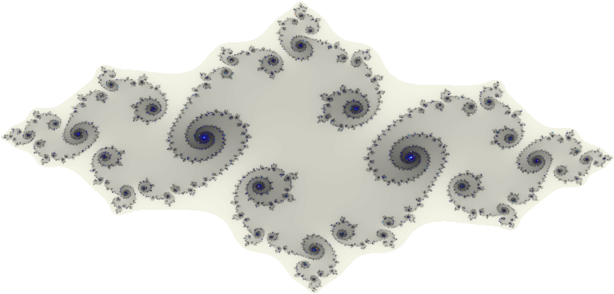
Hartogs' Lemma and Applications to Complex Dynamics

Ivan Chio

IUPUI

April 29th, 2016



Motivation

Montel's Theorem is a powerful tool in one dimensional complex dynamics.

Motivation

Montel's Theorem is a powerful tool in one dimensional complex dynamics.

However, in higher dimensions, Montel's Theorem requires much stronger hypotheses, and is not useful for dynamics.

Motivation

Montel's Theorem is a powerful tool in one dimensional complex dynamics.
However, in higher dimensions, Montel's Theorem requires much stronger hypotheses, and is not useful for dynamics.

Nessim Sibony: "I observed in 1981 that one could treat some aspects of the elementary theory of dynamics of polynomials without recourse to the theorem of Montel mentioned above. What replaces it is a compactness theorem for subharmonic functions that are locally bounded above."

Motivation

Montel's Theorem is a powerful tool in one dimensional complex dynamics.
However, in higher dimensions, Montel's Theorem requires much stronger hypotheses, and is not useful for dynamics.

Nessim Sibony: "I observed in 1981 that one could treat some aspects of the elementary theory of dynamics of polynomials without recourse to the theorem of Montel mentioned above. What replaces it is a compactness theorem for subharmonic functions that are locally bounded above."

We will try to understand this in the case of one dimensional complex dynamics.

Plan for my talk

Complex Analysis - Montel's Theorem.

Plan for my talk

Complex Analysis - Montel's Theorem.

Montel's Theorem and complex dynamics.

Plan for my talk

Complex Analysis - Montel's Theorem.

Montel's Theorem and complex dynamics.

Compactness theorem for subharmonic functions and Hartogs' Lemma.

Plan for my talk

Complex Analysis - Montel's Theorem.

Montel's Theorem and complex dynamics.

Compactness theorem for subharmonic functions and Hartogs' Lemma.

Subharmonic functions and complex dynamics.

Plan for my talk

Complex Analysis - Montel's Theorem.

Montel's Theorem and complex dynamics.

Compactness theorem for subharmonic functions and Hartogs' Lemma.

Subharmonic functions and complex dynamics.

Comparison.

Plan for my talk

Complex Analysis - Montel's Theorem.

Montel's Theorem and complex dynamics.

Compactness theorem for subharmonic functions and Hartogs' Lemma.

Subharmonic functions and complex dynamics.

Comparison.

Complex Analysis - Montel's Theorem

Let S be a Riemann surface and $\mathcal{F}$ be a family of holomorphic functions from S to $\widehat{\mathbb{C}}$.

Complex Analysis - Montel's Theorem

Let S be a Riemann surface and $\mathcal{F}$ be a family of holomorphic functions from S to $\widehat{\mathbb{C}}$.

Definition

$\mathcal{F}$ is called a normal family if every infinite sequence of maps from $\mathcal{F}$ has a subsequence which converges locally uniformly.

Complex Analysis - Montel's Theorem

Let S be a Riemann surface and $\mathcal{F}$ be a family of holomorphic functions from S to $\widehat{\mathbb{C}}$.

Definition

$\mathcal{F}$ is called a normal family if every infinite sequence of maps from $\mathcal{F}$ has a subsequence which converges locally uniformly.

Montel's Theorem

If there are 3 distinct points $\{a, b, c\} \subset \widehat{\mathbb{C}}$ such that,
$f(S) \subseteq \widehat{\mathbb{C}} \setminus \{a, b, c\}$ for every $f \in \mathcal{F}$, then $\mathcal{F}$ is a normal family.

Setting

Let $p(z)$ be a polynomial of degree ≥ 2.

Setting

Let $p(z)$ be a polynomial of degree ≥ 2.

Definition

The Fatou set F_p is the maximal open set where $\{p^n(z) : n = 1, 2, 3, ...\}$ form a normal family.

Setting

Let $p(z)$ be a polynomial of degree ≥ 2.

Definition

The Fatou set F_p is the maximal open set where $\{p^n(z) : n = 1, 2, 3, ...\}$ form a normal family.

The Julia set J_p is the complement of the Fatou set.

Setting

Let $p(z)$ be a polynomial of degree ≥ 2.

Definition

The Fatou set F_p is the maximal open set where $\{p^n(z) : n = 1, 2, 3, ...\}$ form a normal family.

The Julia set J_p is the complement of the Fatou set.

Informally, p behaves stably on F_p, and chaotically on J_p.

Setting

Let $p(z)$ be a polynomial of degree ≥ 2.

Definition

The Fatou set F_p is the maximal open set where $\{p^n(z) : n = 1, 2, 3, ...\}$ form a normal family.

The Julia set J_p is the complement of the Fatou set.

Informally, p behaves stably on F_p, and chaotically on J_p.

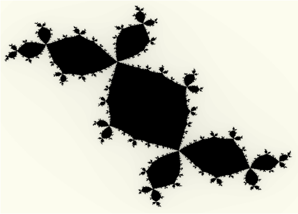


Figure: $z^2 + (-.122 + .745i)$

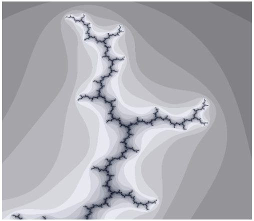


Figure: $z^2 + i$

Setting

Definition

The filled Julia set K_p is the set of $z \in \mathbb{C}$ such that the orbit of z is bounded.

Setting

Definition

The filled Julia set K_p is the set of $z \in \mathbb{C}$ such that the orbit of z is bounded.

Lemma

K_p is compact with connected complement, and $\partial K_p = J_p$.

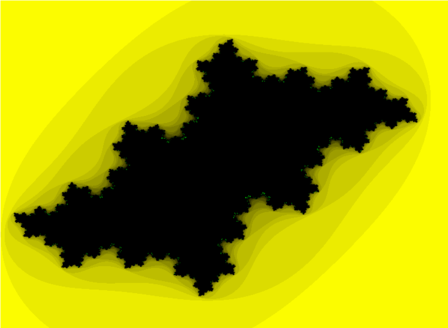
Setting

Definition

The filled Julia set K_p is the set of $z \in \mathbb{C}$ such that the orbit of z is bounded.

Lemma

K_p is compact with connected complement, and $\partial K_p = J_p$.



Exceptional Point

Definition

Given a polynomial $p(z)$, a point $z_0 \in \widehat{\mathbb{C}}$ is called exceptional for p if the set of preimages of all forward images of z_0 is finite.

Exceptional Point

Definition

Given a polynomial $p(z)$, a point $z_0 \in \widehat{\mathbb{C}}$ is called exceptional for p if the set of preimages of all forward images of z_0 is finite.

Example: $p(z) = z^2 + c$ always has an exceptional point at ∞.
It has an additional exceptional point iff $c = 0$.
When $c = 0$, the other exceptional point is 0.

Exceptional Point

Definition

Given a polynomial $p(z)$, a point $z_0 \in \widehat{\mathbb{C}}$ is called exceptional for p if the set of preimages of all forward images of z_0 is finite.

Example: $p(z) = z^2 + c$ always has an exceptional point at ∞.
It has an additional exceptional point iff $c = 0$.
When $c = 0$, the other exceptional point is 0.

Theorem

If $p : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a polynomial of degree ≥ 2, then there can be at most 2 exceptional points. Moreover, if they exist, they must be (super)attracting, hence belong to the Fatou set.

