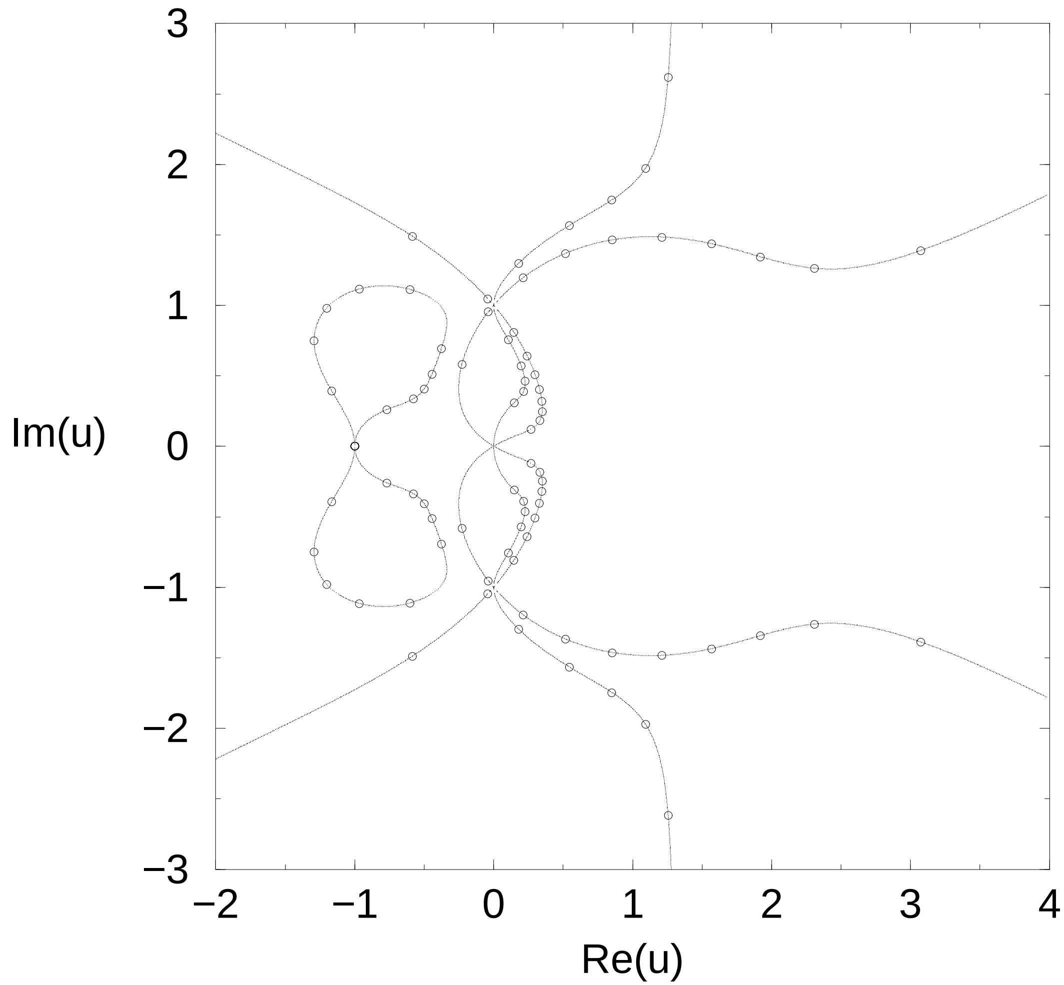


Figure 17: Locus $\mathcal{B}_u$ for the $L_x \to \infty$ limit of the cyclic or Möbius strip of the square lattice with width $L_y = 3$, in the plane of the variable $u = y^{-1}$, with partition function zeros for a finite L_x shown for comparison. From S.-C. Chang and R. Shrock, Physica A 296, 234 (2001).

Potts Model on Hierarchical Lattices

Examples of hierarchical (lattice) graphs G_m: Sierpinski triangles S_m, Diamond hierarchical lattice (DHL) graphs, D_m. In each case, one starts with a given graph G_m and interatively constructs G_{m+1} by a specified procedure of adding vertices and edges; see figure.

In the theory of phase transitions and critical phenomena, the renormalization group (RG) has played a very important role (K. Wilson, M. Fisher, A. Migdal, L. Kadanoff..).

Real-space RG: apply blocking transformation with blocking factor b; calculate effective Hamiltonian after blocking. Iterate. This generates an RG flow in the space of couplings. Look for fixed points of this RG flow and associated basin(s) of attraction; calculate RG trajectories. A second-order phase transition is a nontrivial fixed point of the RG.

