On the neighbor exclusion model on the Cayley tree

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Interplay between statistical mechanics, graph theory, computational complexity and holomorphic dynamics
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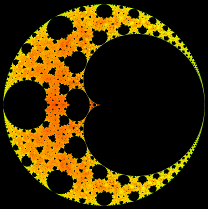


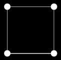
Figure: By Nuria Fagella.

Independence polynomial

$G = (V, E)$: Finite graph.

Independent set: $I \subset V$ such that $v, v' \in I \Rightarrow \{v, v'\} \notin E$;

Indenpendence polynomial: $Z_G(\lambda) := \sum_{\substack{I \subset V \\ \text{independent}}} \lambda^{\# I}$.



Independence polynomial: $1 + 4\lambda + 2\lambda^2$.

Statistical mechanics

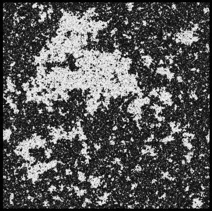


Figure: Ising model configuration at critical temperature, Mario Ullrich.

Statistical mechanics

Phase transitions: Failure of analytic dependence.

$G = (V, E)$: Finite graph;

Ω: Finite space.

Spin: Element of Ω;

Configuration: Element of Ω^V;

Interaction: $X \mapsto \Phi(X) \colon \Omega^X \to \mathbb{R} \cup \{+\infty\}$.

(X: Finite nonempty subset of V)

Statistical mechanics

Hamiltonian:

$$H_G(\Phi) := \sum_{X \subset G} \Phi(X);$$

Partition function:

$$Z_G(\Phi) := \sum_{\omega \in \Omega^V} \exp\left(-H_G(\Phi)(\omega)\right);$$

Pressure, $P(\Phi) :=$ "$\lim_{G\to\infty}$" $\frac{1}{|G|} \log Z_G(\Phi)$.

Gibbs measure:

$$\frac{1}{Z_G(\Phi)} \exp\left(-H_G(\Phi)\right) \cdot \mu_0.$$

$\mu_0 :=$ normalized counting measure on Ω^V.

Ising model

$$\Omega = \{+1, -1\};$$

$$\Phi(X)(\omega) = \begin{cases} -J\omega_v\omega_{v'} & \text{if } X = \{v, v'\} \in E; \\ -h\omega_v & \text{if } X = \{v\}; \\ 0 & \text{otherwise.} \end{cases}$$

J = "interaction" constant;
h = "magnetization" constant.

Z_G is a Laurent polynomial in $\lambda := \exp(-h)$.

Theorem (Lee-Yang, 1952)
Assume $J > 0$. Then all zeros of $Z_G(\lambda)$ are in the unit circle.

Fisher's predictions on the limit distribution of zeros and the phase transition.

Neighbor exclusion model

$$\Omega = \{0, 1\};$$

$$\Phi(X)(\omega) = \begin{cases} +\infty & \text{if } X = \{v, v'\} \in E \text{ and } \omega_v = \omega_{v'} = 1; \\ h\omega_v & \text{if } X = \{v\}; \\ 0 & \text{otherwise.} \end{cases}$$

h = "magnetization" constant.

Partition function = Independence polynomial;

$\lambda = \exp(-h)$ in $(0, +\infty)$.

$Z_G(\lambda)$ may vanish outside the unit circle.

The Lee-Yang theorem **does not** apply.

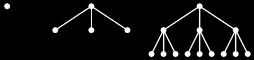
Regular trees

Figure: Bonsai Diurno - estado II, by Jorge Martínez García 2009.

Regular trees

d: Integer ≥ 2.

$\mathcal{T}_{d,k}$: d – regular tree of depth $k \geq 0$;
v_k : root of $\mathcal{T}_{d,k}$.



Phase transition

Phase transition at

$$\lambda_{\mathrm{cr}} := \frac{d^d}{(d-1)^{d+1}}.$$

Birth of a "period 2" Gibbs state;
Clustering of independent sets in odd/even sites;
Also detected computationally:
Polynomial time algorithm for approximating $Z_G(\lambda)$, for $\lambda \in (0, \lambda_{\mathrm{cr}})$;
NP-hard to approximate $|Z_G(\lambda)|$, for $\lambda > \lambda_{\mathrm{cr}}$.

Pressure function:

$$\mathscr{P} := \lim_{k \to \infty} \frac{1}{|\mathcal{T}_{d,k}|} \log |Z_{\mathcal{T}_{d,k}}|.$$

Thermodynamic limit as $k \to \infty$,
with free boundary conditions.