Exceptional Point

Definition

Given a polynomial $p(z)$, a point $z_0 \in \widehat{\mathbb{C}}$ is called exceptional for p if the set of preimages of all forward images of z_0 is finite.

Example: $p(z) = z^2 + c$ always has an exceptional point at ∞.
It has an additional exceptional point iff $c = 0$.
When $c = 0$, the other exceptional point is 0.

Theorem

If $p : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a polynomial of degree ≥ 2, then there can be at most 2 exceptional points. Moreover, if they exist, they must be (super)attracting, hence belong to the Fatou set.

The proof is an application of Montel's Theorem.

Transitivity of Julia set

Theorem (Transitivity)

Let z be an arbitrary point of the Julia set J_p and let U be an arbitrary neighborhood of z. Then the union $V = \bigcup_{n=0}^{\infty} p^n(U)$ contains the whole Riemann sphere except possibly the 2 exceptional points, if they exist.

Transitivity of Julia set

Theorem (Transitivity)

Let z be an arbitrary point of the Julia set J_p and let U be an arbitrary neighborhood of z. Then the union $V = \bigcup_{n=0}^{\infty} p^n(U)$ contains the whole Riemann sphere except possibly the 2 exceptional points, if they exist.

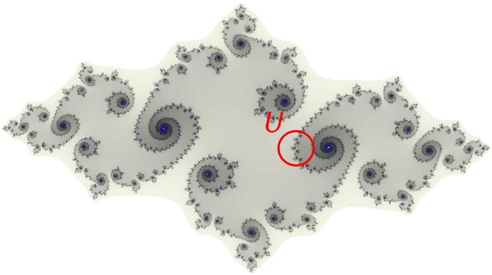


Figure: Iterating the neighborhood U.

Transitivity of Julia set

Theorem (Transitivity)

Let z be an arbitrary point of the Julia set J_p and let U be an arbitrary neighborhood of z. Then the union $V = \bigcup_{n=0}^{\infty} p^n(U)$ contains the whole Riemann sphere except possibly the 2 exceptional points, if they exist.

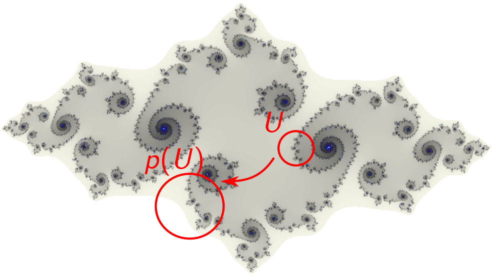


Figure: Iterating the neighborhood U.

Transitivity of Julia set

Theorem (Transitivity)

Let z be an arbitrary point of the Julia set J_p and let U be an arbitrary neighborhood of z. Then the union $V = \bigcup_{n=0}^{\infty} p^n(U)$ contains the whole Riemann sphere except possibly the 2 exceptional points, if they exist.

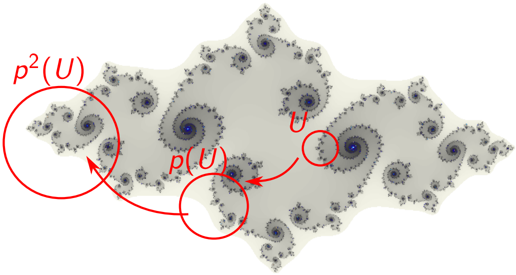


Figure: Iterating the neighborhood U.

Transitivity of Julia set

Theorem (Transitivity)

Let z be an arbitrary point of the Julia set J_p and let U be an arbitrary neighborhood of z. Then the union $V = \bigcup_{n=0}^{\infty} p^n(U)$ contains the whole Riemann sphere except possibly the 2 exceptional points, if they exist.

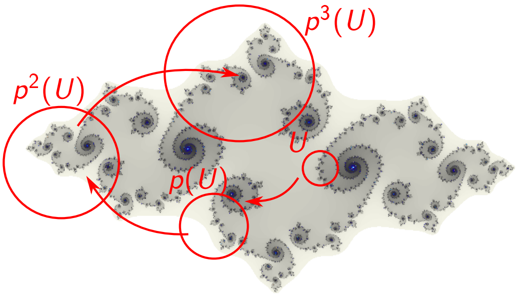


Figure: Iterating the neighborhood U.

Transitivity of Julia set

Theorem (Transitivity)

Let z be an arbitrary point of the Julia set J_p and let U be an arbitrary neighborhood of z. Then the union $V = \bigcup_{n=0}^{\infty} p^n(U)$ contains the whole Riemann sphere except possibly the 2 exceptional points, if they exist.

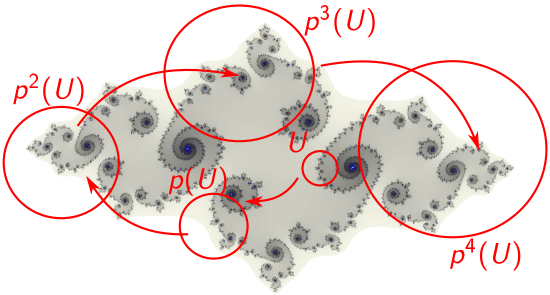


Figure: Iterating the neighborhood U.

Transitivity of Julia set

Theorem (Transitivity)

Let z be an arbitrary point of the Julia set J_p and let U be an arbitrary neighborhood of z. Then the union $V = \bigcup_{n=0}^{\infty} p^n(U)$ contains the whole Riemann sphere except possibly the 2 exceptional points, if they exist.

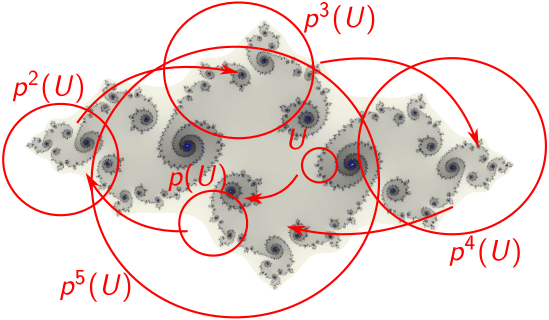


Figure: Iterating the neighborhood U.

Transitivity of Julia set

Theorem (Transitivity)

Let z be an arbitrary point of the Julia set J_p and let U be an arbitrary neighborhood of z. Then the union $V = \bigcup_{n=0}^{\infty} p^n(U)$ contains the whole Riemann sphere except possibly the 2 exceptional points, if they exist.

Transitivity of Julia set

Theorem (Transitivity)

Let z be an arbitrary point of the Julia set J_p and let U be an arbitrary neighborhood of z. Then the union $V = \bigcup_{n=0}^{\infty} p^n(U)$ contains the whole Riemann sphere except possibly the 2 exceptional points, if they exist.

$\mathcal{F} := \{p^n : U \to \widehat{\mathbb{C}}\}$ is a family of holomorphic functions.

Transitivity of Julia set

Theorem (Transitivity)

Let z be an arbitrary point of the Julia set J_p and let U be an arbitrary neighborhood of z. Then the union $V = \bigcup_{n=0}^{\infty} p^n(U)$ contains the whole Riemann sphere except possibly the 2 exceptional points, if they exist.

$\mathcal{F} := \{p^n : U \to \widehat{\mathbb{C}}\}$ is a family of holomorphic functions.

Notice $V = \bigcup_{n=0}^{\infty} p^n(U)$ is forward invariant.

Transitivity of Julia set

Theorem (Transitivity)

Let z be an arbitrary point of the Julia set J_p and let U be an arbitrary neighborhood of z. Then the union $V = \bigcup_{n=0}^{\infty} p^n(U)$ contains the whole Riemann sphere except possibly the 2 exceptional points, if they exist.

$\mathcal{F} := \{p^n : U \to \widehat{\mathbb{C}}\}$ is a family of holomorphic functions.

Notice $V = \bigcup_{n=0}^{\infty} p^n(U)$ is forward invariant.