An appeal of hierarchical lattice graphs is that because of their self-similar nature, one can describe the RG transformation exactly in closed form.

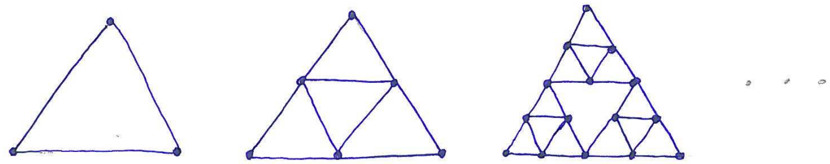
However, spin models on fractals do not exhibit the sort of universality that they do on (thermodynamic limits of) regular lattices.

For example, a discrete spin system (without frustration or disorder) on a regular lattice of dimensionality $d > 1$ exhibits a order-disorder phase transition at $T_c > 0$, but this is not necessarily true on a fractal (Gefen, Mandelbrot, Aharony, 1980). In particular, the $m \to \infty$ limit of the Sierpinski triangular graphs, S_∞, has fractal (Hausdorff) dimension $D = (\ln 3)/(\ln 2) = 1.585 > 1$, but $T_c = 0$.

Nevertheless, it is of interest to study the zeros of Z(S_m, q, v) in the q plane for various v and in the v plane for various q, and we have done this (Chang and Shrock, Phys. Lett. A377, 671-675 (2013))

Among other results, we infer that $q_c(S_\infty) = 3$, which means, from the $\mathbb{C}^2$ perspective, that just as the singular locus $\mathcal{B}_q$ for the $T = 0$ Potts AFM on S_∞ crosses the real q axis at the maximal point $q = 3$, equivalently, the singular locus $\mathcal{B}_y$ passes through $y = 0$, i.e., the $q = 3$ Potts AFM on S_∞ has a zero-temperature critical point.

Sierpinski graphs :



S_1 S_2 S_3

$$n(S_m) = \frac{3(3^{m-1}+1)}{2} \; ; \; e(S_m) = 3^m \; ; \; \chi(S_m) = 3 \; ;$$

average vertex degree $= \kappa_{eff}$:

$$\kappa_{eff} = \frac{2e(G)}{n(G)} \; ; \quad \lim_{m\to\infty} \kappa_{eff}(S_m) = 4$$

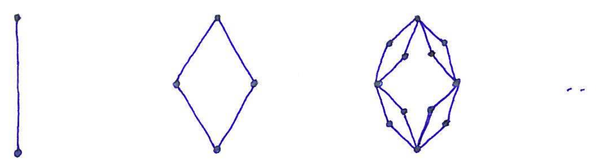
$$P(S_m, 3) = 3!$$
(tripartite)

Iteration : scale factor b, # "copies" $= N$, Hausdorff dimension :

$$N = b^D \; ; \; b=2, \; N=3 \Rightarrow \quad D = \frac{\ln N}{\ln b} = \frac{\ln 3}{\ln 2} = \frac{\log_e 2 \log_2 3}{\log_e 2}$$

$$= \log_2 3 = 1.585$$

Diamond Hierarchical graphs



D_1 D_2 D_3

$$n(D_m) = \frac{2(4^{m-1}+2)}{3} \; ; \; e(D_m) = 4^{m-1}$$

$$\lim_{m\to\infty} \kappa_{eff} = 3 \; ; \; \chi(D_m) = 2 \; ; \; P(D_m, 2) = 2 \text{ (bipartite)}$$

Iteration : $b = 2$, $N = 4 \Rightarrow D = \frac{\ln 4}{\ln 2} = 2$

Figure 18: Sierpinski graphs $\boldsymbol{S_m}$ and Diamond hierarchical lattice graphs $\boldsymbol{D_m}$

