Lee-Yang zeros for the Diamond Hierarchical Lattice.

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Olivier Remy - IUPUI

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Outline

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- Ising model
  1. Partition Function, Lee-Yang zeros, and thermodynamic limit
  2. Expected properties for the $\mathbb{Z}^2$ lattice.
  3. Hierarchical lattices and the Migdal-Kadanoff RG equations
  4. Renormalization Mapping of the Lee-Yang cylinder
- Statement of the main results
  1. Dynamical results
  2. Physical results
- Proof of horizontal expansion

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For any configuration of spins $\sigma : V \to \{\pm 1\}$, we have:

$$I(\sigma) = \sum_{(v,w)\in E} \sigma(v)\sigma(w) \qquad M(\sigma) = \sum_{v\in V} \sigma(v)$$

$I(\sigma)$ is interaction of σ along edges, and
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The energy of state σ exposed to an external magnetic field h is:

$$H(\sigma) = -J \cdot I(\sigma) - h \cdot M(\sigma),$$

where $J > 0$.

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$Z(h, T)$ is called the Partition function.
It governs the physical properties of the Ising model on Γ.

Change of variables

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Since $I(-\sigma) = I(\sigma)$ and $M(-\sigma) = -M(\sigma)$ we have that Z is symmetric under $z \mapsto 1/z$:

$$a_i(t) = a_{-i}(t)$$

Physical values of $T > 0$ correspond to $t \in (0,1)$, and the physical values of $h \in \mathbb{R}$ correspond to $z \in (0, \infty)$.

Thermodynamic quantities in terms of zeros of $Z(z, t)$.

For each $t \in \mathbb{C}^*$ $Z(z, t) = 0$ has $2|E|$ zeros $z_i(t) \in \mathbb{C}$.

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Free energy:

$$F(z,t) := -T \log Z(z,t) = -T \sum \log |z - z_i(t)| + |E| T (\log |z| + \frac{1}{2} \log |t|)$$

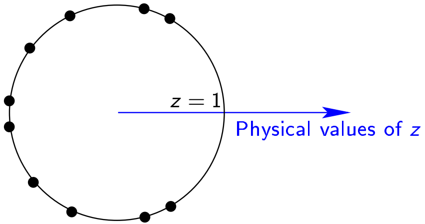
Magnetization:

$$M(z,t) := \sum_\sigma M(\sigma) P(\sigma) = z \sum \frac{1}{z - z_i(t)} - |E|$$

The Lee-Yang Theorem

Theorem (Lee-Yang, 1952)

At any fixed $t \in [0,1]$, *then all complex zeros of* $Z(z,t)$ *lie on the unit circle* $|z| = 1$.



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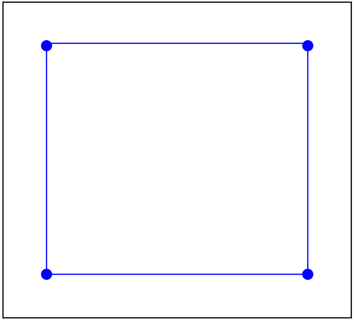

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Model magnetic material: DHL

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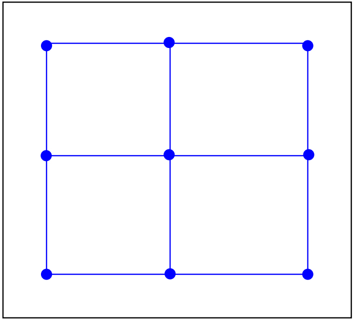

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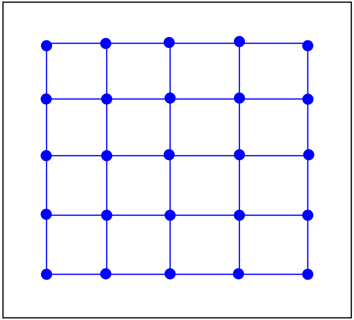

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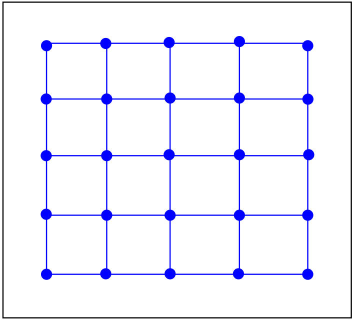

