# Zeros, chaotic ratios and the computational complexity of approximating the independence polynomial

David de Boer
Joint work with: Pjotr Buys, Lorenzo Guerini, Han Peters and Guus Regts

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University of Amsterdam
Korteweg de Vries Institute for Mathematics

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# Very short summary

$$\overline{\{\lambda \in \mathbb{C} : Z_G(\lambda) = 0 \text{ for some } G \in \mathcal{G}_\Delta\}}$$

$$=$$

$$\{\lambda_0 \in \mathbb{C} \mid \{\lambda \mapsto R_{G,v}(\lambda) \mid (G,v) \in \mathcal{G}_\Delta\} \text{ is not locally normal at } \lambda_0\}$$

$$=$$

$$\overline{\{\lambda \in \mathbb{C} \mid \{R_{G,v}(\lambda) \mid (G,v) \in \mathcal{G}_\Delta\} \text{ is dense in } \mathbb{C}\}}$$

$$\subseteq$$

$$\overline{\{\lambda \in \mathbb{Q}[i] : \text{ approximating } Z_G(\lambda) \text{ is } \#P\text{-hard}\}}$$

# Independent sets

**Definition**

Let $G = (V, E)$ a finite graph. A subset $I \subseteq V$ is called an independent set if there are no edges between vertices in $I$. Denote $\mathcal{I}(G)$ the set of independents sets of $G$.

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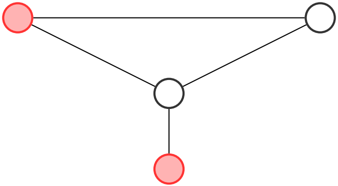

Figure: Independent set of size 2

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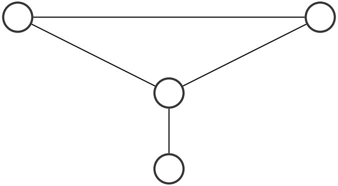

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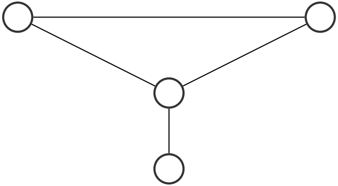
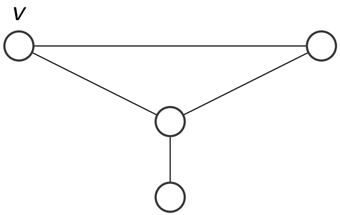


Figure: Independence polynomial $1 + 4\lambda + 2\lambda^2$.

# In and out

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**Example**

We define $Z^{in}_{G,v}(\lambda) =$

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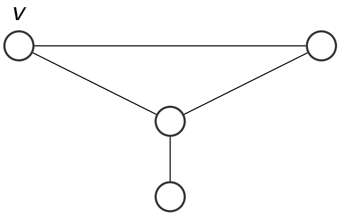


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**Example**

We define $Z^{in}_{G,v}(\lambda) = \lambda + \lambda^2$ and $Z^{out}_{G,v}(\lambda) =$

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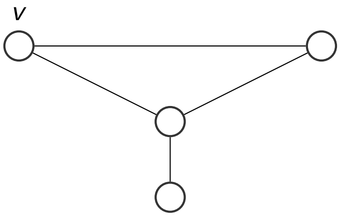


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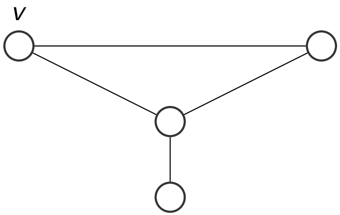


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**Remark**

We have $Z_G(\lambda) = Z^{in}_{G,v}(\lambda) + Z^{out}_{G,v}(\lambda)$.

# Ratio

**Definition**

Let $G = (V, E)$ a graph, and $v \in V$ a vertex. We define the rational function

$$R_{G,v}(\lambda) = \frac{Z^{in}_{G,v}(\lambda)}{Z^{out}_{G,v}(\lambda)}.$$

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## Remark

Note that if $R_{G,v}(\lambda) = -1$, then we have $Z^{in}_{G,v}(\lambda) = -Z^{out}_{G,v}(\lambda)$ and thus $0 = Z^{in}_{G,v}(\lambda) + Z^{out}_{G,v}(\lambda) = Z_G(\lambda)$.