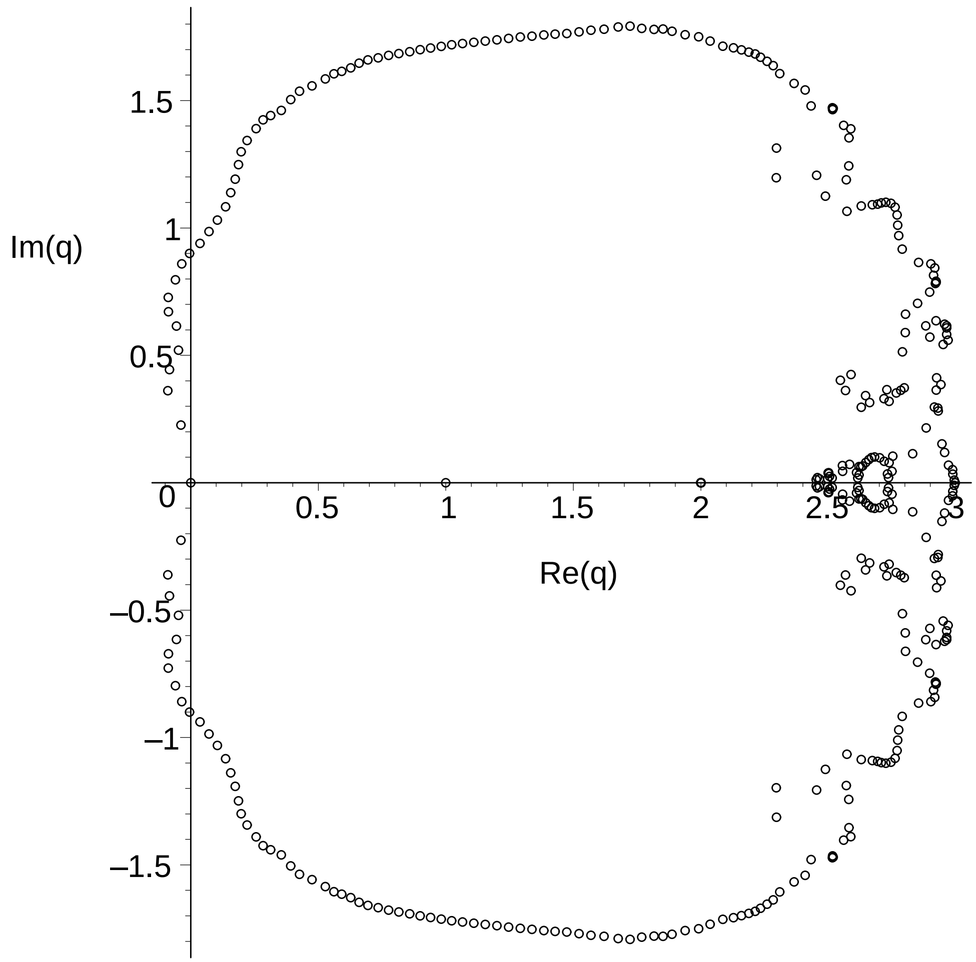


Figure 19: Zeros of the chromatic polynomial $P(S_6, q)$ in the q plane (366 zeros).

