Def Space and the Topology of Moduli Spaces

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Introduction

Cubic polynomials can be parametrized by

$$p_{a,b}(z) = z^3 - 3a^2 z + b,$$

with critical points at $\pm a$.
The parameter space is $\mathbb{C}^2$, parametrized by a, b.
So the cubic polynomials with one critical point periodic of period dividing k are formed by the plane curve of equation

$$p_{a,b}^{\circ k}(a) = a.$$

This is a plane curve of degree 3^k, and is reducible, as it has irreducible components corresponding to periods dividing k.
Let C_k be the union of the irreducible components where generically the critical point a has period exactly k.

The topology of moduli spaces

Is C_k connected? Irreducible? What is its genus?
One can imagine many similar questions about the topology of spaces of dynamical systems,
(polynomials or rational functions).
But they have proved inaccessible to algebraic methods.
I will propose an alternative transcendental approach, based on Teichmüller spaces.
There are many contributors to these ideas:
Adam Epstein, Bill Thurston, Mary Rees, Sarah Koch,
Eriko Hironaka, Tanya Firsova, Jeremy Kahn, Xavier Buff,
Ahmad Rafiqi, Jan Kiwi, Harry Baik,
Xiaoguang Wang, Nikita Selinger
and I hope I haven't forgotten too many names.

Teichmüller spaces

Let S be a compact oriented surface (usually a sphere in this lecture) and $Z \subset S$ a finite subset. We will suppose that the Euler characteristic of $S - Z$ is negative. The *Teichmüller space*

$$\mathcal{T}_{S,Z}$$

modeled on (S, Z) is the space of pairs (X, ϕ), where

- X is a Riemann surface, and
- $\phi : S \to X$ is a homeomorphism

up to the equivalence that $(X_1, \phi_1) \sim (X_2, \phi_2)$ if

- there exists an analytic isomorphism $\alpha : X_1 \to X_2$ such that
- $\alpha \circ \phi_1$ and ϕ_2 coincide on Z, and
- $\alpha \circ \phi_1$ is isotopic to ϕ_2 rel Z.

Some properties of Teichmüller spaces

The space $\mathcal{T}_{S,Z}$ is a complex manifold of dimension $3g(S)-3+|Z|$. The cotangent space $T^*_{(X,\phi)}\mathcal{T}_{S,Z}$ is the space

$$Q^1(X-\phi(Z))$$

of integrable holomorphic quadratic differentials on $X-\phi(Z)$.

It is also a metric space (in many ways, but only one will concern us here):

$$d((X_1,\phi_1),(X_2,\phi_2)) = \log\inf_f K(f),$$

where $f : X_1 \to X_2$ is a C^1 diffeomorphism where

- $f\circ\phi_1$ and ϕ_2 coincide on Z,
- $f\circ\phi_1$ is isotopic to ϕ_2 rel Z, and

$K(f)$ is the maximum over $x\in X_1$ of the ratio $\mathrm{Jac}\, f(x)/\|Df(x)\|^2$.

Some properties of Teichmüller spaces II

Thus a point of Teichmüller space consists of two kinds of data:

- Analytic data: the Riemann surface X, the positions of the marked points $\phi(z),\ z \in Z$;
- Topological: the marking ϕ, that is only defined up to homotopy rel Z.

Example

Let S is a sphere, and $|Z| \geqslant 3$. Then $\mathcal{T}_{S,Z}$ is the universal covering space of the subset of $\mathbb{C}^{|Z|-3}$ where no coordinate is 0 or 1, and all coordinates are distinct. Indeed, we can put 3 points of Z at $0, 1, \infty$ by a Moebius transformation, and then the positions of the others give the coordinates in $\mathbb{C}^{|Z|-3}$.

There is a whole field of mathematics devoted understanding these spaces: Enumerative Geometry.

Why study Teichmüller spaces?

The definition is pretty elaborate. Why carry along ϕ?
Why not just study moduli space,
the space of isomorphism classes of Riemann surfaces,
perhaps with marked points.

The topology of moduli spaces is apparently of interest to physicists.
I can't claim to understand why, but the problem is central to conformal field theory.

The direct study of moduli spaces has proved intractable,
and what is understood about them is largely due
to the study of Teichmüller spaces. These have a rich geometry
with many questions solved, and many others still open.

The geometry of Teichmüller spaces

$\mathcal{T}_{(S,Z)}$ is complete under the Teichmüller metric.
$\mathcal{T}_{(S,Z)}$ is contractible, homeomorphic to $\mathbb{R}^{6g(S)-6+2|Z|}$.

The metric is Finsler, defined by a norm on the tangent spaces.
Here the norm is dual to the L^1-norm

$$\|q\| = \int_{X-\phi(Z)} |q| \qquad \text{on the cotangent space.}$$

The Teichmüller metric is also the Kobayashi metric:
the unit ball $T_{X,\phi}\mathcal{T}_{(S,Z)}$ is the set of $f'(0)$
for all analytic $f : \mathcal{D} \to \mathcal{T}_{S,Z}$ with $f(0) = (X, \phi)$.