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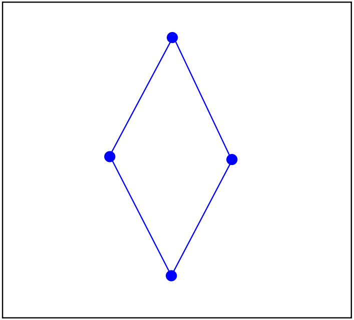

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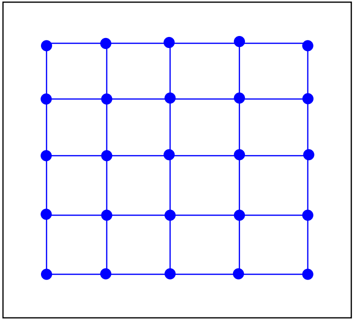

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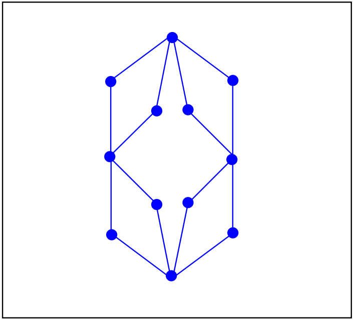

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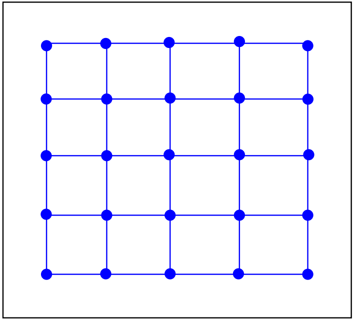

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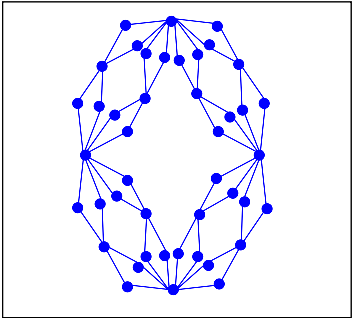

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For each $t \in [0,1]$ there is a measure μ_t on $\mathbb{T}$ describing the asymptotic distribution of Lee-Yang zeros.

Phase transitions in terms of Lee-Yang distribution

If the thermodynamic limit exists, one can define the physical quantities for the limiting model.

$$F(z,t) \quad = \quad -2T\int_{\mathbb{T}} \log|z-\zeta|d\mu_t(\zeta) + T\log|z| + \frac{1}{2}\log|t|$$

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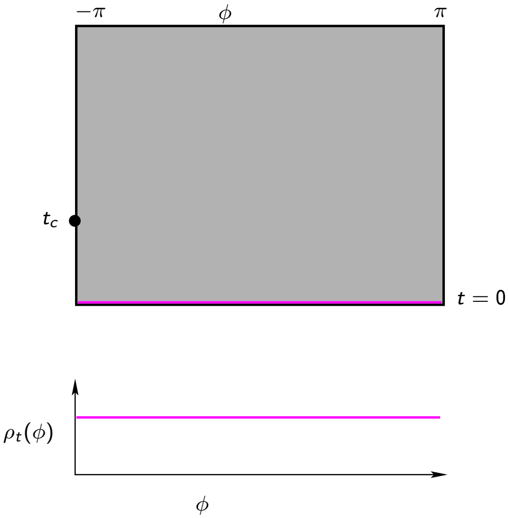
Understanding how the Lee-Yang distributions $\mu_t(\phi)$ vary with t and ϕ is essential to understanding phase transitions of the model.

Expected limiting distributions of Lee-Yang zeros for $\mathbb{Z}^2$



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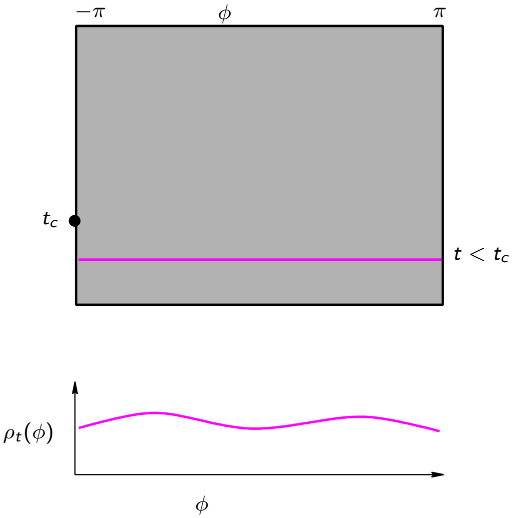