If $\widehat{\mathbb{C}} - V$ has more than two points,

Transitivity of Julia set

Theorem (Transitivity)

Let z be an arbitrary point of the Julia set J_p and let U be an arbitrary neighborhood of z. Then the union $V = \bigcup_{n=0}^{\infty} p^n(U)$ contains the whole Riemann sphere except possibly the 2 exceptional points, if they exist.

$\mathcal{F} := \{p^n : U \to \widehat{\mathbb{C}}\}$ is a family of holomorphic functions.

Notice $V = \bigcup_{n=0}^{\infty} p^n(U)$ is forward invariant.

If $\widehat{\mathbb{C}} - V$ has more than two points,

then since $p^n(U) \subset V$, Montel $\Rightarrow$ $U \subset F_p$.

Transitivity of Julia set

Theorem (Transitivity)

Let z be an arbitrary point of the Julia set J_p and let U be an arbitrary neighborhood of z. Then the union $V = \bigcup_{n=0}^{\infty} p^n(U)$ contains the whole Riemann sphere except possibly the 2 exceptional points, if they exist.

$\mathcal{F} := \{p^n : U \to \widehat{\mathbb{C}}\}$ is a family of holomorphic functions.

Notice $V = \bigcup_{n=0}^{\infty} p^n(U)$ is forward invariant.

If $\widehat{\mathbb{C}} - V$ has more than two points,

then since $p^n(U) \subset V$, Montel $\Rightarrow U \subset F_p$.

But U contains a point in J_p.

Transitivity of Julia set

Theorem (Transitivity)

Let z be an arbitrary point of the Julia set J_p and let U be an arbitrary neighborhood of z. Then the union $V = \bigcup_{n=0}^{\infty} p^n(U)$ contains the whole Riemann sphere except possibly the 2 exceptional points, if they exist.

$\mathcal{F} := \{p^n : U \to \widehat{\mathbb{C}}\}$ is a family of holomorphic functions.

Notice $V = \bigcup_{n=0}^{\infty} p^n(U)$ is forward invariant.

If $\widehat{\mathbb{C}} - V$ has more than two points,

then since $p^n(U) \subset V$, Montel $\Rightarrow U \subset F_p$.

But U contains a point in J_p.

Contradiction. $\square$

Transitivity - rephrased

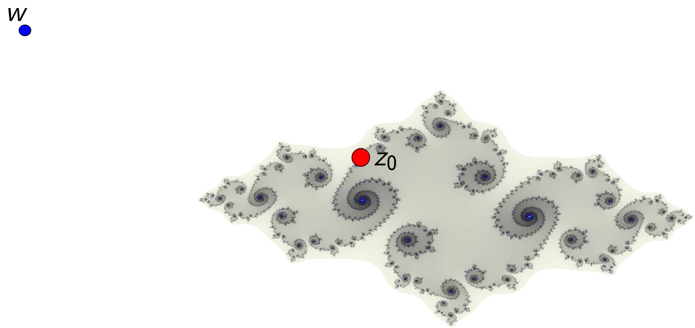
Theorem (Transitivity)

Let $p(z)$ be a polynomial and let $z_0 \in J_p$. Then for any $w \in \widehat{\mathbb{C}}$ which is not an exceptional point, there exists a sequence of preimages of w that converges to z_0.

Transitivity - rephrased

Theorem (Transitivity)

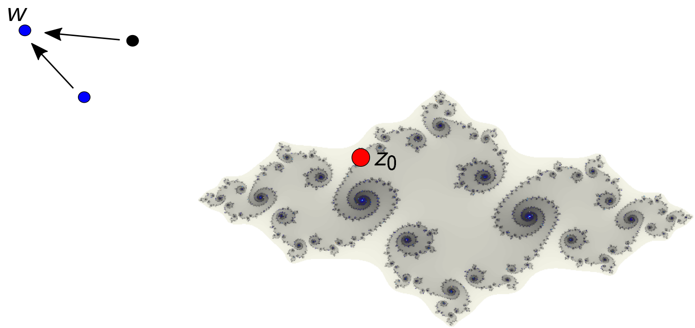
Let $p(z)$ be a polynomial and let $z_0 \in J_p$. Then for any $w \in \widehat{\mathbb{C}}$ which is not an exceptional point, there exists a sequence of preimages of w that converges to z_0.



Transitivity - rephrased

Theorem (Transitivity)

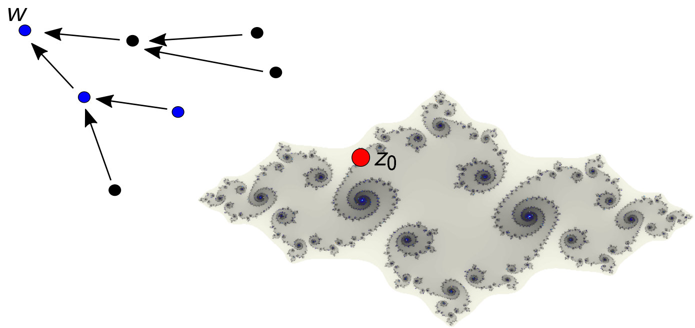
Let $p(z)$ be a polynomial and let $z_0 \in J_p$. Then for any $w \in \widehat{\mathbb{C}}$ which is not an exceptional point, there exists a sequence of preimages of w that converges to z_0.



Transitivity - rephrased

Theorem (Transitivity)

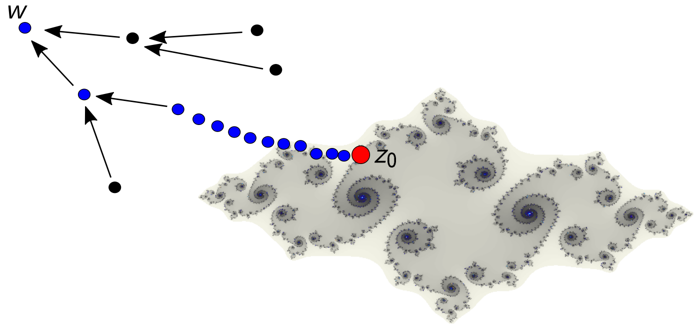
Let $p(z)$ be a polynomial and let $z_0 \in J_p$. Then for any $w \in \widehat{\mathbb{C}}$ which is not an exceptional point, there exists a sequence of preimages of w that converges to z_0.



Transitivity - rephrased

Theorem (Transitivity)

Let $p(z)$ be a polynomial and let $z_0 \in J_p$. Then for any $w \in \widehat{\mathbb{C}}$ which is not an exceptional point, there exists a sequence of preimages of w that converges to z_0.



Subharmonic functions

Let Ω be a domain in $\mathbb{C}$.

Subharmonic functions

Let Ω be a domain in $\mathbb{C}$.

Definition: $u : \Omega \to [-\infty, \infty)$ is subharmonic (SH) if:

Subharmonic functions

Let Ω be a domain in $\mathbb{C}$.

Definition: $u : \Omega \to [-\infty, \infty)$ is subharmonic (SH) if:

1. u is upper semicontinuous, i.e
   $\forall z_0 \in \Omega, \limsup_{z \to z_0} u(z) \leq u(z_0),$

Subharmonic functions

Let Ω be a domain in $\mathbb{C}$.
Definition: $u : \Omega \to [-\infty, \infty)$ is subharmonic (SH) if:

1. u is upper semicontinuous, i.e
   $\forall z_0 \in \Omega, \limsup_{z \to z_0} u(z) \leq u(z_0)$,
2. $\forall z \in \Omega, \forall r > 0$ so that $\mathbb{D}(z, r) \subset \Omega$, we have
   $u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta$ (submean value property),

Subharmonic functions

Let Ω be a domain in $\mathbb{C}$.

Definition: $u : \Omega \to [-\infty, \infty)$ is subharmonic (SH) if:

1. u is upper semicontinuous, i.e
   $\forall z_0 \in \Omega, \limsup_{z \to z_0} u(z) \leq u(z_0)$,
2. $\forall z \in \Omega, \forall r > 0$ so that $\mathbb{D}(z, r) \subset \Omega$, we have
   $u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta$ (submean value property),
3. $u \not\equiv -\infty$.