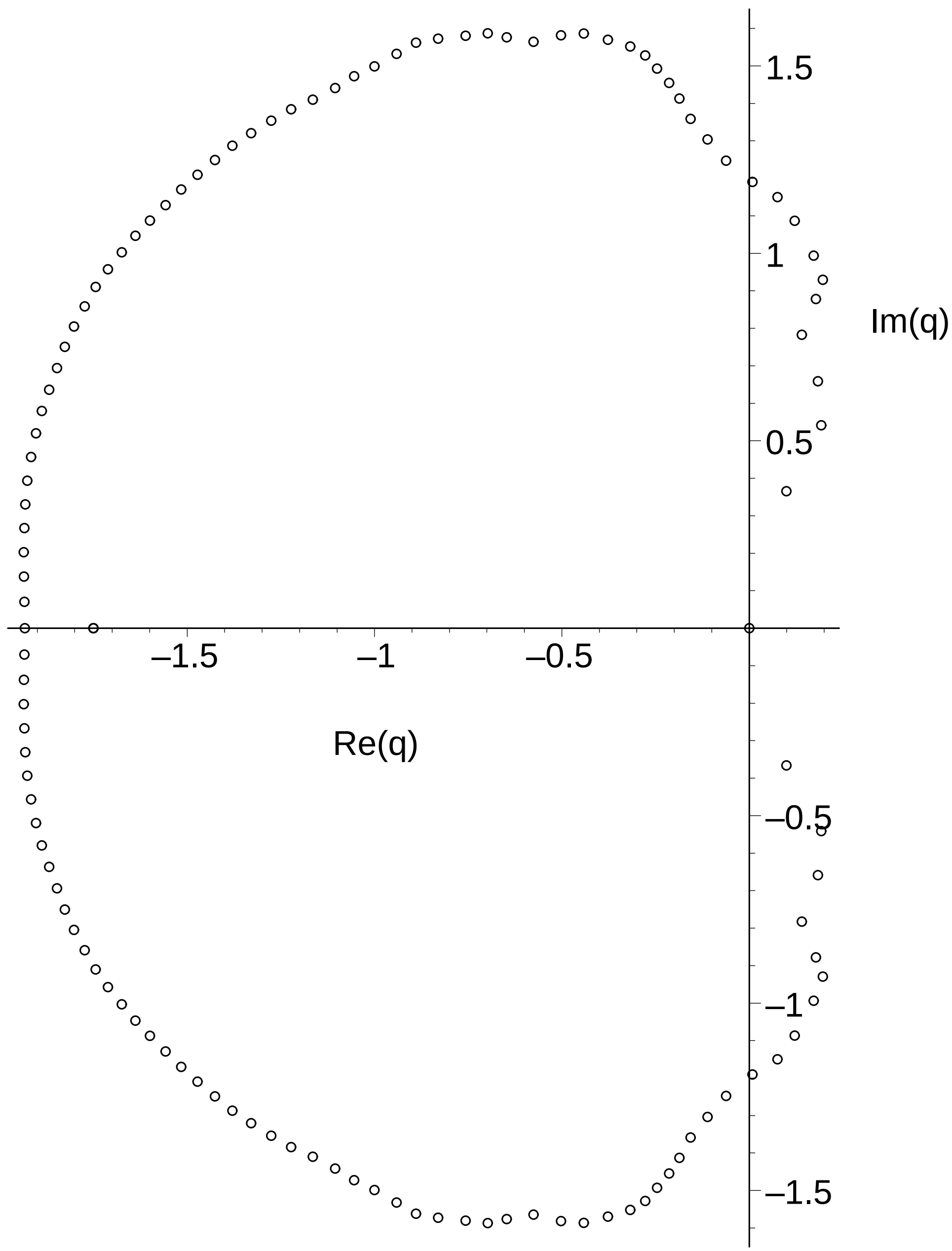


Figure 20: Zeros of $Z(S_5, q, v)$ in the q plane for $v = 0.5$.