There is a unique geodesic joining any pair of points of $\mathcal{T}_{(S,Z)}$.

The deformation space

From here on, S will be a sphere.
We will write $\mathcal{T}_Z$ instead of $\mathcal{T}_{S,Z}$.

Suppose $f : S \to S$ is a ramified covering map, with critical set $\mathrm{Crit}(f)$ for the critical set of f.
Let A and B be finite subsets of S such that

$$B \supset A \cup f(A) \cup f(\mathrm{Crit}(f)).$$

In particular, B must contain all the critical values.
There are then two analytic maps

$$\mathcal{T}_B \underset{\sigma_f}{\overset{i_{A,B}}{\rightrightarrows}} \mathcal{T}_A.$$

The deformation space, II

Let $\phi(S, B) \to \mathbb{P}^1$ represent a point in $\mathcal{T}_B$.
The forgetful map $i_{A,B}$ simply forgets $\phi(x)$ when $x \in B - A$.
The pull-back map σ_f requires the uniformization theorem: restrictions of $\phi \circ f$ are an atlas on S, making it isomorphic to P^1; let $\phi' : (S, A) \to \mathbb{P}^1$ be such an isomorphism. Then $\sigma_f(\phi) = \phi'$.

It isn't absolutely clear that we have a complex structure on S near a critical point $x \in \mathrm{Crit}(f)$, but it is easy to show using an appropriate dth root, where d is the local degree of f at x. The map ϕ' is only defined up to post composition with an automorphism of $\mathbb{P}^1$, but that is allowed in our definition of Teichmüller space.

$\mathrm{Def}_B^A(f)$is the equalizer of these two maps:

$$\mathrm{Def}_B^A(f = \{\tau \in \mathcal{T}_B \mid i_{(A,B)}(\tau) = \sigma_f(\tau)\}.)$$

The deformation space, III

Let ϕ represent an element of $\mathcal{T}_B$.
Another way to say that $\mathrm{Def}_B^A(f)$is the equalizer is to say that there exists a commuting diagram

$$\begin{array}{ccc} (S, A) & \xrightarrow{\phi'} & \mathbb{P}^1 \\ f \downarrow & & \downarrow f_\phi \\ (S, B) & \xrightarrow{\phi} & \mathbb{P}^1 \end{array}$$

with f_ϕ analytic, and ϕ' a homeomorphism agreeing with ϕ on A.

The relation to dynamical systems

It isn't clear that $\mathrm{Def}_B^A(f)$has any relation to dynamics.
But it does: it parametrizes a set of rational functions which have finite sets with the dynamics exhibited by A under f.

Denote by Rat_d the space of rational maps of degree d, and by $[\mathrm{Rat}]_d$ the quotient by $\mathrm{Aut}\,\mathbb{P}^1$ acting by conjugation.

Theorem

There exists a unique analytic map $\Phi : \mathrm{Def}_B^A(f) \to [\mathrm{Rat}]_d$ *and a map* $\widetilde{\Phi} : B \times \mathrm{Def}_B^A(f \to \mathbb{P}^1 \times [\mathrm{Rat}]_d)$*such that the diagram*

$$\begin{array}{ccc} A \times \mathrm{Def}_B^A(f) & \xrightarrow{\widetilde{\Phi}|_A} & \mathbb{P}^1 \times [\mathrm{Rat}]_d \\ \downarrow & & \downarrow \widetilde{F}_d \\ B \times \mathrm{Def}_B^A(f) & \xrightarrow{\widetilde{\Phi}} & \mathbb{P}^1 \times [\mathrm{Rat}]_d \end{array}$$

commutes.

Thurston's theorem

The inspiration for the definition of $\mathrm{Def}_B^A(f)$is Thurston's theorem, which concerns the case $A = B$.

The condition $B \supset A \cup f(A) \cup f(\mathrm{Crit}(f)$ implies that the post-critical set

$$P_f = \cup_{n>0} f^{\circ n}(\mathrm{Crit}\, f)$$

is finite, and that $A = B$ is the union of P_n and some cycles and their preimages.

Theorem (Thurston)

In that case either $\mathrm{Def}_A^A(f)$*consists of a single point, or it is empty. In that case, there is an* f*-invariant multicurve* Γ *on* $S - A$ *such that the linear transformation*

$$f^\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$$

has leading eigenvalue $\lambda_\Gamma \geqslant 1$.

Thurston's theorem,II

A *multicurve* Γ on $S - A$ is a collection of disjoint, non-peripheral, non-homotopic simple closed curves.

It is f-invariant if every component of $f^{-1}(\gamma),\ \gamma \in \Gamma$ is homotopic rel A to an element of Γ or peripheral.