Subharmonic functions

Let Ω be a domain in $\mathbb{C}$.
Definition: $u : \Omega \to [-\infty, \infty)$ is subharmonic (SH) if:

1. u is upper semicontinuous, i.e
   $\forall z_0 \in \Omega, \limsup_{z \to z_0} u(z) \leq u(z_0)$,
2. $\forall z \in \Omega, \forall r > 0$ so that $\mathbb{D}(z, r) \subset \Omega$, we have
   $u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta$ (submean value property),
3. $u \not\equiv -\infty$.

Three properties of subharmonic functions which will be used:

1. max of two SH functions is SH.
2. Uniform limit of SH functions is SH.
3. if u is SH, p is holomorphic, then $u \circ p$ is SH.

Subharmonic functions

Let Ω be a domain in $\mathbb{C}$.
Definition: $u : \Omega \to [-\infty, \infty)$ is subharmonic (SH) if:

1. u is upper semicontinuous, i.e
   $\forall z_0 \in \Omega, \limsup_{z \to z_0} u(z) \leq u(z_0)$,
2. $\forall z \in \Omega, \forall r > 0$ so that $\mathbb{D}(z, r) \subset \Omega$, we have
   $u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta$ (submean value property),
3. $u \not\equiv -\infty$.

Three properties of subharmonic functions which will be used:

1. max of two SH functions is SH.
2. Uniform limit of SH functions is SH.
3. if u is SH, p is holomorphic, then $u \circ p$ is SH.

Example: $u(z) = \log |z|$,

Subharmonic functions

Let Ω be a domain in $\mathbb{C}$.

Definition: $u : \Omega \to [-\infty, \infty)$ is subharmonic (SH) if:

1. u is upper semicontinuous, i.e
   $\forall z_0 \in \Omega, \limsup_{z \to z_0} u(z) \leq u(z_0)$,
2. $\forall z \in \Omega, \forall r > 0$ so that $\mathbb{D}(z, r) \subset \Omega$, we have
   $u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta$ (submean value property),
3. $u \not\equiv -\infty$.

Three properties of subharmonic functions which will be used:

1. max of two SH functions is SH.
2. Uniform limit of SH functions is SH.
3. if u is SH, p is holomorphic, then $u \circ p$ is SH.

Example: $u(z) = \log |z|$,
$v(z) = \log_+ |z| := \max\{0, \log |z|\}$.

Distributions and subharmonic functions

Definition

A continuous linear functional on $C_0^\infty(\Omega)$ is called a distribution. The space of distribution is denoted by $D'(\Omega)$.

Distributions and subharmonic functions

Definition

A continuous linear functional on $C_0^\infty(\Omega)$ is called a distribution. The space of distribution is denoted by $D'(\Omega)$.

A subharmonic function u is locally integrable.

Distributions and subharmonic functions

Definition

A continuous linear functional on $C_0^\infty(\Omega)$ is called a distribution. The space of distribution is denoted by $D'(\Omega)$.

A subharmonic function u is locally integrable.

Hence we can define a distribution by $\langle u, \phi \rangle := \int \phi \cdot u \; dLeb$.

Distributions and subharmonic functions

Definition

A continuous linear functional on $C_0^\infty(\Omega)$ is called a distribution. The space of distribution is denoted by $D'(\Omega)$.

A subharmonic function u is locally integrable.

Hence we can define a distribution by $\langle u, \phi \rangle := \int \phi \cdot u \, dLeb$.

We can take the Laplacian of a SH function (in the distributional sense) by $\langle \Delta u, \phi \rangle := \int \Delta\phi \cdot u \, dLeb$.

Distributions and subharmonic functions

Definition

A continuous linear functional on $C_0^\infty(\Omega)$ is called a distribution. The space of distribution is denoted by $D'(\Omega)$.

A subharmonic function u is locally integrable.

Hence we can define a distribution by $\langle u, \phi \rangle := \int \phi \cdot u \, dLeb$.

We can take the Laplacian of a SH function (in the distributional sense) by $\langle \Delta u, \phi \rangle := \int \Delta\phi \cdot u \, dLeb$.

If u is in fact a smooth function, then the distributional Laplacian coincides with the classical Laplacian by Stoke's theorem.

$$\langle \Delta u, \phi \rangle = \int \Delta\phi \cdot u \, dLeb = \int \phi \cdot \Delta u \, dLeb.$$

Distributions and subharmonic functions

Definition

A continuous linear functional on $C_0^\infty(\Omega)$ is called a distribution. The space of distribution is denoted by $D'(\Omega)$.

A subharmonic function u is locally integrable.

Hence we can define a distribution by $\langle u, \phi \rangle := \int \phi \cdot u \, dLeb$.

We can take the Laplacian of a SH function (in the distributional sense) by $\langle \Delta u, \phi \rangle := \int \Delta\phi \cdot u \, dLeb$.

If u is in fact a smooth function, then the distributional Laplacian coincides with the classical Laplacian by Stoke's theorem.

$$\langle \Delta u, \phi \rangle = \int \Delta\phi \cdot u \, dLeb = \int \phi \cdot \Delta u \, dLeb.$$

Notice if $u \in L^1_{loc}(\mathbb{R}^2)$, then $u \, dLeb$ gives a signed measure.

Distributions and subharmonic functions

Example

Distributions and subharmonic functions

Example

$\langle \Delta \log |z|, \phi \rangle = \langle \log |z|, \Delta\phi \rangle = \int \Delta\phi \log |z| \; dLeb = 2\pi\phi(0).$

Distributions and subharmonic functions

Example

$$\langle \Delta \log |z|, \phi \rangle = \langle \log |z|, \Delta\phi \rangle = \int \Delta\phi \log |z| \; dLeb = 2\pi\phi(0).$$

The last equality is a consequence of Gauss' theorem.

Distributions and subharmonic functions

Example

$\langle \Delta \log |z|, \phi \rangle = \langle \log |z|, \Delta\phi \rangle = \int \Delta\phi \log |z| \; dLeb = 2\pi\phi(0)$.

The last equality is a consequence of Gauss' theorem.

So $\Delta \log |z| = 2\pi\delta_0$, where δ_0 is the Dirac mass at 0.

Distributions and subharmonic functions

Example

$\langle \Delta \log|z|, \phi \rangle = \langle \log|z|, \Delta\phi \rangle = \int \Delta\phi \log|z| \; dLeb = 2\pi\phi(0).$

The last equality is a consequence of Gauss' theorem.

So $\Delta \log|z| = 2\pi\delta_0$, where δ_0 is the Dirac mass at 0.

For a monic polynomial $p(z) = \prod_{i=1}^{n}(z - z_i)$, we have $\Delta \log|p(z)| = 2\pi(\sum_{i=1}^{n} \delta_{z_i})$.

Distributions and subharmonic functions

Example

$\langle \Delta \log |z|, \phi \rangle = \langle \log |z|, \Delta\phi \rangle = \int \Delta\phi \log |z| \ dLeb = 2\pi\phi(0)$.

The last equality is a consequence of Gauss' theorem.

So $\Delta \log |z| = 2\pi\delta_0$, where δ_0 is the Dirac mass at 0.

For a monic polynomial $p(z) = \prod_{i=1}^{n}(z - z_i)$, we have $\Delta \log |p(z)| = 2\pi(\sum_{i=1}^{n} \delta_{z_i})$.

Theorem (Fundamental Equivalence)

If h is SH, then Δh is a non-negative measure. Conversely, if μ is a distribution such that $\Delta\mu$ is a non-negative measure, then μ is given by a SH function. i.e there is a SH function v such that $\langle \mu, \phi \rangle = \int \phi \cdot v \ dLeb$.

Hartogs' Lemma: an analogy of Montel's Theorem

Compactness theorem and Hartogs' Lemma

Let v_j be a sequence of subharmonic functions on a domain $\Omega \in \mathbb{C}$, which have a uniform upper bound on any compact set. Then

Hartogs' Lemma: an analogy of Montel's Theorem

Compactness theorem and Hartogs' Lemma

Let v_j be a sequence of subharmonic functions on a domain $\Omega \in \mathbb{C}$, which have a uniform upper bound on any compact set. Then

(a) if v_j does not converge to $-\infty$ uniformly on every compact set in Ω then there is a subsequence v_{j_k} which is convergent in $L^1_{loc}(\Omega)$.