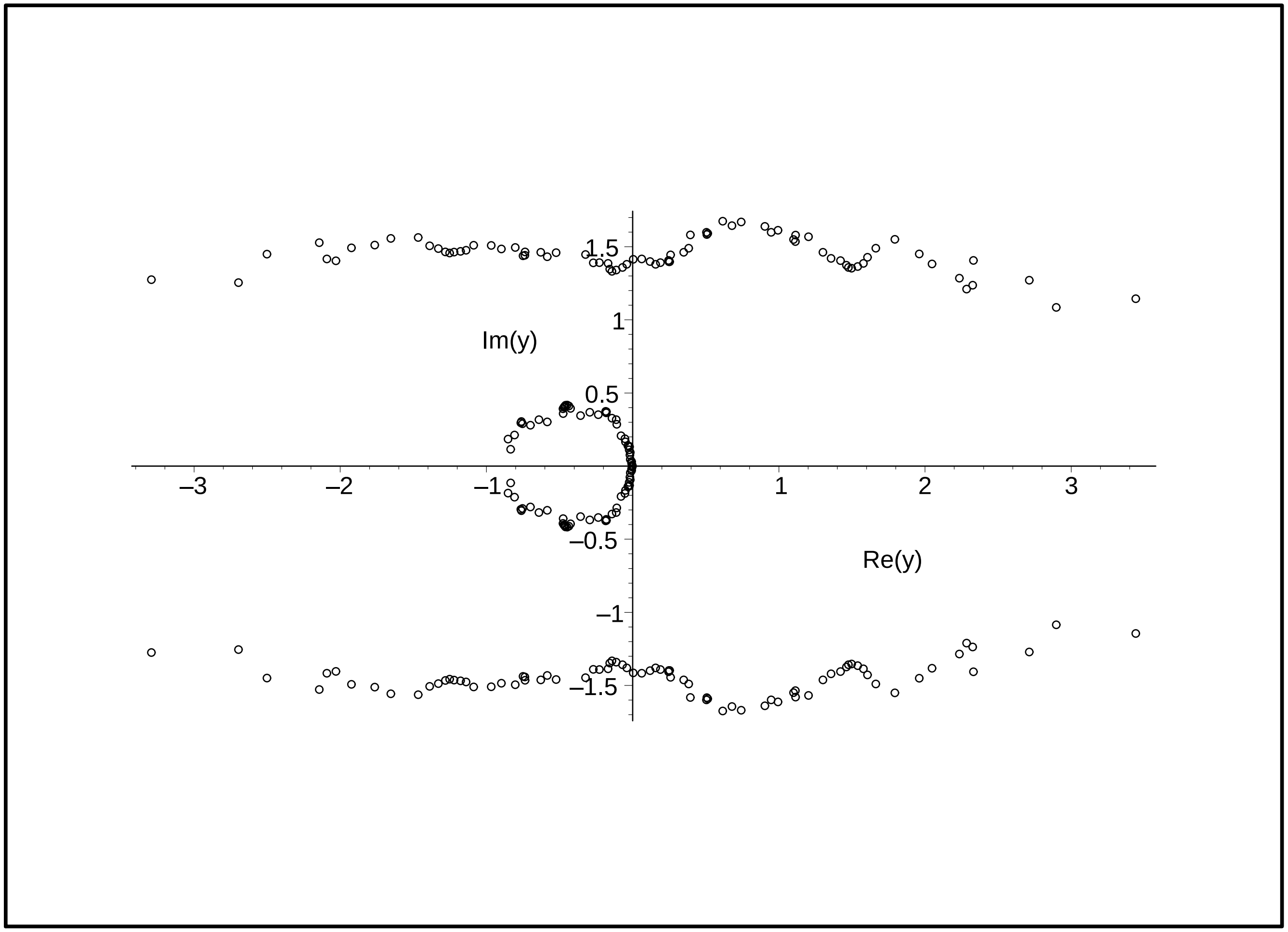


Figure 21: Zeros of $Z(S_5, q, v)$ in the y plane for $q = 3$.

Diamond hierarchical lattice: many studies of zeros in v and μ (complex-temp. and complex-field), including Gefen, Mandelbrot, and Aharony, 1979-1984; Griffiths and Kaufman, 1981-1982; Derrida, De Seze, Itzykson, 1983; Bleher, Zalys, 1989; Bleher, Lyubich, 1991; Qin- Z. Yang, 19921; Qiao, 2001-2011...; Gao-Qiao; Bleher, Lyubich, Roeder: BLR1, Lee-Yang Zeros for DHL, arXiv:1009.4691; BLR2, Lee-Yang-Fisher Zeros for DHL, arXiv:1107.5764.

By carrying out the summation over the spins at intermediate vertices at each stage, one finds the RG transformation

$$Z(D_{m+1}, q, v) = Z(D_m, q, v')(q + 2v)^{2\cdot 4^m}$$

where

$$v' = \frac{v^2(2q + 4v + v^2)}{(q + 2v)^2}$$

or equivalently,

$$y' = \left[\frac{q + y^2 - 1}{q + 2(y - 1)}\right]^2$$

The RG has fixed points where $v' = v$, i.e., $v(q^2 + 2qv - v^3) = 0$; sols.: $v = 0$, i.e., the trivial $\beta = 0$ ($T = \infty$) fixed point, and the PM-FM critical point of the Potts model, v_{PM-FM}, given by physical sol. of cubic eq. $q^2 + 2qv - v^3$:

$$v_{PM-FM} = y_{PM-FM} - 1 = 2^{-1/3}\Big[q^2 + \sqrt{q^3(q-(32/27))}\ \Big]^{1/3} + \frac{2^{4/3}\,q}{3\,\Big[q^2 + \sqrt{q^3(q-(32/27))}\ \Big]^{1/3}}$$

For example, the $q = 2$ Potts (Ising) model has PM-FM phase transition at $y_{PM-FM} = 3.3830$ and, since D_∞ is bipartite, also PM-AFM phase transition at $y_{PM-AFM} = 1/y_{PM-FM} = 0.2956$.

The locus $\mathcal{B}_v$ (equiv. $\mathcal{B}_y$) has been extensively studied (Julia set of RG transformation) by physicists and mathematicians.