In that case, $f^\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ is defined by

$$f^\Gamma([\gamma]) = \sum_{\delta \in \Gamma} \left(\sum_{\eta \text{ component of } f^{-1}(\gamma) \text{ homotopic to } \delta} \frac{1}{\deg f : \eta \to \gamma} \right) [\delta].$$

This is a non-negative matrix, so it has a unique leading eigenvalue λ_Γ.

When $\lambda_\Gamma \geqslant 1$, we say that Γ is a *Thurston obstruction*.

$\mathrm{Def}_B^A(f)$ is a manifold

Theorem (Adam Epstein)

The space $\mathrm{Def}_B^A(f \subset \mathcal{T}_B)$*is a complex submanifold of dimension* $|B - A|$.
Its cotangent space is

$$T^*_{(\mathbb{P}^1,\phi)} \mathrm{Def}_B^A(f) = \mathrm{coker}\ \nabla_f$$

where

$$\nabla_f = I - (f_\phi)_* : Q^1(\mathbb{P}^1 - \sigma_f(\phi)(A)) \to Q^1(\mathbb{P}^1 - \phi(B)).$$

Here $(f_\phi)_*$ *is the direct image of an integrable quadratic differential.*

$\mathrm{Def}_B^A(f)$ is a manifold,II

Note that even if a quadratic differential q is holomorphic at a critical point of f_Φ, the quadratic differential $(f_\Phi)_* q$ will likely have a simple pole at the corresponding critical value. Thus the critical values had to be in B for this to make sense.

The second part is more or less obviously the coderivative of "$I - \sigma_f$", the equation of the equalizer. Then the fact that $\mathrm{Def}_B^A(f)$is a manifold follows from $\|(f_\Phi)_*\| < 1$, which is true except for some Lattès examples.

Why is this interesting?

Then $\mathrm{Def}_B^A(f)$maps analytically and surjectively to the curve of monic centered cubic polynomials for which both critical points are ordinary (hence distinct), one is periodic of period k and the other is not in the orbit of the first.

Theorem

$\mathrm{Def}_B^A(f)$*connected implies* C_k *irreducible.*

Why is this interesting II?

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Theorem

$\mathrm{Def}_B^A(f)$*connected implies* C_k *irreducible.*

One can easily imagine that this is just one case where and appropriate choice of f, A, B will relate the topology of $\mathrm{Def}_B^A(f)$to the algebraic geometry of loci of rational functions whose critical points have specified dynamical properties.

The topology of $\mathrm{Def}_B^A(f)$

What hope is there of understanding whether $\mathrm{Def}_B^A(f)$is connected, or contractible, or anything else?

Answer: quite a lot, because there is a vectorfield $\xi_B^A(f)$ on $\mathcal{T}_B$ vanishing only on $\mathrm{Def}_B^A(f.)$

I am not absolutely sure, but I believe that Mary Rees proved that $\xi_B^A(f)$ is Lipshitz, so it has a well-defined local flow. Because the Teichmüller metric is the Kobayashi metric, along flow lines the vector field gets shorter and shorter. It follows that the flow is defined for all time, and all solutions must converge to $\mathrm{Def}_B^A(f)$or to infinity.

So either $\mathrm{Def}_B^A(f)$is contractible, or there are loci at infinity that are attracting.

The vector field $\xi_B^A(f)$

Choose $\tau \in \mathcal{T}_B$ Then $i_{A,B}(\tau)$ and $\sigma_f(\tau)$ are two points of $\mathcal{T}_A$, and if they coincide if and only if $\tau \in \mathrm{Def}_B^A(f.)$If they do not coincide, they are joined by a unique geodesic P_τ in $\mathcal{T}_A$. Parametrize $l_\tau : \mathbb{R} \to P_\tau$ so that

$$l_\tau(0) = i_{A,B}(\tau) \quad \text{and} \quad l_\tau(1) = \sigma_f(\tau).$$

The tangent vector $l'_\tau(0)$ to $\mathcal{T}_A$ can be uniquely lifted to the tangent vector $\xi_B^A(f)(\tau)$ to $\mathcal{T}_B$ at τ.
In fact, you can lift the whole geodesic isometrically to a map $\tilde{l}_\tau$ such that

$$i_{A,B}(\tilde{l}_\tau(t)) = l_\tau(t).$$

The map $\tau \mapsto \tilde{l}_\tau(1)$ is probably just as interesting as the flow of $\xi_B^A(f)$.

Loci at infinity

In the Thurston case where $A = B$, the space $\mathrm{Def}_A^A(f)$is either a point or empty.
In the case where $\mathrm{Def}_A^A(f,)$the flow of $\xi_B^A(f)$ tends to infinity, more specifically to a component of the extended Teichmüller space corresponding to a Thurston obstruction Γ.
The eigenvalue λ_Γ measures whether the component is attracting or repelling.

In the general case where $A \neq B$, if the space $\mathrm{Def}_B^A(f)$is not empty it is not compact, so must accumulate at infinity.
Where? Is this locus attracting or repelling?

I think there are people who know about these things, but it is mainly
WORK IN PROGRESS.