Hartogs' Lemma: an analogy of Montel's Theorem

Compactness theorem and Hartogs' Lemma

Let v_j be a sequence of subharmonic functions on a domain $\Omega \in \mathbb{C}$, which have a uniform upper bound on any compact set. Then

(a) if v_j does not converge to $-\infty$ uniformly on every compact set in Ω then there is a subsequence v_{j_k} which is convergent in $L^1_{loc}(\Omega)$.

(b) if v is a subharmonic function and $v_j \to v$ in $\mathcal{D}'(\Omega)$, then $v_j \to v$ in $L^1_{loc}(\Omega)$.

Hartogs' Lemma: an analogy of Montel's Theorem

Compactness theorem and Hartogs' Lemma

Let v_j be a sequence of subharmonic functions on a domain $\Omega \in \mathbb{C}$, which have a uniform upper bound on any compact set. Then

(a) if v_j does not converge to $-\infty$ uniformly on every compact set in Ω then there is a subsequence v_{j_k} which is convergent in $L^1_{loc}(\Omega)$.

(b) if v is a subharmonic function and $v_j \to v$ in $\mathcal{D}'(\Omega)$, then $v_j \to v$ in $L^1_{loc}(\Omega)$. Moreover,

$$\limsup_{j\to\infty} v_j(z) \leq v(z).$$

With the two sides equal and finite a.e.

Hartogs' Lemma: an analogy of Montel's Theorem

Compactness theorem and Hartogs' Lemma

Let v_j be a sequence of subharmonic functions on a domain $\Omega \in \mathbb{C}$, which have a uniform upper bound on any compact set. Then

(a) if v_j does not converge to $-\infty$ uniformly on every compact set in Ω then there is a subsequence v_{j_k} which is convergent in $L^1_{loc}(\Omega)$.

(b) if v is a subharmonic function and $v_j \to v$ in $\mathcal{D}'(\Omega)$, then $v_j \to v$ in $L^1_{loc}(\Omega)$. Moreover,

$$\limsup_{j\to\infty} v_j(z) \leq v(z).$$

With the two sides equal and finite a.e.

Part (b) of above is the classical Hartogs' Lemma.

Montel's Theorem and Hartogs' Lemma are both compactness theorems for family of functions.

Montel's Theorem and Hartogs' Lemma are both compactness theorems for family of functions.

Let us re-prove transitivity of the Julia set using Hartogs' Lemma.

Green Function for $p(z)$

Let us focus on $p(z) = z^2 + c$, where $c \in \mathbb{C} \setminus \{0\}$.

Green Function for $p(z)$

Let us focus on $p(z) = z^2 + c$, where $c \in \mathbb{C} \setminus \{0\}$.

Let $G_n(z) = \frac{1}{2^n} \log_+ |p^n(z)|$.

Green Function for $p(z)$

Let us focus on $p(z) = z^2 + c$, where $c \in \mathbb{C} \setminus \{0\}$.

Let $G_n(z) = \frac{1}{2^n} \log_+ |p^n(z)|$.

G_n is SH because it is a composition of $\log_+ |z|$ (SH) and $p^n(z)$ (holomorphic).

Green Function for $p(z)$

Let us focus on $p(z) = z^2 + c$, where $c \in \mathbb{C} \setminus \{0\}$.

Let $G_n(z) = \frac{1}{2^n} \log_+ |p^n(z)|$.

G_n is SH because it is a composition of $\log_+ |z|$ (SH) and $p^n(z)$ (holomorphic).

Proposition

$G(z) := \lim_{n\to\infty} G_n(z)$ converges uniformly on $\mathbb{C}$ and is SH (because it is the uniform limit of SH functions). G is called the Green Function of p.

Green Function for $p(z)$

Let us focus on $p(z) = z^2 + c$, where $c \in \mathbb{C} \setminus \{0\}$.

Let $G_n(z) = \frac{1}{2^n} \log_+ |p^n(z)|$.

G_n is SH because it is a composition of $\log_+ |z|$ (SH) and $p^n(z)$ (holomorphic).

Proposition

$G(z) := \lim_{n\to\infty} G_n(z)$ converges uniformly on $\mathbb{C}$ and is SH (because it is the uniform limit of SH functions). G is called the Green Function of p.

ΔG is zero everywhere except on the Julia set.

Green Function for $p(z)$

Let us focus on $p(z) = z^2 + c$, where $c \in \mathbb{C} \setminus \{0\}$.

Let $G_n(z) = \frac{1}{2^n} \log_+ |p^n(z)|$.

G_n is SH because it is a composition of $\log_+ |z|$ (SH) and $p^n(z)$ (holomorphic).

Proposition

$G(z) := \lim_{n\to\infty} G_n(z)$ converges uniformly on $\mathbb{C}$ and is SH (because it is the uniform limit of SH functions). G is called the Green Function of p.

ΔG is zero everywhere except on the Julia set.

$\mu := \frac{1}{2\pi}\Delta G$ is a measure supported on the Julia set. (In fact, μ has important dynamical meaning: it is the unique invariant measure for p that gives the maximal entropy.)

Green Function for $z^2 + (-.122 + .745i)$

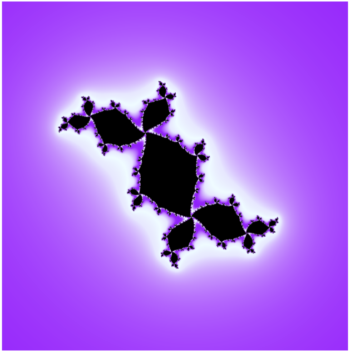


Figure: colors indicate rates of escape

Green Function for $z^2 + (-.122 + .745i)$

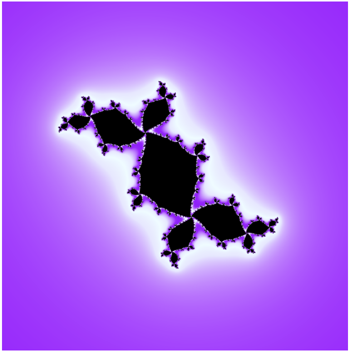


Figure: colors indicate rates of escape

Green Function for $z^2 + (-.122 + .745i)$

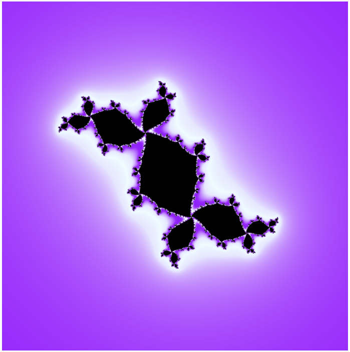


Figure: colors indicate rates of escape

μ assigns mass to any neighborhood of any point of the Julia set.

Transitivity of Julia set revisited

Recall transitivity of Julia set

Transitivity of Julia set revisited

Recall transitivity of Julia set

Theorem (Transitivity)

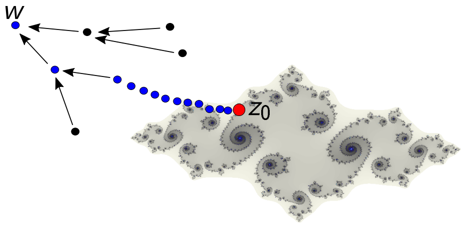
Let $p(z)$ be a polynomial and let $z_0 \in J_p$. Then for any $w \in \widehat{\mathbb{C}}$ which is not an exceptional point, there exists a sequence of preimages of w that converges to z_0.

Transitivity of Julia set revisited

Recall transitivity of Julia set

Theorem (Transitivity)

Let $p(z)$ be a polynomial and let $z_0 \in J_p$. Then for any $w \in \widehat{\mathbb{C}}$ which is not an exceptional point, there exists a sequence of preimages of w that converges to z_0.