# Ratio of a point and an edge

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$$R_G\left(\begin{smallmatrix}\circ\\ | \\ \circ\end{smallmatrix}\right) = \frac{\begin{smallmatrix}\color{green}{\bullet}\\ | \\ \circ\end{smallmatrix}}{\begin{smallmatrix}\color{red}{\bullet}\\ | \\ \circ\end{smallmatrix}} = \frac{\lambda\cdot\color{red}{\bullet}}{\color{red}{\bullet} + \color{green}{\bullet}} = \frac{\lambda\cdot 1}{1+\lambda}$$

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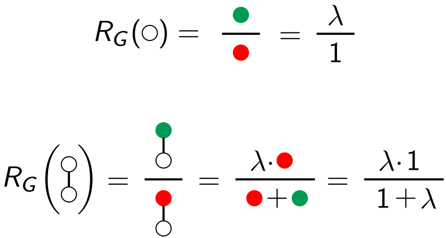
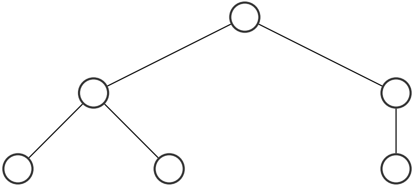



**Remark**

If we define $f_\lambda(z) = \frac{\lambda}{1+z}$, we see the ratio of an edge is $f_\lambda$ applied to the ratio of a point.

# Trees instead of graphs



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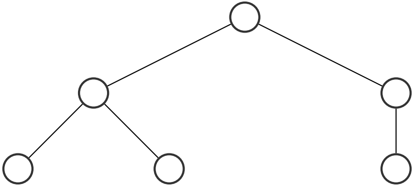


The following result due to Bencs [Ben18], building on Weitz [Wei06].

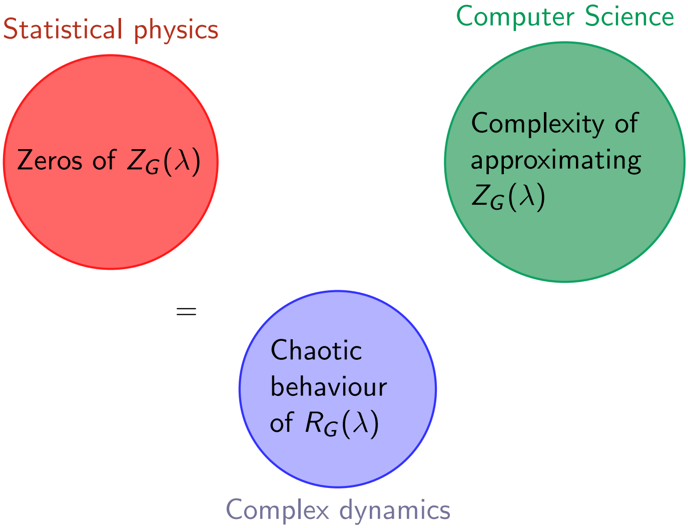
**Theorem**

*Let $(G, v) \in \mathcal{G}_\Delta$ be a rooted connected graph. Then there is a rooted tree $(T, u) \in \mathcal{G}_\Delta$ with $\deg_T(u) = \deg_G(v)$ such that*

*(i)* $Z_G$ *divides* $Z_T$,

*(ii)* $R_{G,v} = R_{T,u}$.

# Zeros, chaotic ratios and approximation



# Zeros and approximation

**Notation**

We denote

$$\mathcal{Z}_\Delta = \{\lambda \in \mathbb{C} : Z_G(\lambda) = 0 \text{ for some } G \in \mathcal{G}_\Delta\}.$$

We call $\overline{\mathcal{Z}_\Delta}$ the zero-locus.

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## Remark

Suppose $U$ is a simply connected open set containing 0 such that $\mathcal{Z}_\Delta \cap U = \emptyset$. Then for each $\lambda \in U$ there exist a polynomial time algorithm to approximate $Z_G(\lambda)$ for each $G \in \mathcal{G}_\Delta$ by Patel and Regts [PR17].

# Normal Family

Let $\mathcal{F}$ a family of rational maps $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

**Definition**

For an open $U \subseteq \hat{\mathbb{C}}$ the family $\mathcal{F}$ is normal on $U$ if each sequence $\{f_n\}_{n \geq 0} \subseteq \mathcal{F}$ has a subsequence that converges to a holomorphic $g : U \to \hat{\mathbb{C}}$, uniformly on compact $K \subseteq U$.