Much less attention has been paid to the zeros in the q plane. We are studying these (work in progress with S.-C. Chang and R. Roeder). Example of zeros of $P(D_5, q)$:

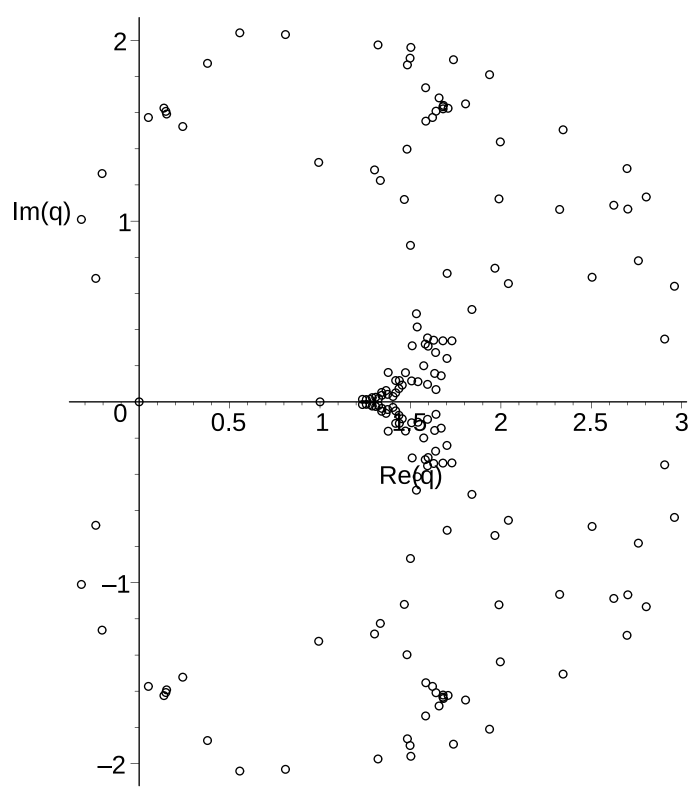


Figure 22: Zeros of the chromatic polynomial $P(D_5, q)$ in the q plane.

Potts Model with General External Magnetic Field

Recall from Lecture 1 that for the $q = 2$ Potts (i.e. Ising) model (with no external magnetic field H), the locus of complex-temperature phase boundaries (accumulation set of zeros) $\mathcal{B}_y$ is the set of two circles $|y \pm 1| = \sqrt{2}$, or equivalently, given the $y \to 1/y$ invariance of this locus, $|z \pm 1| = \sqrt{2}$, where $z = 1/y$ and the limcçon-like curve in the $\eta = y^2$ or $u = z^2 = 1/\eta$ plane.

The free energy f of the square-lattice Ising model can also be calculated for the nonzero value $h = \beta H = \pm i\pi/2$, i.e., $\mu = -1$, where $\mu = e^{2h}$. We have determined the locus $\mathcal{B}_u$ for this case (Matveev and Shrock, J. Phys. A 28, 4859-4882 (1995)). It is the union of the unit circle $|u| = 1$ and the line segment on the negative real u axis with endpoints at u_e and $1/u_e$: $1/u_e < u < u_e$, where $u_e = -(3 - 2\sqrt{2})$:

$$\mu = -1 \ \Rightarrow \ \mathcal{B}_u : \ \{|u| = 1\} \cup \{-(3 + 2\sqrt{2}) < u < -(3 - 2\sqrt{2})\}$$

It is also of interest to study the zeros of the Ising partition function for general nonzero values of H and we have done this (Matveev and Shrock, Phys. Rev. $\mathrm{E}53$, 254-267 (1996); J. Phys. A 41, 135002 (2008); McCoy..) These are not known exactly.

It was known that, in the $n \to \infty$ limit, the continuous accumulation set of zeros in μ (Lee-Yang zeros) of the Ising partition function form a closed unit circle $|\mu| = 1$ as T decreases below the critical temp. T_c, in the interval $0 \le u \le 3 - 2\sqrt{2}$ (sq lattice) and $0 \le u \le 1/3$ (tri lattice).

We have shown that the Lee-Yang zeros remain on the unit circle in the μ plane as $u = e^{-4K_I}$ passes through zero to an interval of negative real values and have determined this interval:

$$-u_{PM-FM,sq} < u \le 0 \quad \text{(sq lattice)}$$

where $u_{PM-FM,sq} = (3 - 2\sqrt{2})$, and

$$-u_{PM-FM,tri} < u \le 0 \quad \text{(tri lattice)}$$

where $u_{PM-FM,tri} = 1/3$ for the triangular lattice. (Matveev and Shrock, Phys. Lett. $\mathbf{A215}$, 271-279 (1996)).

As with zeros of Z and their accumulation sets in the q and v plane, it is also of interest to study zeros of Z in the v (or z) and μ planes in a unified manner: zeros in a $\mathbb{C}^2$ space.

A further generalization is to consider the Potts model in a generalized external magnetic field that favors or disfavors spin values in a subset $I_s = \{1, ..., s\}$ of the total set of q possible spin values. Thus, let $H_p = H$ for $1 \le p \le s$ and $H_p = 0$ for $s+1 \le p \le q$. Define $h = \beta H$ and $w = e^h$.