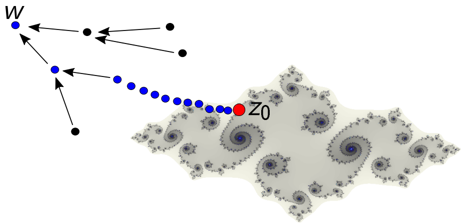


Transitivity of Julia set revisited

Recall transitivity of Julia set

Theorem (Transitivity)

Let $p(z)$ be a polynomial and let $z_0 \in J_p$. Then for any $w \in \widehat{\mathbb{C}}$ which is not an exceptional point, there exists a sequence of preimages of w that converges to z_0.



Let us prove the analog of transitivity in the measure theoretical sense using compactness theorem and Hartogs' Lemma, instead of Montel's Theorem.

Brolin's Theorem

Brolin's Theorem

For all $w \in \mathbb{C}$ except at most 2 (exceptional) points,

$$\frac{1}{2^n}(p^n)^*\delta_w \to \frac{1}{2\pi}\Delta G.$$

Brolin's Theorem

Brolin's Theorem

For all $w \in \mathbb{C}$ except at most 2 (exceptional) points,

$$\frac{1}{2^n}(p^n)^*\delta_w \to \frac{1}{2\pi}\Delta G.$$

Notice LHS $\frac{1}{2\pi}\Delta\frac{1}{2^n}\log|p^n(z) - w|$ is a probability measure with mass equally distributed on the roots of $p^n(z) = w$.

Brolin's Theorem

Brolin's Theorem

For all $w \in \mathbb{C}$ except at most 2 (exceptional) points,

$$\frac{1}{2^n}(p^n)^*\delta_w \to \frac{1}{2\pi}\Delta G.$$

Notice LHS $\frac{1}{2\pi}\Delta\frac{1}{2^n}\log|p^n(z)-w|$ is a probability measure with mass equally distributed on the roots of $p^n(z)=w$.

In other words, $\frac{1}{2\pi}\Delta\frac{1}{2^n}\log|p^n(z)-w| \to \frac{1}{2\pi}\Delta G$ in the distributional sense.

Brolin's Theorem

Brolin's Theorem

For all $w \in \mathbb{C}$ except at most 2 (exceptional) points,

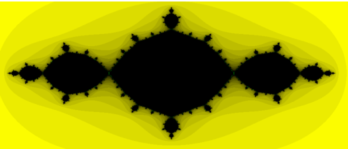
$$\frac{1}{2^n}(p^n)^*\delta_w \to \frac{1}{2\pi}\Delta G.$$

Notice LHS $\frac{1}{2\pi}\Delta\frac{1}{2^n}\log|p^n(z) - w|$ is a probability measure with mass equally distributed on the roots of $p^n(z) = w$.

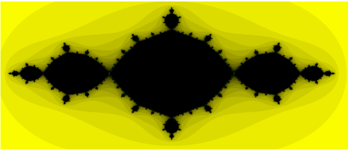
In other words, $\frac{1}{2\pi}\Delta\frac{1}{2^n}\log|p^n(z) - w| \to \frac{1}{2\pi}\Delta G$ in the distributional sense.

Warning: now we have log instead of $\log_+$

Brolin's Theorem For $z^2 - 1$



Brolin's Theorem For $z^2 - 1$



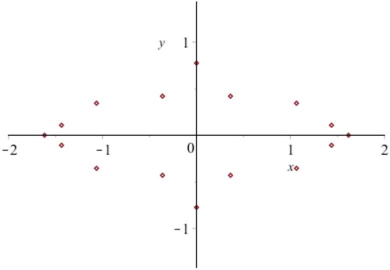
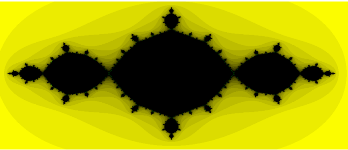


Figure: 4th pullback of $w = 0$.

Brolin's Theorem For $z^2 - 1$



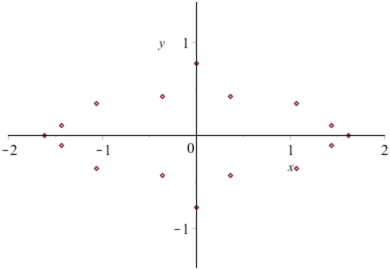


Figure: 4th pullback of $w = 0$.

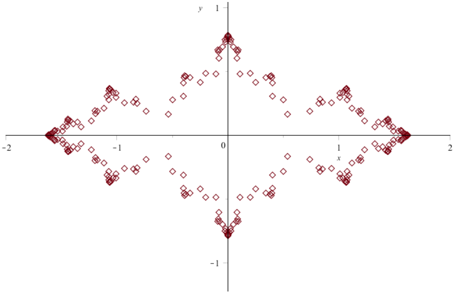
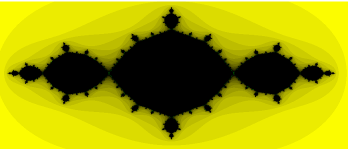
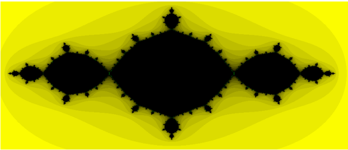


Figure: 8th pullback of $w = 0$.

Brolin's Theorem For $z^2 - 1$



Brolin's Theorem For $z^2 - 1$



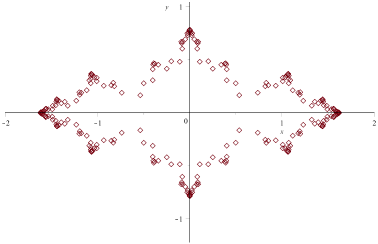
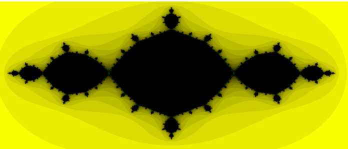


Figure: 8th pullback of $w = 0$.

Brolin's Theorem For $z^2 - 1$



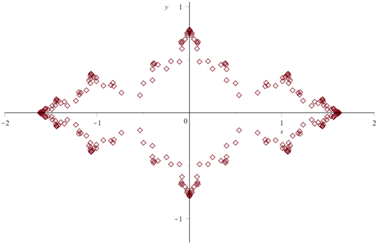


Figure: 8th pullback of $w = 0$.

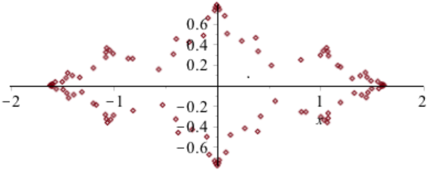


Figure: 8th pullback of $w = 1 + i$.

Transitivity vs Brolin's Theorem

Theorem (Transitivity)

Let $p(z)$ be a polynomial and let $z_0 \in J_p$. Then for any $w \in \widehat{\mathbb{C}}$ which is not an exceptional point, there exists a sequence of preimages of w that converges to z_0.

Transitivity vs Brolin's Theorem

Theorem (Transitivity)

Let $p(z)$ be a polynomial and let $z_0 \in J_p$. Then for any $w \in \widehat{\mathbb{C}}$ which is not an exceptional point, there exists a sequence of preimages of w that converges to z_0.

Brolin's Theorem

For all $w \in \mathbb{C}$ except at most 2 (exceptional) points,

$$\frac{1}{2^n}(p^n)^*\delta_w \to \frac{1}{2\pi}\Delta G.$$

Transitivity vs Brolin's Theorem

Theorem (Transitivity)

Let $p(z)$ be a polynomial and let $z_0 \in J_p$. Then for any $w \in \widehat{\mathbb{C}}$ which is not an exceptional point, there exists a sequence of preimages of w that converges to z_0.