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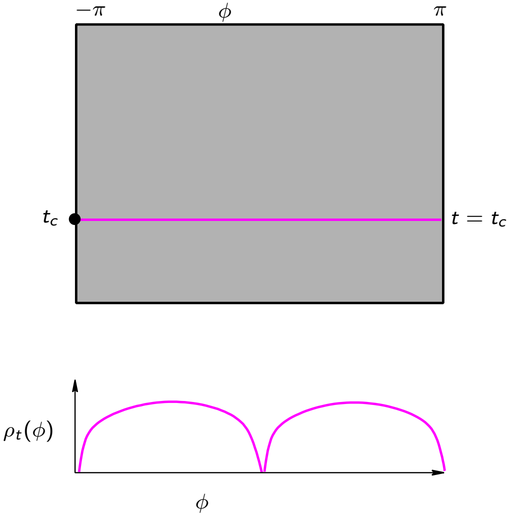

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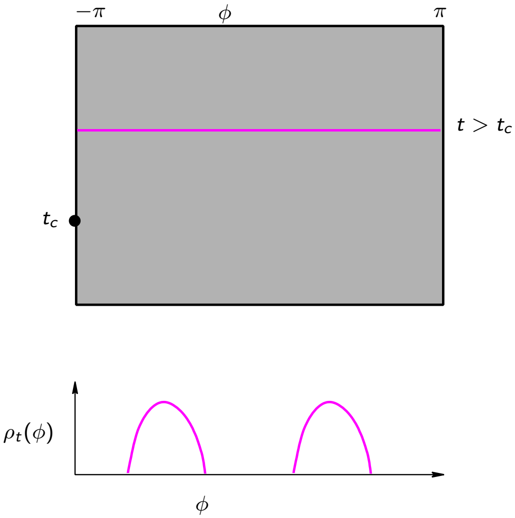

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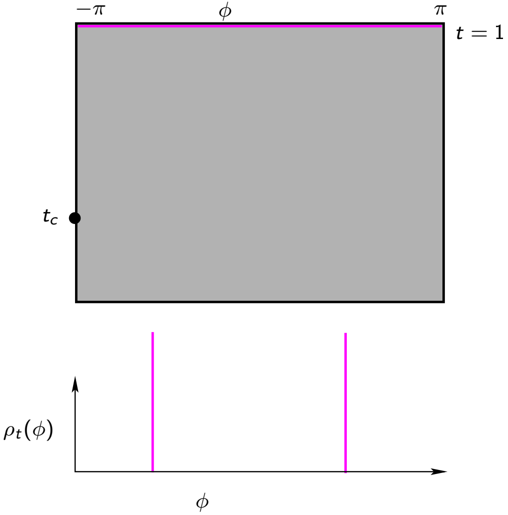
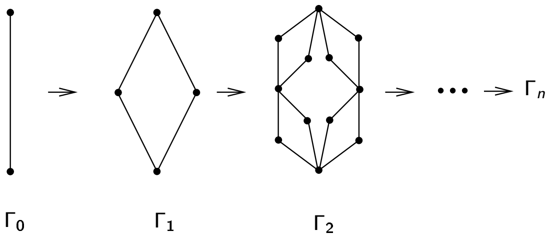
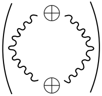
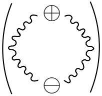
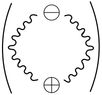
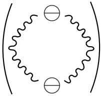


The Diamond Hierarchical Lattice (DHL)



Migdal-Kadanoff Renormalization[1][2][3]

Consider the conditional partition functions:

$U_n := Z_n$ (), $V_n := Z_n$ () $= Z_n$ (), $W_n := Z_n$ ()

The total partition function is equal to $Z_n = U_n + 2V_n + W_n$.

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[2] L. P. Kadanoff. Notes on Migdal's recursion formulae. *Ann. Phys.*, (1976).