The partition function is $Z = \sum_{\sigma_i} \exp(-\beta \mathcal{H})$, where

$$-\beta \mathcal{H} = K \sum_{e_{ij}} \delta_{\sigma_i \sigma_j} + h \sum_i \sum_{p=1}^{s} \delta_{\sigma_i, p}$$

It is desirable to obtain a generalization that expresses the partition function of the Potts model in a general magnetic field partition function in a purely graph-theoretic manner as a sum over spanning subgraphs without reference to the spin configurations. We have done this (Chang and Shrock, J. Phys. A42, 385004 (2009); J. Stat. Phys. 138, 496 (2010); Shrock and Xu, J. Stat. Phys. 139, 27 (2010)).

We obtain

$$Z(G,q,s,v,w) = \sum_{G' \subseteq G} v^{e(G')} \prod_{i=1}^{k(G')} (q - s + sw^{n(G'_i)})$$

where G' has $k(G')$ components, denoted G'_i, $i = 1, ...k(G')$. The factor $q - s + sw^m$ for each connected component $G'_i \in G'$ can be understood since each of the spins must have the same value σ in this term in the cluster sum; for the s cases where $\sigma \in I_s$, this gives $sw^{n(G'_i)}$, while for the $q - s$ cases where $\sigma \notin I_s$, this gives $q - s$.

In the quantity $u_m \equiv q - s + sw^m = q + s(w^m - 1)$ note that

$$w^m - 1 = (w - 1) \sum_{j=0}^{m-1} w^j$$

so s occurs in Z only in the combination $s(w - 1)$. Hence, $s = 0$ is equivalent to $w = 1$, i.e., $H = 0$, and for these values, $Z(G,q,s,v,w)$ reduces to the zero-field Potts model:

$$Z(G,q,0,v,w) = Z(G,q,s,v,1) = Z(G,q,v)$$

Clearly, if $s = q$, then all spin states are treated equally, so

$$Z(G, q, q, v, w) = w^n Z(G, q, v)$$

For $H \to -\infty$, i.e., $w \to 0$,

$$Z(G, q, s, v, 0) = Z(G, q - s, v)$$

Switching $s \to q - s$ yields the identity

$$Z(G, q, s, v, w) = w^n Z(G, q, q - s, v, w^{-1})$$

The special case $v = -1$ is a weighted-set generalization of the chromatic polynomial, $Ph(G, q, s, w)$, describing a proper q-coloring of the vertices of G with a vertex weighting w that either favors (for $w > 1$) or disfavors (for $0 \le w < 1$) the colors in the set I_s.

We have used the cluster formula to compute $Z(G, q, s, v, w)$ and $Ph(G, q, s, w)$ for various families of graphs (Chang and Shrock, op. cit. Shrock and Xu, J. Stat. Phys. 139, 27 (2010); J. Stat. Phys., 141, 909 (2010); Chang and Shrock, J. Stat. Phys. 161, 915 (2015)).

We have given results for various boundary conditions. In particular, we have found that for a cyclic strip graph of a lattice Λ of width L_y and length $L_x = m$ vertices,

$$Z([\Lambda, L_y \times m, cyc.], q, s, v, w) = \sum_{d=0}^{L_y} \tilde{c}^{(d)}(q) \sum_{j=1}^{n_{Zh}(L_y,d,s)} [\lambda_{Z,\Lambda,L_y,d,j}(q,s,v,w)]^m \ ,$$

where

$$\tilde{c}^{(d)}(q) \equiv c^{(d)}(\tilde{q}) = \sum_{j=0}^{d} (-1)^j \binom{2d-j}{j} \tilde{q}^{\,d-j}$$

and $\tilde{q} = q - s$, so, e.g., $\tilde{c}^{(1)}(q) = \tilde{q} - 1 = q - s - 1$,
$\tilde{c}^{(2)}(q) = \tilde{q}^{\,2} - 3\tilde{q} + 1 = q^2 - 2qs + s^2 + 3(s-q) + 1$, etc.

These results provide exact solutions of the Potts model in a general magnetic field for arbitrary q on these families of graphs.