Brolin's Theorem

For all $w \in \mathbb{C}$ except at most 2 (exceptional) points,

$$\frac{1}{2^n}(p^n)^*\delta_w \to \frac{1}{2\pi}\Delta G.$$

Brolin's theorem is the measure theoretic analog:

Transitivity vs Brolin's Theorem

Theorem (Transitivity)

Let $p(z)$ be a polynomial and let $z_0 \in J_p$. Then for any $w \in \widehat{\mathbb{C}}$ which is not an exceptional point, there exists a sequence of preimages of w that converges to z_0.

Brolin's Theorem

For all $w \in \mathbb{C}$ except at most 2 (exceptional) points,

$$\frac{1}{2^n}(p^n)^*\delta_w \to \frac{1}{2\pi}\Delta G.$$

Brolin's theorem is the measure theoretic analog:

If we take an arbitrary point w and start pulling back by p^n,

Transitivity vs Brolin's Theorem

Theorem (Transitivity)

Let $p(z)$ be a polynomial and let $z_0 \in J_p$. Then for any $w \in \widehat{\mathbb{C}}$ which is not an exceptional point, there exists a sequence of preimages of w that converges to z_0.

Brolin's Theorem

For all $w \in \mathbb{C}$ except at most 2 (exceptional) points,

$$\frac{1}{2^n}(p^n)^* \delta_w \to \frac{1}{2\pi} \Delta G.$$

Brolin's theorem is the measure theoretic analog:

If we take an arbitrary point w and start pulling back by p^n,

then the pull back measure assigns mass the same way as μ does.

Proof of Brolin's Theorem for $p(z) = z^2 + c, c \neq 0$

To simply notation, let $z_n := p^n(z)$ and $u_n(z) := \frac{1}{2^n} \log |z_n - w|$.

Proof of Brolin's Theorem for $p(z) = z^2 + c, c \neq 0$

To simply notation, let $z_n := p^n(z)$ and $u_n(z) := \frac{1}{2^n} \log |z_n - w|$.

Notice $\frac{1}{2\pi} \Delta u_n = \frac{1}{2^n} (p^n)^* \delta_w$

Proof of Brolin's Theorem for $p(z) = z^2 + c, c \neq 0$

To simply notation, let $z_n := p^n(z)$ and $u_n(z) := \frac{1}{2^n} \log |z_n - w|$.

Notice $\frac{1}{2\pi} \Delta u_n = \frac{1}{2^n} (p^n)^* \delta_w$

Outside of the filled Julia set, u_n converges locally uniformly to G.

Proof of Brolin's Theorem for $p(z) = z^2 + c, c \neq 0$

To simply notation, let $z_n := p^n(z)$ and $u_n(z) := \frac{1}{2^n} \log |z_n - w|$.

Notice $\frac{1}{2\pi} \Delta u_n = \frac{1}{2^n} (p^n)^* \delta_w$

Outside of the filled Julia set, u_n converges locally uniformly to G.

Take a large enough disc containing K_p. We need to prove convergence in the disc.

Proof of Brolin's Theorem for $p(z) = z^2 + c, c \neq 0$

To simply notation, let $z_n := p^n(z)$ and $u_n(z) := \frac{1}{2^n} \log |z_n - w|$.

Notice $\frac{1}{2\pi} \Delta u_n = \frac{1}{2^n}(p^n)^* \delta_w$

Outside of the filled Julia set, u_n converges locally uniformly to G.

Take a large enough disc containing K_p. We need to prove convergence in the disc.

u_n does not converge locally uniformly to $-\infty$.

Proof of Brolin's Theorem for $p(z) = z^2 + c, c \neq 0$

To simply notation, let $z_n := p^n(z)$ and $u_n(z) := \frac{1}{2^n} \log |z_n - w|$.

Notice $\frac{1}{2\pi} \Delta u_n = \frac{1}{2^n} (p^n)^* \delta_w$

Outside of the filled Julia set, u_n converges locally uniformly to G.

Take a large enough disc containing K_p. We need to prove convergence in the disc.

u_n does not converge locally uniformly to $-\infty$.

By the compactness theorem, for any sequence u_{n_j} there is a further subsequence that converges in L^1_{loc} to some SH function v.

Proof of Brolin's Theorem for $p(z) = z^2 + c, c \neq 0$

To simply notation, let $z_n := p^n(z)$ and $u_n(z) := \frac{1}{2^n} \log |z_n - w|$.

Notice $\frac{1}{2\pi}\Delta u_n = \frac{1}{2^n}(p^n)^*\delta_w$

Outside of the filled Julia set, u_n converges locally uniformly to G.

Take a large enough disc containing K_p. We need to prove convergence in the disc.

u_n does not converge locally uniformly to $-\infty$.

By the compactness theorem, for any sequence u_{n_j} there is a further subsequence that converges in L^1_{loc} to some SH function v.

We will abuse notation and just denote the subsequence as u_n.

Proof of Brolin's Theorem for $p(z) = z^2 + c, c \neq 0$

To simply notation, let $z_n := p^n(z)$ and $u_n(z) := \frac{1}{2^n} \log |z_n - w|$.

Notice $\frac{1}{2\pi} \Delta u_n = \frac{1}{2^n} (p^n)^* \delta_w$

Outside of the filled Julia set, u_n converges locally uniformly to G.

Take a large enough disc containing K_p. We need to prove convergence in the disc.

u_n does not converge locally uniformly to $-\infty$.

By the compactness theorem, for any sequence u_{n_j} there is a further subsequence that converges in L^1_{loc} to some SH function v.

We will abuse notation and just denote the subsequence as u_n.

Contradiction hypothesis: Suppose $v \neq G$. Then they differ on K_p. We will derive a contradiction using control of how much iterates of p can contract the radius of a disc.

Proof of Brolin's Theorem for $p(z) = z^2 + c, c \neq 0$

Recall $G(z) := \frac{1}{2^n} \log_+ |p^n(z)|$ measures the rate of escape of z.

Proof of Brolin's Theorem for $p(z) = z^2 + c, c \neq 0$

Recall $G(z) := \frac{1}{2^n} \log_+ |p^n(z)|$ measures the rate of escape of z.

Since $G \equiv 0$ on K, there is some $\delta > 0$ such that $\{v < -2\delta\}$ is a nonempty open set W.

Proof of Brolin's Theorem for $p(z) = z^2 + c, c \neq 0$

Recall $G(z) := \frac{1}{2^n} \log_+ |p^n(z)|$ measures the rate of escape of z.

Since $G \equiv 0$ on K, there is some $\delta > 0$ such that $\{v < -2\delta\}$ is a nonempty open set W.

By Hartogs' Lemma, $\limsup_{j \to \infty} u_n(x) \leq v(x)$, so there is a precompact open set $W_0 \subset W$ such that $\frac{1}{2^n} \log |z_n - w| < -\delta$ for all $z_0 \in W_0$.

Proof of Brolin's Theorem for $p(z) = z^2 + c, c \neq 0$

Recall $G(z) := \frac{1}{2^n} \log_+ |p^n(z)|$ measures the rate of escape of z.

Since $G \equiv 0$ on K, there is some $\delta > 0$ such that $\{v < -2\delta\}$ is a nonempty open set W.

By Hartogs' Lemma, $\limsup_{j\to\infty} u_n(x) \leq v(x)$, so there is a precompact open set $W_0 \subset W$ such that $\frac{1}{2^n} \log |z_n - w| < -\delta$ for all $z_0 \in W_0$.

So $|z_n - w| < e^{-\delta 2^n}$.

Proof of Brolin's Theorem for $p(z) = z^2 + c, c \neq 0$

Recall $G(z) := \frac{1}{2^n} \log_+ |p^n(z)|$ measures the rate of escape of z.

Since $G \equiv 0$ on K, there is some $\delta > 0$ such that $\{v < -2\delta\}$ is a nonempty open set W.

By Hartogs' Lemma, $\limsup_{j \to \infty} u_n(x) \leq v(x)$, so there is a precompact open set $W_0 \subset W$ such that $\frac{1}{2^n} \log |z_n - w| < -\delta$ for all $z_0 \in W_0$.

So $|z_n - w| < e^{-\delta 2^n}$.

Geometrically, $p^n(z)$ will map W_0 inside a disc of radius $e^{-\delta 2^n}$.

