Iterated Monodromy Groups and Self-similar Groups

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Reference

- V. Nekrashevych, Iterated Monodromy Groups.
  http://www.math.tamu.edu/ nekrash/Preprints/img.pdf

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Some reasons to study iterated monodromy groups:

- algebraic invariants of topological dynamical systems;
- encoding combinatorial information about the dynamical systems;
- interesting from the point of group theory.

Iterated Monodromy Groups

- Definition
- Examples
- Computation
- Examples

Setup:

1. Let M be a path connected and locally path connected topological space.
2. Let $M_1 \subset M$ and $p : M_1 \to M$ be a degree $d > 1$ covering map (partial self-covering of M).
3. Consider the n-th iterates $p^n : M_n \to M$.
4. Choose a point $t \in M$ and define the rooted tree of preimages T_t. The vertex set is $\bigsqcup_{n \geq 0} p^{-n}(t)$ and a vertex $z \in p^{-(n+1)}(t)$ is connected by an edge with the vertex $p(z) \in p^{-n}(t)$.

Iterated Monodromy Groups

The fundamental group $\pi_1(M, t)$ acts on the levels $p^{-n}(t)$ of the tree T_t by the monodromy action $F_n : \pi_1(M, t) \to Sym(p^{-n}(t))$ such that for $z \in p^{-n}(t)$ and $\gamma \in \pi_1(M, t)$, $F_n(\gamma)(z)$ is the endpoint of the unique lift of γ by p^n starting at z.
It induces an action of $\pi_1(M, t)$ on the tree T_t. This action is called the iterated monodromy action.

Iterated Monodromy Groups

Definition

The iterated monodromy group $IMG(p)$ of p is the quotient of $\pi_1(M, t)$ by the kernel of the iterated monodromy action, that is,

$$IMG(p) = \frac{\pi_1(M, t)}{\bigcap_{n \in \mathbb{N}} Ker(F_n)}.$$

$IMG(p)$, as an action, does not depend on the choice of the base point $t \in M$.

Example 1: double self-covering of the circle.
$p : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ such that $p(x) = 2x$.
Pick $t = 0$.
$\pi_1(\mathbb{R}/\mathbb{Z}, t) =< \gamma >\cong \mathbb{Z}$.
$\bigcap\limits_{n\in\mathbb{N}} Ker(F_n) = \{1\}$, since γ acts as a transitive cycle on every level of T_t.
So, $IMG(p) \cong \mathbb{Z}$.

Introduce some "coordinates" on the tree T_t.
Want:

1. Vertices of T_t are encoded by finite words over an alphabet X.
2. The root is the empty word $\emptyset$.
3. A vertex represented by a word v is connected to the vertices of the form vx for $x \in X$.

Denote by X^* the set of all finite words over X seen in T_t.

Encoding:
Let $|X| = \deg p$. Choose a bijection $\Lambda : X \to p^{-1}(t)$ and a path $\ell(x)$ from t to $\Lambda(x)$ for every $x \in X$.
Set $\Lambda(\emptyset) = t$ and the map $\Lambda : X^* \to T_t$ inductively by the rule:
$\Lambda(xv)$ is the end of the $p^{|v|}$-lift of $\ell(x)$ starting at $\Lambda(v)$.

Identify the tree T_t and X^* by Λ.

Theorem

Let $\gamma \in \pi_1(M, t)$. For $x \in X$, let γ_x be the lift of γ by p starting at x and y be the end of γ_x. Then for every $v \in X^*$, we have