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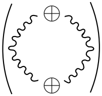
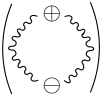
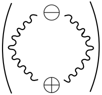
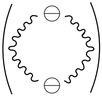
$U_n := Z_n$ , $V_n := Z_n$  $= Z_n$ , $W_n := Z_n$ 

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Migdal-Kadanoff RG Equations:

$$U_{n+1} = (U_n^2 + V_n^2)^2, \quad V_{n+1} = V_n^2(U_n + W_n)^2, \quad W_{n+1} = (V_n^2 + W_n^2)^2.$$

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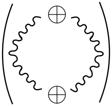
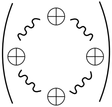
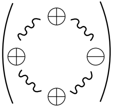
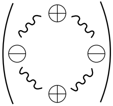
$$U_{n+1} = Z_{n+1}$$ 

$$= Z_{n+1}$$  $$+ 2Z_{n+1}$$  $$+ Z_{n+1}$$ 

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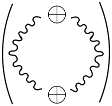
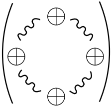
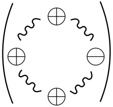
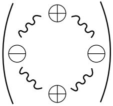
$U_{n+1} = Z_{n+1}$ ()

$= Z_{n+1}$ () $+ 2Z_{n+1}$ () $+ Z_{n+1}$ ()

$$= \quad U_n^4 \quad + \quad 2U_n^2\, V_n^2 \quad + \quad V_n^4 .$$

$$R : \mathbb{C}^3 \to \mathbb{C}^3,\ (U, V, W) \mapsto ((U^2+V^2)^2, V^2(U+W)^2, (V^2+W^2)^2)$$

MK renormalization in the (z, t) coordinates:

We can lift R from the $[U : V : W]$ coordinates (downstairs) to the (z, t) coordiantes upstairs.

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The (z, t) coordinates can be seen as affine coordinates of $[z : t : 1]$.

$$\begin{array}{ccc} \mathbb{CP}^2 & \xrightarrow{\mathcal{R}} & \mathbb{CP}^2 \\ \downarrow \Psi & & \downarrow \Psi \\ \mathbb{CP}^2 & \xrightarrow{R} & \mathbb{CP}^2 \end{array} \tag{2}$$

and Ψ is a degree 2 rational map.

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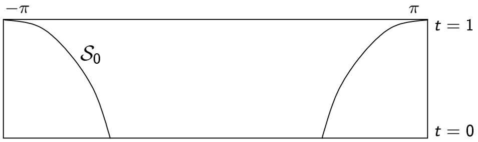
It is this recursive relationship between $\mathcal{S}_{n+1}$ and $\mathcal{S}_n$ that makes this problem become a dynamical systems problem.

Lee-Yang zeros as pull-backs under $\mathcal{R}$



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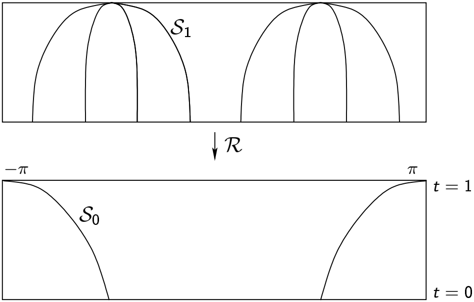

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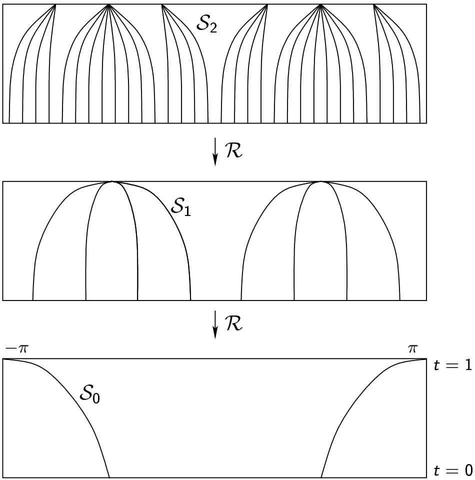

Geometry of $\mathcal{R} : \mathcal{C} \to \mathcal{C}$, part I

$\mathcal{R}$ has two points of indeterminacy $\alpha_\pm = (\pm i, 1) \in \mathcal{T}$.