Proof of Brolin's Theorem for $p(z) = z^2 + c, c \neq 0$

Recall $G(z) := \frac{1}{2^n} \log_+ |p^n(z)|$ measures the rate of escape of z.

Since $G \equiv 0$ on K, there is some $\delta > 0$ such that $\{v < -2\delta\}$ is a nonempty open set W.

By Hartogs' Lemma, $\limsup_{j\to\infty} u_n(x) \leq v(x)$, so there is a precompact open set $W_0 \subset W$ such that $\frac{1}{2^n} \log |z_n - w| < -\delta$ for all $z_0 \in W_0$.

So $|z_n - w| < e^{-\delta 2^n}$.

Geometrically, $p^n(z)$ will map W_0 inside a disc of radius $e^{-\delta 2^n}$.

How is that a contradiction?

Proof of Brolin's Theorem for $p(z) = z^2 + c, c \neq 0$

For any disc of radius r small, $p^n(\mathbb{D}(z, r))$ always contains a disc of radius $Ar^{\sqrt{2}^n}$ for some constant $A > 0$.

Proof of Brolin's Theorem for $p(z) = z^2 + c, c \neq 0$

For any disc of radius r small, $p^n(\mathbb{D}(z, r))$ always contains a disc of radius $Ar^{\sqrt{2}^n}$ for some constant $A > 0$.

Idea of proof:

Proof of Brolin's Theorem for $p(z) = z^2 + c, c \neq 0$

For any disc of radius r small, $p^n(\mathbb{D}(z,r))$ always contains a disc of radius $Ar^{\sqrt{2}^n}$ for some constant $A > 0$.

Idea of proof:

Near the critical point, image of disc with radius r contains a disc of radius $b_1 r^2$.

Proof of Brolin's Theorem for $p(z) = z^2 + c, c \neq 0$

For any disc of radius r small, $p^n(\mathbb{D}(z,r))$ always contains a disc of radius $Ar^{\sqrt{2}^n}$ for some constant $A > 0$.

Idea of proof:

Near the critical point, image of disc with radius r contains a disc of radius $b_1 r^2$.

Away from the critical point, image of a disc with radius r contains a disc of radius $b_2 r$.

Proof of Brolin's Theorem for $p(z) = z^2 + c, c \neq 0$

For any disc of radius r small, $p^n(\mathbb{D}(z, r))$ always contains a disc of radius $Ar^{\sqrt{2}^n}$ for some constant $A > 0$.

Idea of proof:

Near the critical point, image of disc with radius r contains a disc of radius $b_1 r^2$.

Away from the critical point, image of a disc with radius r contains a disc of radius $b_2 r$.

Since $c \neq 0$, a disc can only get mapped near the critical point every other iterate.

Proof of Brolin's Theorem for $p(z) = z^2 + c, c \neq 0$

From previous slide, we have

$$\mathbb{D}(z_\bullet, Ar^{\sqrt{2}^n}) \subset p^n(W_0).$$

Proof of Brolin's Theorem for $p(z) = z^2 + c, c \neq 0$

From previous slide, we have

$$\mathbb{D}(z_\bullet, Ar^{\sqrt{2}^n}) \subset p^n(W_0).$$

By contradiction hypothesis and Hartogs' lemma, we have

$$p^n(W_0) \subset \mathbb{D}(w, e^{-\delta 2^n}).$$

Proof of Brolin's Theorem for $p(z) = z^2 + c, c \neq 0$

From previous slide, we have

$$\mathbb{D}(z_\bullet, Ar^{\sqrt{2}^n}) \subset p^n(W_0).$$

By contradiction hypothesis and Hartogs' lemma, we have

$$p^n(W_0) \subset \mathbb{D}(w, e^{-\delta 2^n}).$$

$$\mathbb{D}(z_\bullet, Ar^{\sqrt{2}^n}) \subset p^n(W_0) \subset \mathbb{D}(w, e^{-\delta 2^n}).$$

Proof of Brolin's Theorem for $p(z) = z^2 + c, c \neq 0$

From previous slide, we have

$$\mathbb{D}(z_\bullet, Ar^{\sqrt{2}^n}) \subset p^n(W_0).$$

By contradiction hypothesis and Hartogs' lemma, we have

$$p^n(W_0) \subset \mathbb{D}(w, e^{-\delta 2^n}).$$

$$\mathbb{D}(z_\bullet, Ar^{\sqrt{2}^n}) \subset p^n(W_0) \subset \mathbb{D}(w, e^{-\delta 2^n}).$$

For n large, $e^{-\delta 2^n} < Ar^{\sqrt{2}^n}$.

Proof of Brolin's Theorem for $p(z) = z^2 + c, c \neq 0$

From previous slide, we have

$$\mathbb{D}(z_\bullet, Ar^{\sqrt{2}^n}) \subset p^n(W_0).$$

By contradiction hypothesis and Hartogs' lemma, we have

$$p^n(W_0) \subset \mathbb{D}(w, e^{-\delta 2^n}).$$

$$\mathbb{D}(z_\bullet, Ar^{\sqrt{2}^n}) \subset p^n(W_0) \subset \mathbb{D}(w, e^{-\delta 2^n}).$$

For n large, $e^{-\delta 2^n} < Ar^{\sqrt{2}^n}$.

CONTRADICTION!

Proof of Brolin's Theorem for $p(z) = z^2 + c, c \neq 0$

From previous slide, we have

$$\mathbb{D}(z_\bullet, Ar^{\sqrt{2}^n}) \subset p^n(W_0).$$

By contradiction hypothesis and Hartogs' lemma, we have

$$p^n(W_0) \subset \mathbb{D}(w, e^{-\delta 2^n}).$$

$$\mathbb{D}(z_\bullet, Ar^{\sqrt{2}^n}) \subset p^n(W_0) \subset \mathbb{D}(w, e^{-\delta 2^n}).$$

For n large, $e^{-\delta 2^n} < Ar^{\sqrt{2}^n}$.

CONTRADICTION!

QED

Why Hartogs' Lemma?

In higher dimensions, Montel's Theorem requires stronger hypotheses. For example, see

Why Hartogs' Lemma?

In higher dimensions, Montel's Theorem requires stronger hypotheses. For example, see

M. L. Green. The hyperbolicity of the complement of 2n+1 hyperplanes in general position in P^n and related results. Proc. Amer. Math. Soc., 66(1):p. 109-113, 1977.

Why Hartogs' Lemma?

In higher dimensions, Montel's Theorem requires stronger hypotheses. For example, see

M. L. Green. The hyperbolicity of the complement of 2n+1 hyperplanes in general position in P^n and related results. Proc. Amer. Math. Soc., 66(1):p. 109-113, 1977.

M. L. Green. Some Picard theorems for holomorphic maps to algebraic varieties. Amer. J. Math., 97:p.43-75, 1975.

Why Hartogs' Lemma?

In higher dimensions, Montel's Theorem requires stronger hypotheses. For example, see

M. L. Green. The hyperbolicity of the complement of 2n+1 hyperplanes in general position in P^n and related results. Proc. Amer. Math. Soc., 66(1):p. 109-113, 1977.

M. L. Green. Some Picard theorems for holomorphic maps to algebraic varieties. Amer. J. Math., 97:p.43-75, 1975.

Hartogs' Lemma still holds with the same hypotheses.

Why Hartogs' Lemma?

In higher dimensions, Montel's Theorem requires stronger hypotheses. For example, see

M. L. Green. The hyperbolicity of the complement of 2n+1 hyperplanes in general position in P^n and related results. Proc. Amer. Math. Soc., 66(1):p. 109-113, 1977.

M. L. Green. Some Picard theorems for holomorphic maps to algebraic varieties. Amer. J. Math., 97:p.43-75, 1975.

Hartogs' Lemma still holds with the same hypotheses.

Applications: We can prove analogs of Brolin's theorem to holomorphic endomorphisms of $\mathbb{CP}^2$, or automorphisms of $\mathbb{C}^2$. For example the complex Hénon map.