$$\gamma(xv) = y(\ell(x)\gamma_x\ell(y)^{-1})(v).$$

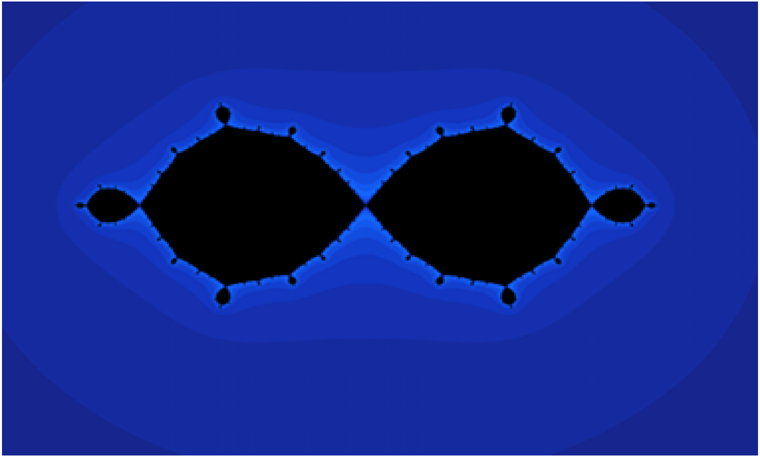
Example 1: The adding machine.
$p : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ $p(z) = z^2$.
Choose basepoint $t = 1$. Then $p^{-1}(t) = \{1, -1\}$.
Let $X = \{\mathbb{0}, \mathbb{1}\}$.
Let $\ell(\mathbb{0})$ be the trivial path at the basepoint and $\ell(\mathbb{1})$ be the unit upper half-circle.
Let γ be the unit circle based at t with the positive orientation.
Then

$$\gamma(\mathbb{0}v) = \mathbb{1}v,$$

$$\gamma(\mathbb{1}v) = \mathbb{0}\gamma(v).$$

Iterated Monodromy Groups

Example 2: The polynomial $f(z) = -\frac{z^3}{2} + \frac{3z}{2}$.
$Crit(f) = \{1, -1, \infty\}$, post-critical set $P_f = \{1, -1, \infty\}$.
Then $f : \mathbb{C} \setminus f^{-1}(\{\pm 1\}) \to \mathbb{C} \setminus \{\pm 1\}$ is a partial self-covering.



Pick basepoint $t = 0$.Then $f^{-1}(t) = \{0, \sqrt{3}, -\sqrt{3}\}$.
Take $X = \{\mathbb{0}, \mathbb{1}, \mathbb{2}\}$.
Choose a, b the generators of $\pi_1(\mathbb{C} \setminus \{\pm 1\}, 0)$.
We can get
$a(\mathbb{0}v) = \mathbb{1}v, a(\mathbb{1}v) = \mathbb{0}a(v), a(\mathbb{2}v) = \mathbb{2}v,$
$b(\mathbb{0}v) = \mathbb{2}v, b(\mathbb{1}v) = \mathbb{1}v, b(\mathbb{2}v) = \mathbb{0}b(v),$

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Self-similar Groups

Definition

A group G acting faithfully on the set X^* is called self-similar if for every $g \in G$ and every $x \in X$ there exists $h \in G$ such that

$$g(xw) = g(x)h(w)$$

for all $w \in X^*$.

Remarks:

1. If G is self-similar, then for every $v \in X^*$ and every $g \in G$, there exists $h \in G$ such that $g(vw) = g(v)h(w)$ for all $w \in X^*$.
2. The element h is uniquely defined, is called section (or restriction) of g in v and is denoted $g|_v$.
3. $g|_{v_1 v_2} = g|_{v_1} g|_{v_2}$, $(g_1 g_2)|_v = g_1|_{g_2(v)} g_2|_v$.

Self-similar Groups

Take $X = \{1, \cdots, d\}$ and let G be self-similar. Consider the map $\Phi : G \to S_d \ltimes G^d = S_d \wr G$ with $\Phi(g) = \pi(g|_1, \cdots, g|_d)$, where π is the action of g on X. The map Φ is a homomorphism and is called the wreath recursion associated with the self-similar group G.
In general, a wreath recursion on a group G is any homomorphism $\Phi : G \to S_d \wr G$.

Self-similar Groups

Remarks:

1. Two wreath recursions $\Phi_1, \Phi_2 : G \to S_d \wr G$ are equivalent if there exists an inner automorphism τ of $S_d \wr G$ such that $\Phi_2 = \tau \circ \Phi_1$.
2. Every wreath recursion defines an action on the tree $\{1, \cdots, d\}^*$. If $\Phi(g) = \pi(g_1, \cdots, g_d)$, then we put $g(xv) = \pi(x)g_x(v)$ for all $v \in \{1, \cdots, d\}^*$ and $x \in \{1, \cdots, d\}$.
3. The faithful self-similar group defined by the wreath recursion Φ is the quotient of G by the kernel K_Φ of the associated action.
4. If G is finitely generated, then the wreath recursion is determined by its values on the generators.
5. It is not known if every finitely generated self-similar group has solvable word problem.

Thank you