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**Definition**

We say a parameter $\lambda_0$ is active for $\mathcal{F}$ if for any open neighborhood of $\lambda_0$ the family $\mathcal{F}$ is not a normal family. The set of all active parameters is called the activity-locus of $\mathcal{F}$.

# Example Normality

Recall the map $f_\lambda(z) = \frac{\lambda}{1+z}$. We will determine when the family $\{\lambda \mapsto f_\lambda^n(0)\}$ is normal.

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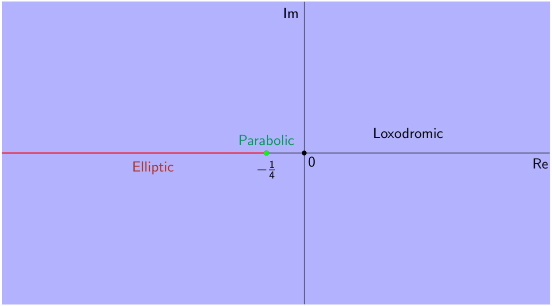


Elliptic $z \mapsto e^{i\theta} \cdot z$

Parabolic $z \mapsto z + 1$

Loxodromic $z \mapsto \xi \cdot z$ for some $\xi \in \mathbb{C}^*$ with $|\xi| < 1$

# Chaotic behaviour of ratios

**Definition**

We define the activity-locus $\mathcal{A}_\Delta$ to be the activity-locus of the family

$$\{\lambda \mapsto R_{G,v}(\lambda) : G \in \mathcal{G}_\Delta \text{ and } v \in V(G)\}.$$

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If the set

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is dense in $\hat{\mathbb{C}}$, we say $\lambda_0$ is a density parameter. We denote $\mathcal{D}_\Delta$ for the set of density parameters and define the density-locus to be the closure of $\mathcal{D}_\Delta$.

# Main theorem

**Definition**

We informally define the *$\#\mathcal{P}$-locus* as the closure of the collection of $\lambda$ for which approximating $Z_G(\lambda)$ is #P-hard for $G \in \mathcal{G}_\Delta$.

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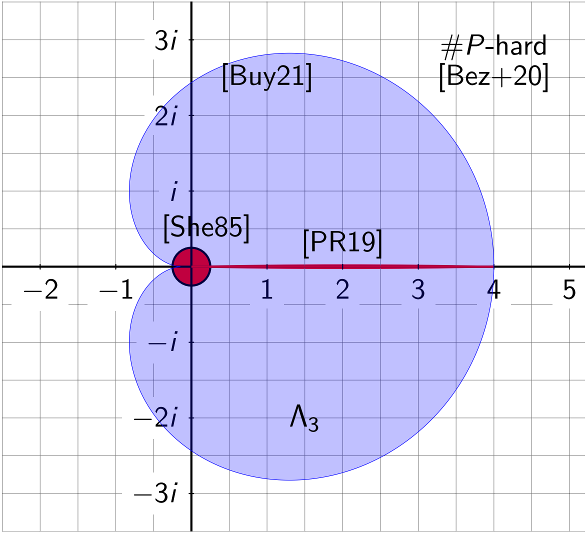
**Main Theorem**

For any integer $\Delta \geq 3$ the zero-locus, the activity-locus and the density-locus are equal and contained in the $\#\mathcal{P}$-locus. In other words:

$$\overline{\mathcal{Z}_\Delta} = \mathcal{A}_\Delta = \overline{\mathcal{D}_\Delta} \subseteq \overline{\#\mathcal{P}_\Delta}.$$

# Related results



# A useful lemma

**Lemma**

*Let* $\lambda \in \mathbb{C} \setminus \{0\}$*. If there exists a rooted graph* $(G, v) \in \mathcal{G}_\Delta$ *for which* $R_{G,v}(\lambda) \in \{-1, 0, \infty\}$*, then there exists a graph* $H$ *of maximum degree at most* $\Delta$ *for which* $Z_H(\lambda) = 0$*.*

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Now look at $\frac{\lambda \cdot Z_{G-N[v]}(\lambda)}{Z_{G-v}(\lambda)}$

□

# Active parameters have zeros nearby

**Corollary**

*For all* $\Delta \geq 2$ *the activity-locus is contained in the zero-locus, i.e.* $\mathcal{A}_\Delta \subseteq \overline{\mathcal{Z}_\Delta}$.

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**Montel's theorem**

Let $\mathcal{F}$ a family of holomorphic functions $f : U \to \hat{\mathbb{C}}$ for which there are three distinct values that each $f \in \mathcal{F}$ omits. Then $\mathcal{F}$ is normal on $S$.