These results are also of interest for mathematical graph theory. We have shown that $Z(G,q,s,v,w)$ can distinguish between graphs that yield the same zero-field Z (Tutte-equivalent graphs), such as different n-vertex tree graphs, T_n, for which

$$Z(T_n,q,v) = q(q+v)^{n-1}, \quad \text{equiv.} \;\; T(T_n,x,y) = x^{n-1}$$

e.g., there are two types of tree graphs with $n = 4$: the line graph L_4 and the star graph S_4. These have the same Potts/Tutte polynomial. But our $Z(G,q,s,v,w)$ can distinguish between them (recall: $u_m = q - s + sw^m$):

$$Z(S_4,q,s,v,w) = u_1^4 + 3vu_2u_1^2 + 3v^2u_3u_1 + v^3u_4 \ ,$$

$$Z(L_4,q,s,v,w) = u_1^4 + 3vu_2u_1^2 + v^2(2u_3u_1 + u_2^2) + v^3u_4 \ .$$

so

$$Z(S_4,q,s,v,w) - Z(L_4,q,s,v,w) = v^2(u_3u_1 - u_2^2) = v^2s(q-s)w(w-1)^2 \ .$$

Some Open Problems and Directions for Further Research

There are many open problems and directions for further research. These include the following:

1. Further calculations of $Z(G, q, v)$ and $P(G, q)$ for various families of graphs, with analysis of zeros and the accumulation set $\mathcal{B}_q$ for fixed v.
2. We have calculated S_0 for various infinite-length, finite-width lattice strips; extend these calculations to larger widths and study the approach to $L_y = \infty$ further.
3. Study properties of the $\mathcal{B}_q$, e.g., topology and q_c values for various infinite-length limits of lattice strips.
4. We have found that, for the infinite-length limit of a lattice strip graph with arbitrary transverse boundary conditions (or more generally, a necklace graph), a sufficient condition for $\mathcal{B}_q$ to separate the complex q plane into regions and cross the real axis is that the longitudinal b.c. should be periodic or twisted periodic (cyclic, Möbius, toroidal, or Klein-bottle). Prove this in general.
5. It may happen for some families of graphs that even when the longitudinal boundary conditions are not periodic or twisted periodic, $\mathcal{B}_q$ still separates the complex q plane into regions. Carry out a general characterization of these families of graphs.

6. In all of the infinite-length limits of families of graphs with cyclic boundary conditions, we have found that q_c is a nondecreasing function of L_y. Prove this in general. Note that we have shown that this monotonicity does not hold for infinite-length limits of graphs with toroidal or Klein-bottle boundary conditions.
7. For the infinite-length limits of families of graphs for which $\mathcal{B}_q$ does separate the complex q plane into several regions, give a precise characterization of the number of regions. This amounts to a problem in algebraic geometry, but entails considering not just one algebraic equation, but a set of them, since $\mathcal{B}_q$ is determined only by the dominant λ's, and these change from point to point in the q plane.
8. Extend calculations of $\mathcal{B}_v$ for general q and $\mathcal{B}$ with $\phi(q, v) = 0$.
9. Extend the study of $Z(G, q, s, v, w)$ for Potts model partition function in a generalized magnetic field.
10. The zeros in μ do not lie on the unit circle for the Ising AFM and hence are not described by the Lee-Yang circle theorem. Determine what can be said in general concerning their location.
11. Give a rigorous proof of our inference that $q_c(S_\infty) = 3$ for the $m \to \infty$ limit of the Sierpinski triangle family of graphs.
12. Elucidate properties of zeros and their accumulation sets in q, y, and μ planes for Potts model on $m \to \infty$ limits of hierarchical graphs.

Conclusions

The equivalence between the Potts model partition function and the Tutte polynomial, and the identity of the partition function of the $T = 0$ Potts antiferromagnet with the chromatic polynomial show deep connections between statistical physics, combinatorics, and graph theory.

We have discussed how one can obtain exact calculations of the Potts model partition function $Z(G, q, v)$ for arbitrary q and temperature variable v and equivalent Tutte polynomial $T(G, x, y)$ for arbitrary x and y on recursive families of graphs, and have presented some of the results.

Similarly, we have presented results for the chromatic polynomial on various families of graphs and discussed the asymptotic limiting W function, with connection to ground-state entropy of the Potts antiferromagnet.

Since our calculations apply for arbitrarily great strip length L_x, we have used them to obtain limiting functions for $L_x \to \infty$ and have studied analytic properties of these functions. We have studied patterns of zeros of these polynomials and how they merge to form loci $\mathcal{B}_q$ and $\mathcal{B}_v$.

We have presented a generalization for the Potts model in an external magnetic field.

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