Pointwise Dimension of Bifurcation Measures and Critical Exponent of the Free Energy

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Interplay Between Statistical Mechanics, Graph Theory, Computational Complexity, and Holomorphic Dynamics
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Some Definitions

Let Λ be a complex manifold.

- *A holomorphic family of rational maps of degree d* is a holomorphic function $f : \Lambda \times \mathbb{P}^1 \to \mathbb{P}^1$, such that $f_\lambda(z) := f(\lambda, z)$ is a rational map in $\mathbb{P}^1$ of degree d for all $\lambda \in \Lambda$.
- A *marked critical point* is a holomorphic function $c : \Lambda \to \mathbb{P}^1$ such that $f'_\lambda c(\lambda) = 0$ for all λ.

Example to have in mind:

1. $f_\lambda(z) = z^2 + \lambda$.
2. $c(\lambda) = 0$.

The Mandelbrot Set

Main cardioid:

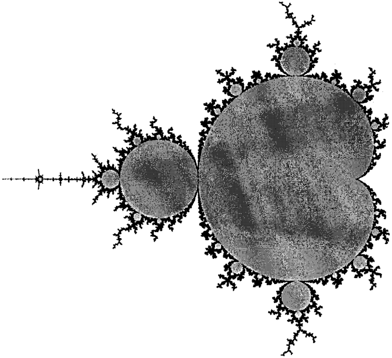
* $f_\lambda(z)$ has an attracting fixed point.

Left 'bulb':

* $f_\lambda(z)$ has a period-2 attracting cycle, etc.

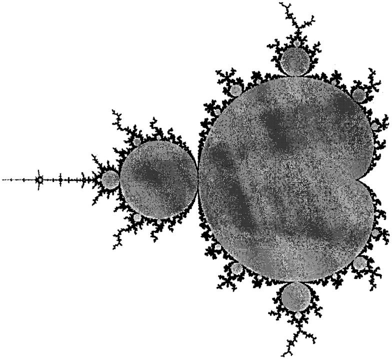
Interpretation (many equivalences):

* Behavior of critical point 0 changes as parameter crosses boundary.
* ∂M is the *bifurcation locus* for (f_λ, c).



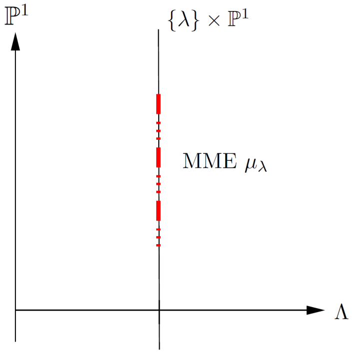
Bifurcation Measure

* Measure in the parameter space.
* Supported on ∂M.
* Introduced by L. DeMarco.
* Closely related to the measures of maximal entropy (MME) μ_λ (in dynamics plane). How?



Fiberwise Green current

For each parameter λ, get a measure μ_λ (MME) in $\mathbb{P}^1$, support on Julia set J_λ.



We can "concatenate" these μ_λ to get a so-called *fiberwise Green current* T.

T is an object in the product space $\Lambda \times \mathbb{P}^1$.

Fiberwise Green current

T can be "sliced" in different ways to get useful dynamical information.

* T sliced at a vertical fiber $\{\lambda\} \times \mathbb{P}^1$ is μ_λ.
* T sliced along the graph $\{(\lambda, c(\lambda)) : \lambda \in \Lambda\}$ is the bifurcation measure supported on ∂M.

Motivated by statistical physics, we will modify this approach.

Set Up

$\mathbb{D}$ unit disk (focus: local property in parameter space).

Let $f_\lambda(z)$ be a holomorphic family of rational maps of degree d, parameterized by $\mathbb{D}$.

A *marked point* is a holomorphic function $a : \Lambda \to \mathbb{P}^1$.

*Marked point is not necessarily critical (motivated by physics).

Slice the fiberwise Green current T along the graph $\{(\lambda, a(\lambda)) : \lambda \in \mathbb{D}\}$,

resulting μ is called the activity/bifurcation measure of the marked point a.

Behavior of $a(\lambda)$ changes as parameter crosses boundary.

Set Up

Definition: Pointwise dimension of μ at $\lambda_0 \in \mathrm{supp}(\mu)$ is defined by

$$d_\mu(\lambda_0) := \lim_{\epsilon \to \infty} \frac{\log \mu(\mathbb{D}_\epsilon(\lambda_0))}{\log \epsilon}.$$

(if limit exists)

$d_\mu(\lambda_0)$ measures locally how "sparse" μ is compared with Lebesgue measure.

$$\epsilon^{d_\mu(\lambda_0)} + \delta < \mu(\mathbb{D}_\epsilon) < \epsilon^{d_\mu(\lambda_0)} - \delta$$

Main Result

Suppose 0 is a parameter s.t. $a(0)$ is a repelling fixed point for $f_0(z)$ with multiplier η_0.