□

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You can show that for $\lambda \in B_\Delta$, the family

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## Lemma

*Let $\Delta \geq 2$ be an integer. Then $B_\Delta$ is disjoint from the activity-locus, the density-locus and the zero-locus, so*

$$B_\Delta \cap \mathcal{A}_\Delta = B_\Delta \cap \overline{\mathcal{D}_\Delta} = B_\Delta \cap \overline{\mathcal{Z}_\Delta} = \emptyset.$$

# Activity locus for paths

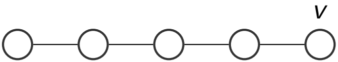


Figure: A path with root $v$ in an endpoint

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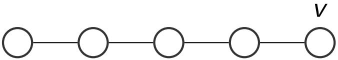


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**Remark**

The family $\{\lambda \mapsto R_{G,v}(\lambda) : G \in \mathcal{G}_\Delta \text{ and } v \in V(G)\}$ contains the family of ratios for these rooted paths with the root chosen in the endpoint.

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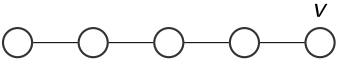


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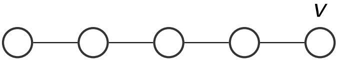


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## Corollary

*We have for each $\Delta \geq 2$ that*

$$(-\infty, -\frac{1}{4}] \subseteq \mathcal{A}_\Delta$$

# Ratio $-1$

**Lemma**

*Let $\lambda \in \mathbb{C}$ and $G \in \mathcal{G}_\Delta$ with $Z_G(\lambda) = 0$. Then there is a rooted tree $(T, u)$ with $\deg(u) = 1$ and $R_{T,u}(\lambda) = -1$.*

**Proof.**

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Note that $Z^{out}_{T,u}(\lambda) = Z_{T-u}(\lambda) \neq 0$, so $R_{T,u}(\lambda) = -1$

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# Zeros imply activity

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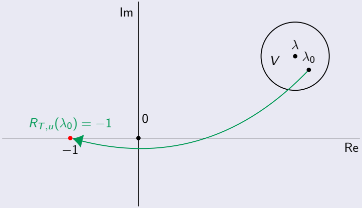


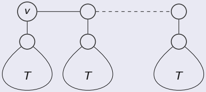

# Zeros imply activity

## Proposition

Let $\Delta \geq 2$. Then $\overline{\mathcal{Z}_\Delta} \subseteq \mathcal{A}_\Delta$.

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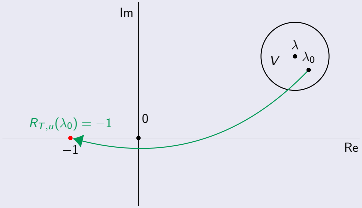


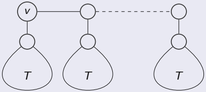


Denote the tree on the right as $(T_n, v)$ and the rooted path $(P_n, v_n)$. Then

$$R_{T_n,v} = R_{P_n,v_n} \circ R_{T,u}.$$

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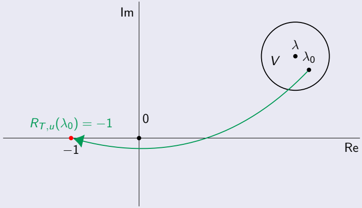


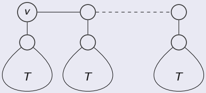


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Now $\{\lambda \mapsto R_{T_n,v}(\lambda)\}_{n\geq 0}$ is not normal on $V$, so $\lambda$ is active. □

# Recap

**Main Theorem**

For any integer $\Delta \geq 3$ the zero-locus, the activity-locus and the density-locus are equal and contained in the $\#\mathcal{P}$-locus. In other words:

$$\overline{\mathcal{Z}_\Delta} = \mathcal{A}_\Delta = \overline{\mathcal{D}_\Delta} \subseteq \overline{\#\mathcal{P}_\Delta}.$$

# Exponentially fast implementation

**Lemma**

*Let $\Delta \geq 3$ be an integer. Let $\lambda_0 \in \mathcal{D}_\Delta \cap \mathbb{Q}[i]$. Then given $P \in \mathbb{Q}[i]$ and rational $\varepsilon > 0$ there exists an algorithm that generates a rooted tree $(T, v)$ such that $|R_{T,v}(\lambda_0) - P| \leq \varepsilon$ and $Z^{out}_{T,v}(\lambda_0) \neq 0$, and outputs $Z^{in}_{T,v}(\lambda_0)$ and $Z^{out}_{T,v}(\lambda_0)$ in time bounded by poly(size($\varepsilon, P$)).*