Phase transition

$$\mathscr{P} := \lim_{k \to \infty} \frac{1}{|\mathcal{T}_{d,k}|} \log |Z_{\mathcal{T}_{d,k}}|.$$

Theorem (RL, Sombra)

1. *The pressure function $\mathscr{P}$ is real analytic on $(0, +\infty) \setminus \{\lambda_{\mathrm{cr}}\}$, and infinitely differentiable at λ_{cr}.*
2. *The limit holds as electrostatic potentials on $\mathbb{C}$, and for every $\alpha > 0$:*

$$\lim_{r \to 0} \frac{\Delta \mathscr{P}(B(\lambda_{\mathrm{cr}}, r))}{r^{\alpha}} = 0.$$

$\Delta \mathscr{P}$ = Laplacian of $\mathscr{P}$;
$\Rightarrow \Delta \mathscr{P}$ has zero pointwise dimension at λ_{cr};
Consistent with Fisher's predictions.

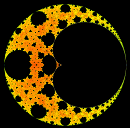


Figure: Support of $\Delta\mathscr{P}$ for $d = 2$, by Nuria Fagella.

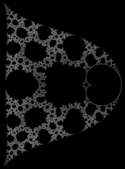


Figure: Support of $\Delta\mathscr{P}$ for $d = 4$, by Bernat Espigule.

Plan

Plan:

1. Recursive relation and complex dynamics;
   Scott–Sokal, 2005.
2. Speed of converge towards the thermodynamic limit;
   Quantative equidistribution.
3. Zero free region and pointwise dimension;
4. Smoothness of the pressure function.

Recursive relation and complex dynamics

$$R_{d,k} := \frac{Z_{\mathcal{T}_{d,k}}}{Z_{\mathcal{T}_{d,k}\setminus\{v_k\}}}; \quad f_{d,\lambda}(z) := 1 + \frac{\lambda}{z^d}.$$

$$\begin{aligned} R_{d,k}(\lambda) &= f_{d,\lambda}(R_{d,k-1}(\lambda)) \\ &= f_{d,\lambda}^k(1+\lambda). \end{aligned}$$

$$f_{d,\lambda}^k = \underbrace{f_{d,\lambda} \circ \cdots \circ f_{d,\lambda}}_{k}.$$

Dynamics of $f_{d,\lambda}$:

- Critical points: 0 and ∞;
- $0 \mapsto \infty \mapsto 1 \mapsto 1 + \lambda$.

$$\boxed{\Lambda_{d,k} := \{\lambda \in \mathbb{C} : Z_{\mathcal{T}_{d,k}}(\lambda) = 0\} = \{\lambda \in \mathbb{C} : f_{d,\lambda}^{k+3}(0) = 0\}.}$$

Centers of hyperbolic components of period $k+3$.

Speed of converge towards the thermodynamic limit

μ_{bif}: Bifurcation measure of $(f_{d,\lambda})_{\lambda\in\mathbb{C}}$.

$\mu_{\mathrm{bif}} := \lim_{k\to\infty} \frac{1}{\deg(F_k)} F_k^*\omega$, $F_k(\lambda) := f_{d,\lambda}^k(0)$, ω uniform measure on $\overline{\mathbb{C}}$.

Proposition

There is $C > 0$ such that for every Lipschitz function $\varphi\colon \mathbb{C} \to \mathbb{R}$, and $k \geq 3$,

$$\left| \frac{1}{|\Lambda_{d,k}|} \sum_{\lambda\in\Lambda_{d,k}} \varphi(\lambda) - \int \varphi \, \mathrm{d}\mu_{\mathrm{bif}} \right| \leq C \, \mathrm{Lip}(\varphi) \left(\frac{\log k}{d^k} \right)^{\frac{1}{2}}.$$

Arithmetic height function of $(f_{d,\lambda})_{\lambda\in\mathbb{C}^*}$;
+ arithmetic equidistribution (Favre–RL).

In particular,

$$\frac{1}{|\Lambda_{d,k}|} \sum_{\lambda\in\Lambda_{d,k}} \delta_\lambda \xrightarrow[k\to\infty]{} \mu_{\mathrm{bif}}.$$

Levin, RL–Favre, Dujardin–Favre, Okuyama, Gauthier–Vigny, …

Speed of converge towards the thermodynamic limit

Equidistribtion $\Leftrightarrow$

$$\mathscr{P} = \lim_{k\to\infty} \frac{1}{|\mathcal{T}_{d,k}|} \log |Z_{\mathcal{T}_{d,k}}|.$$

exists as electrostatic potentials on $\mathbb{C}$.

Not really, but morally true.

$$\boxed{\Delta\mathscr{P} = \mu_{\mathrm{bif}}.}$$

Zero free region and pointwise dimension

At $\lambda = \lambda_{\mathrm{cr}}$:

- $f_{\lambda_{\mathrm{cr}}}$ has $\zeta_0 := \frac{d}{d-1}$ as a fixed point of multiplier -1;
- $f^k_{\lambda_{\mathrm{cr}}}(0) \xrightarrow[k\to\infty]{} \zeta_0$.