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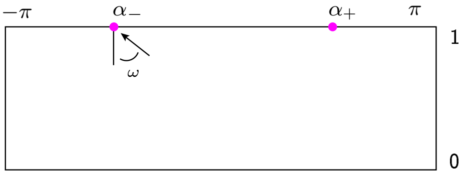

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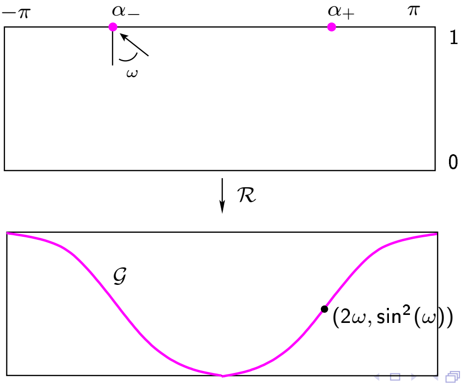

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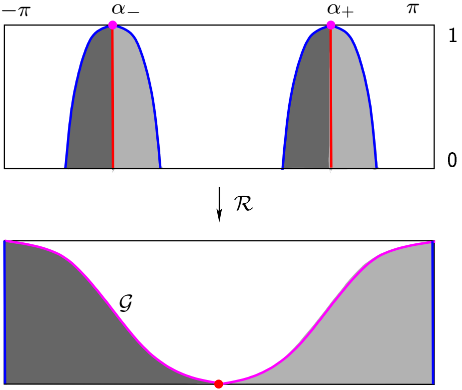

Dynamical results I

Theorem (Bleher, Lyubich, Roeder)

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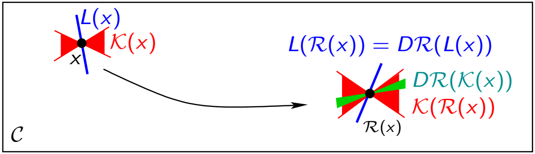
That is:

1. We have a horizontal tangent conefield $\mathcal{K}(x)$ and a vertical linefield $L(x) \subset T_x\mathcal{C}$ depending continuously on x and invariant under $D\mathcal{R}$:



2. Horizontal tangent vectors $v \in \mathcal{K}(x)$ get exponentially stretched under $D\mathcal{R}^n$ at a rate that dominates any occasional expansion of tangent vectors in $L(x)$.

Dynamical results II

Proposition (BLR)

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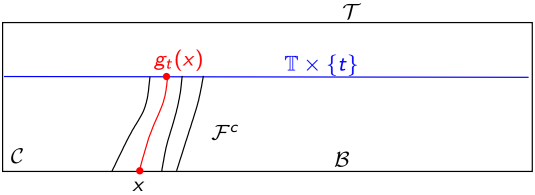
Precisely:

A vertical foliation is a regular family of disjoint vertical paths that cover the cylinder. « Central » means that the foliation is obtained integrating $L(x)$.

One can think of a vertical foliation as a local deformation of the genuinely vertical foliation, $\{I_\phi,\ \phi \in \mathbb{R}/2\pi\mathbb{Z}\}$

Physical Results

For $t \in [0,1)$ the holonomy transformation $g_t : \mathcal{B} \to \mathbb{T} \times \{t\}$ obtained by flowing along $\mathcal{F}^c$.



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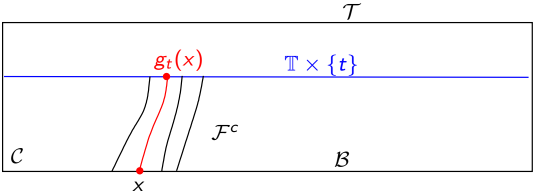
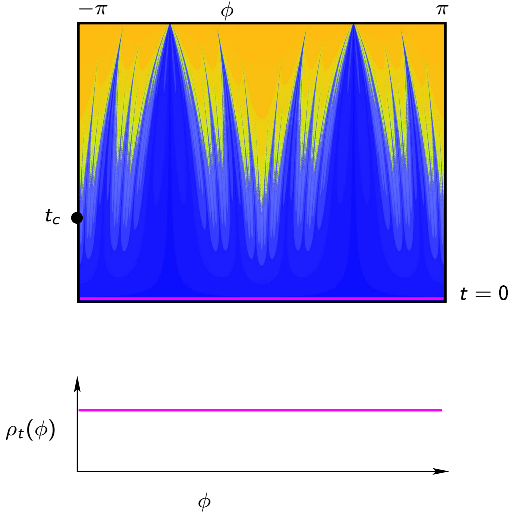


Theorem (BLR)

The asymptotic distribution of Lee-Yang zeros at a temperature $t_0 \in [0,1)$ is given by under holonomy by $\mu_t = (g_t)_*(\mu_0)$ where μ_0 be the Lebesgue measure on $\mathcal{B}$.