# Exponentially fast implementation

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Now in [Bez+20, Section 6] the authors show that a polynomial time algorithm to approximate $|Z_G(\lambda_0)|$ within a constant factor, combined with the Lemma above for $\lambda_0$ yields an algorithm that on input of a graph $G$ of maximum degree at most $\Delta$ exactly computes $Z_G(1)$, the number of independent sets of $G$, in polynomial time in the number of vertices of $G$.

# Final Remarks

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**Question**

Is the region $\mathbb{C} \setminus \overline{\mathcal{Z}_\Delta}$ connected?

# The end

Thank you for your attention!

# Cayley trees

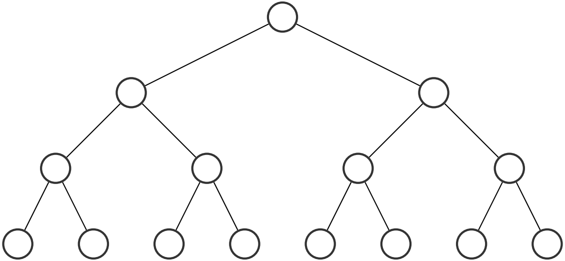


Figure: Cayley tree with down degree $d = 2$

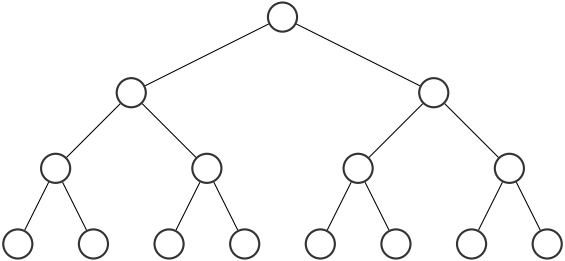
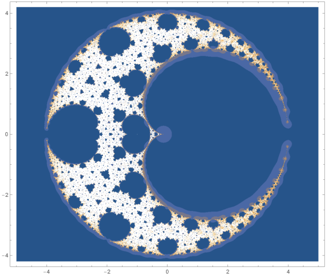


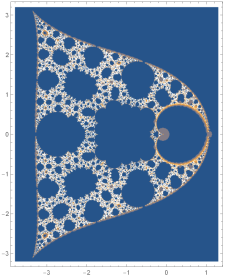
Figure: Cayley tree with down degree $d = 2$

Iteration of the map $f_{\lambda,d}(z) = \frac{\lambda}{(1+z)^d}$

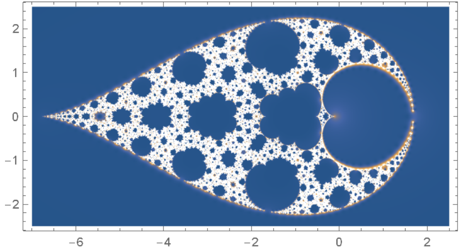
# Activity locus Cayley trees




down-degree 2




down-degree 4




down-degree 3

# Complexity of approximating $Z_G(\lambda)$

Let $\lambda \in \mathbb{Q}[i]$, $\Delta \in \mathbb{N}$ and consider the following computational problems.

*Name* $\#$IndependenceNorm$(\lambda, \Delta, K)$

*Input* A graph $G$ of maximum degree at most $\Delta$.

*Output* If $Z_G(\lambda) \neq 0$ the algorithm must output a rational number $N$ such that $N/K \leq |Z_G(\lambda)| \leq KN$. If $Z_G(\lambda) = 0$ the algorithm may output any rational number.

*Name* $\#$IndependenceArg$(\lambda, \Delta, \rho)$

*Input* A graph $G$ of maximum degree at most $\Delta$.

*Output* If $Z_G(\lambda) \neq 0$ the algorithm must output a rational number $A$ such that $|A - a| \leq \rho$ for some $a \in \arg(Z_G(\lambda))$. If $Z_G(\lambda) = 0$ the algorithm may output any rational number.

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For most $\lambda \in \mathbb{C}$ exact computation of $Z_G(\lambda)$ for large graphs is $\#P$-hard.

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Computing $Z_G(1)$ exactly is $\#P$-hard