Fatou's theorem.

Proposition

There is $C' > 0$ such that for $k \geq 3$,

$$\Lambda_{d,k} \cap B\left(\lambda_{\mathrm{cr}}, \frac{C'}{k}\right) = \emptyset.$$

Approximate Fatou coordinates.

Corollary

There is $\kappa \in (0,1)$, such that for every small $\rho > 0$

$$\mu_{\mathrm{bif}}\left(B(\lambda_{\mathrm{cr}}, \rho)\right) \leq \kappa^{\frac{1}{\rho}}.$$

Zero free region and pointwise dimension

Proof.

$\varphi\colon \mathbb{C} \to [0,1]$ Lipschitz, such that:

- $\mathrm{Lip}(\varphi) \sim \frac{1}{\rho}$;
- $\varphi \equiv 0$ on $B(\lambda_{\mathrm{cr}}, 2\rho)$;
- $\varphi \equiv 1$ on $B(\lambda_{\mathrm{cr}}, \rho)$.

Quantitative equidistribution $\Rightarrow$ For $k \geq 3$ so that $k \sim \frac{C'}{2\rho}$:

$$\begin{aligned}\mu_{\mathrm{bif}}(B(\lambda_{\mathrm{cr}}, \rho)) &\leq \int \varphi \, \mathrm{d}\mu_{\mathrm{bif}} \\ &\leq \frac{1}{|\Lambda_{d,k}|} \sum_{\lambda\in\Lambda_{d,k}} \varphi(\lambda) + C\,\mathrm{Lip}(\varphi)\left(\frac{\log k}{d^k}\right)^{\frac{1}{2}} \\ &= C\,\mathrm{Lip}(\varphi)\left(\frac{\log k}{d^k}\right)^{\frac{1}{2}} \sim C\frac{1}{\rho}\left(\log\frac{1}{\rho}\right)^{\frac{1}{2}}\left(d^{-\frac{C'}{4}}\right)^{\frac{1}{\rho}}.\end{aligned}$$

□

Smoothness of the pressure function

Proposition

$\widehat{\mu}$: probability measure on $\mathbb{C}$ such that

$$\lim_{r\to 0} \frac{\log \widehat{\mu}(B(0,r))}{\log r} = 0,$$

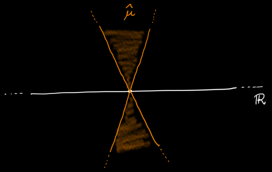
and for some $\eta > 0$ is supported on

$$\{z \in \mathbb{C} : \Im(z) > \eta\Re(z)\}.$$

$$\Rightarrow \widehat{\mathscr{P}}(x) := \int \log|x - \zeta| \, \mathrm{d}\widehat{\mu}(\zeta)$$

is infinitely differentiable on $\mathbb{R}$.

Smoothness of the pressure function



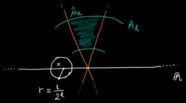
Smoothness of the pressure function

$$\widehat{\mathscr{P}}(\lambda) = \int \log|\lambda - \zeta| \, \mathrm{d}\widehat{\mu}(\zeta).$$

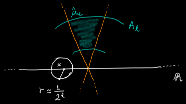
Divide and conquer:

$$\begin{aligned} A_\ell &:= \left\{\lambda \in \mathbb{C} : 2^{-(\ell+1)} < |\lambda| \le 2^{-\ell}\right\}; \\ \widehat{\mu}_\ell &:= \widehat{\mu}|_{A_\ell}; \\ \widehat{\mathscr{P}}_\ell(\lambda) &:= \int \log|\lambda - \zeta| \, \mathrm{d}\widehat{\mu}_\ell(\zeta). \end{aligned}$$

$$\Rightarrow \widehat{\mathscr{P}} = \sum_{\ell=-\infty}^{\infty} \widehat{\mathscr{P}}_\ell.$$



Smoothness of the pressure function



Cauchy estimate:

$$\frac{1}{(m-1)!}|\widehat{\mathscr{P}}_\ell^{(m)}(x)| \le \left| \int \frac{1}{(x-\zeta)^m} \, \mathrm{d}\widehat{\mu}_\ell(x) \right| \lessapprox \frac{1}{(2^{-\ell})^m} \widehat{\mu}_\ell(A_\ell).$$

$$\Rightarrow \frac{1}{(m-1)!} \sum_{\ell=-\infty}^{\infty} |\widehat{\mathscr{P}}_\ell^{(m)}(x)| \le \sum_{\ell=-\infty}^{\infty} 2^{\ell m} \widehat{\mu}\left(B\left(0, 2^{-\ell}\right)\right) < +\infty.$$