Geometric view of Lee-Yang distributions for the DHL



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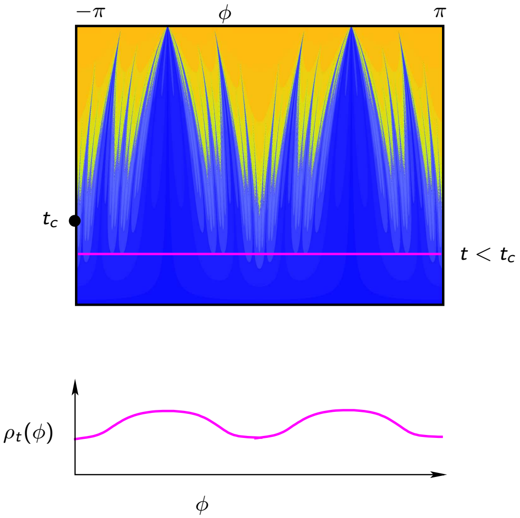

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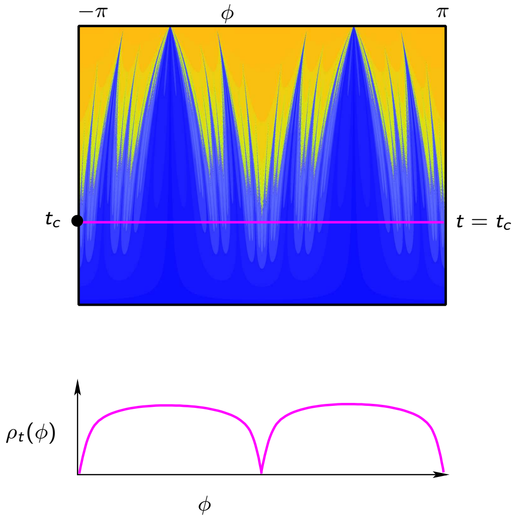

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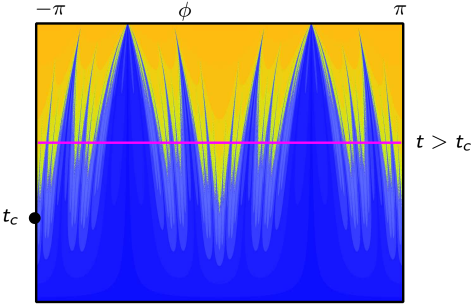

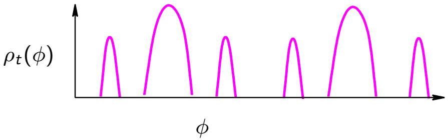

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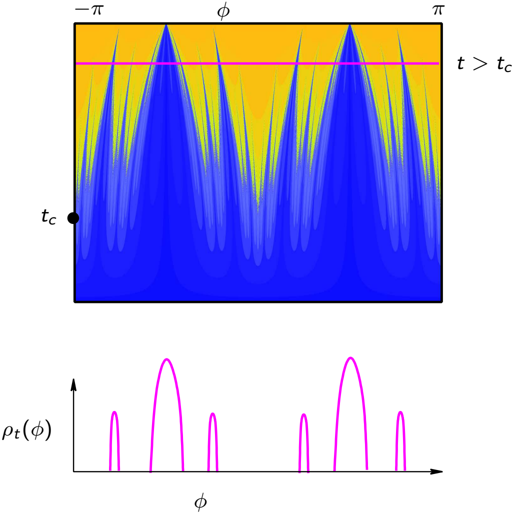

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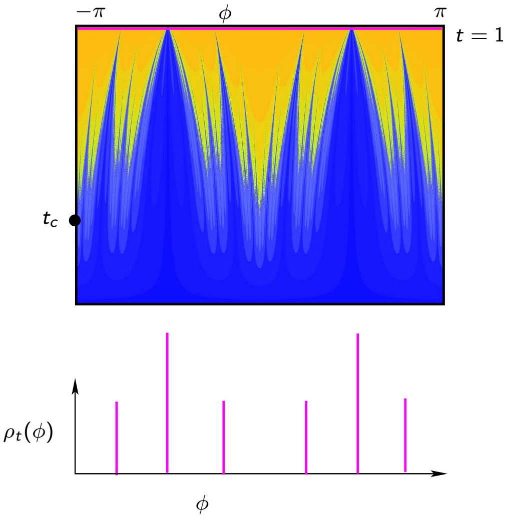

Horizontal expansion (the main part of the proof)

Proposition

$\mathcal{R}$ expands the genuinely horizontal direction by a factor of at least 2. Precisely, there exists $c > 0$ such that:

$$\forall x \in \mathcal{C} \backslash \{\alpha_{\pm}\}, \ \forall n \in \mathbb{N}, \ ||D_x \mathcal{R}^n(h_x)|| \geq c 2^n ||h_x||$$

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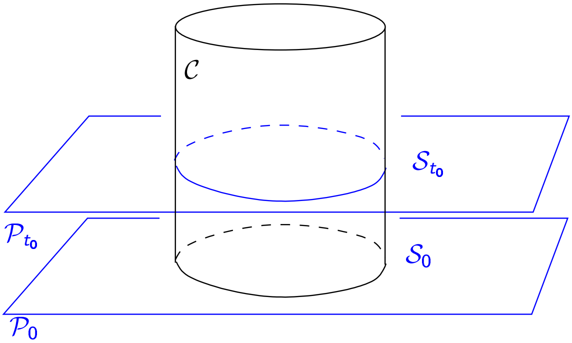
3. A combinatorial proof using a "Lee-Yang Theorem with Boundary conditions" and the fundamental symmetry of the Ising model under $z \mapsto 1/z$.

Combinatorial proof of expansion, part I

Idea: Map forward a horizontal line $\mathcal{P}_{t_0} := \{t = t_0\}$ under $\mathcal{R}^n$, then project vertically onto $\mathcal{P}_0$. Sends the circle $\mathcal{S}_{t_0} := \mathcal{P}_{t_0} \cap \mathcal{C}$ to the circle $\mathcal{S}_0$.



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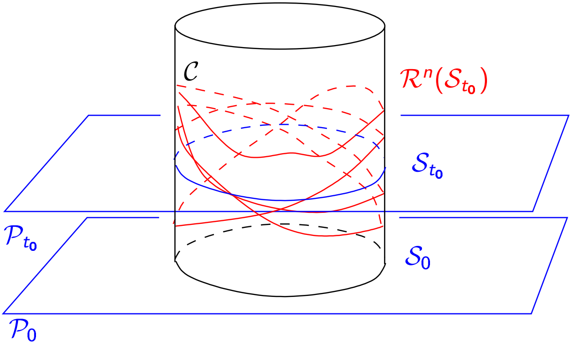

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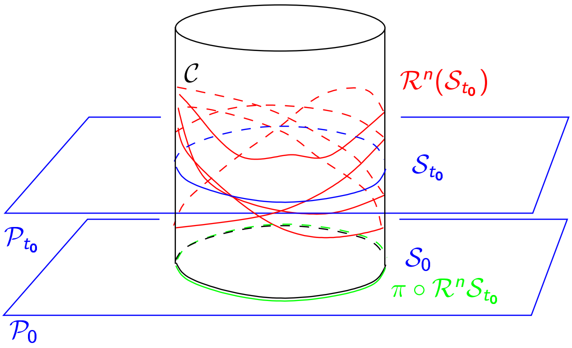

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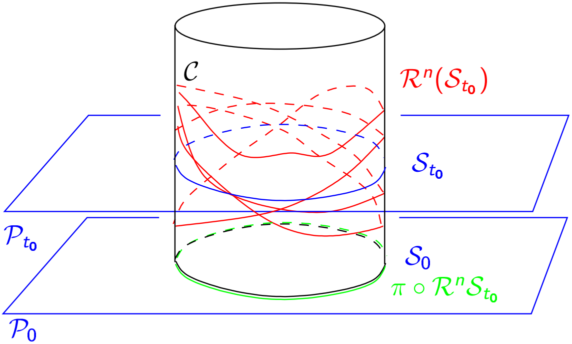

Use complex extension to prove that $\pi \circ \mathcal{R}^n : \mathcal{S}_{t_0} \to \mathcal{S}_0$ is expanding.

Combinatorial proof of expansion, part II

Recall the a semiconjugacy

$$\begin{array}{ccc} \mathbb{CP}^2 & \xrightarrow{\mathcal{R}} & \mathbb{CP}^2 \\ \downarrow \Psi & & \downarrow \Psi \\ \mathbb{CP}^2 & \xrightarrow{R} & \mathbb{CP}^2 \end{array}$$

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Ψ induces a conjugacy[4] between $\mathcal{R} : \mathcal{C} \to \mathcal{C}$ and $R : C \to C$, where $C = \Psi(\mathcal{C})$ is some appropriate Möbius band.

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Horizontal circle $\mathcal{S}_{t_0}$ becomes $S_{t_0} = \{|u| = t_0^{-1}\} = \Psi(\mathcal{S}_{t_0})$.

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Vertical projection π becomes radial projection $\mathrm{pr}(u, w) = w/u$ out to the line at infinity P_0.

Combinatorial proof of expansion, part III

The Mobius band C is the closure of

$$C_0 = \{(u, w) \in \mathbb{C}^2 : \; w = \bar{u}, \; |u| \geq 1\}.$$

in $\mathbb{CP}^2$.

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We will show that $\mathrm{pr} \circ R^n : P_{t_0} \to P_0$ expands that circle S_{t_0}.

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We have:

$$\psi_n(z) := \mathrm{pr} \circ R^n \circ \Psi(z, t_0) = \frac{W_n(z, t_0)}{U_n(z, t_0)},$$

where W_n and U_n are the conditional partition functions from the derivation of R.

Combinatorial proof of the expansion: Blaschke products

A finite Blaschke product is a function of the type:

$$B(z) : \mathbb{C} \to \mathbb{C}, \qquad z \mapsto \prod \frac{z - a_i}{1 - \overline{a_i} z}$$

where the a_i are a finite family of complex numbers.

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Claim: $\psi_n : \mathbb{C} \to \mathbb{C}$ is an Blaschke product preserving the unit disc $\mathbb{D}$, expanding the circle $\mathbb{T} = \partial\mathbb{D}$ by a factor of 2^{n+1}.

Conditional partition functions and their symmetries

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$$\begin{aligned} U_n(z,t) &= \sum_{\sigma(a)=\sigma(b)=+1} W(\sigma) = \sum_{\sigma(a)=\sigma(b)=+1} t^{-I(\sigma)/2} z^{-M(\sigma)} \\ &= a_d^+(t) z^d + \cdots + a_{-d}^+(t) z^{-d}, \end{aligned}$$

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Remarks:

1. Fundamental symmetry of the Ising model under $z \mapsto 1/z$ becomes:

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1. Fundamental symmetry of the Ising model under $z \mapsto 1/z$ becomes:

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2. Since Γ_n has valence 2^n at marked vertices a and b we have

$$a_i^-(t) = 0 \qquad \text{for } i < -4^n + 2^{n+1}$$

Reason for 2: With -1 spins at the marked vertices a, b, we can't get more than $4^n - 2^{n+1}$ edges with $++$, so $M(\sigma) \leq 4^n - 2^{n+1}$.

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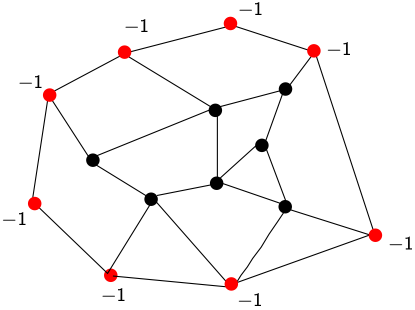
Are the other zeros b_i within the unit disc $\mathbb{D}$?
If yes, then $\psi_n(z)$ is a Blaschke product that expands the circle $\mathbb{T}$ by at least 2^{n+1}

Lee-Yang Theorem with Boundary conditions



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Theorem (Bleher, Lyubich, Roeder)

Consider a ferromagnetic Ising model on a connected graph Γ *and let* $\sigma_S \equiv -1$ *on a nonempty subset* S *of the vertex set* V*.*

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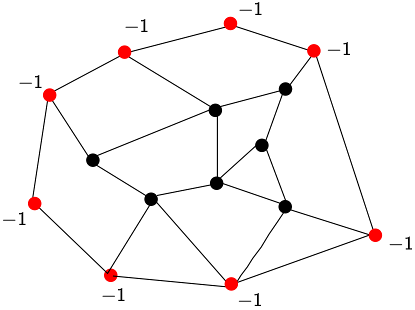

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Theorem (Bleher, Lyubich, Roeder)

Consider a ferromagnetic Ising model on a connected graph Γ and let $\sigma_S \equiv -1$ on a nonempty subset S of the vertex set V.
Then, for any temperature $t \in (0,1)$ the Lee-Yang zeros $z_i^-(t)$ of the conditional partition function $Z_{\Gamma|\sigma_S}$ lie inside the open disc $\mathbb{D}$.