Theorem

Under a mild transversality assumption, the pointwise dimension of μ at 0 is

$$d_\mu(\lambda_0) = \frac{\log d}{\log|\eta_0|}.$$

Assumption:

$$(\eta_0 - 1)^{-1} \frac{\partial}{\partial \lambda} f_0 a(0) + \frac{d}{d\lambda} a(0) \neq 0,$$

Motivation

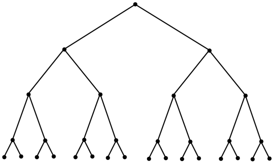
Measures constructed by a family $f_\lambda(z)$ and a marked point a appear in statistical mechanics.

Concerning zeros of partition functions for various models.

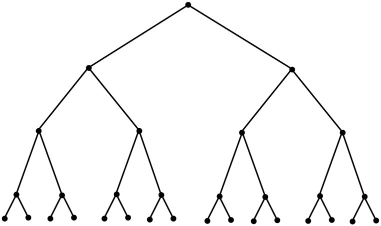
Sequence of recursively defined graphs + Migdal-Kadanoff renormalization

Example 1:
Neighbor exclusion model / Independence Polynomials
(de Boer, Buys, Peters, Guerini, Regts, Rivera–Letelier-Sombra, Sokal, etc)

Cayley trees



Motivation



Family of maps:

$$f_{d,\lambda}(z) = \frac{\lambda}{(1+z)^d}.$$

Marked point:

$$a(\lambda) = \lambda.$$

Motivation

Example 2:

Ising model
(Bleher-Lyubich-Roeder, C-He-Ji-Roeder,
Bencs-Buys-Guerini-Peters, Peters-Regts etc)

On Cayley tree, Family of maps:

$$f_{\lambda,t,d}(z) = \lambda \left(\frac{z+t}{1+tz} \right)^d .$$

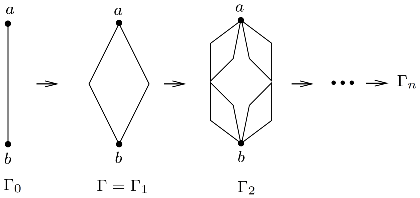
Marked point:

$$a(\lambda) = \lambda.$$

Motivation

Example 3:

(Antiferromagnetic) q-states Potts model
(Chang-Roeder-Shrock, C-Roeder, Royle-Sokal)
on Diamond Hierarchical Lattice



Family of maps:

$$f_q(z) = \left(\frac{z^2 + q - 1}{2z + q - 2}\right)^2.$$

Marked point:

$$a(\lambda) = 0.$$

Motivation

Logarithmic potential of μ:

$$\mathcal{F}(\lambda) := \int_{\mathbb{C}} \log|\lambda - w| d\mu(w),$$

is the **free energy** in statistical mechanics.

Used to characterize **phase transition**.

$\mathcal{F}$ is analytic except possibly at $\lambda \in \operatorname{supp}(\mu)$.

Typically interested in physical region in parameter space, such as $(0, \infty)$ or $\mathbb{R}$.

At such a point λ_0, if there is a real analytic $\mathcal{F}_{\mathrm{reg}}(\lambda)$ such that:

$$\sigma(\lambda_0) := \lim_{\lambda \to \lambda_0} \frac{\log|\mathcal{F}(\lambda) - \mathcal{F}_{\mathrm{reg}}(\lambda)|}{\log|\lambda - \lambda_0|}$$

exists, it is called the **critical exponent**.

Pointwise Dimension vs Critical Exponent

Ising model on Diamond(like) Hierarchical Lattices.

Study MME μ of a single rational map

$$f(t) = \frac{4t^b}{(1+t^b)^2}$$

0 and 1 are superattracting fixed points; a repelling fixed point $t_c \in (0,1)$, called the critical temperature.

Building on the works of Bleher, Derrida, Lyubich, Zalis, etc

Theorem (Ishii (1995))

For ℓ large enough so that $(f'(t_c))^\ell > 2b$,

$$\lim_{t \nearrow t_c} \frac{\log|\mathcal{F}^{(\ell)}(t)|}{-\log|t-t_c|} = \ell - \frac{\log 2b}{\log f'(t_c)}$$

Pointwise Dimension vs Critical Exponent

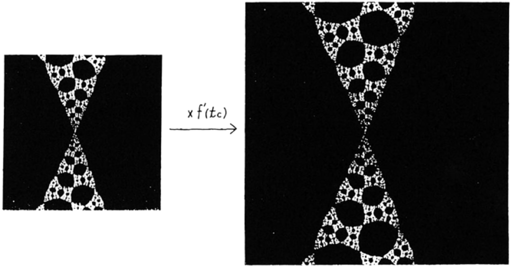


Fig. 4. Enlargement of $J(f)$ near t_c.

Ishii: "When we enlarge the size of a neighborhood V of t_c by $f'(t_c)$ times, the measure $\mu(V)$ becomes about $2b$ times larger. So the critical exponent reflects the local similarity of the maximal entropy measure"

Remark: although similar in spirit, our bifurcation measure is in the parameter space, while Ishii's MME is in the dynamical space.

Pointwise Dimension vs Critical Exponent

Ising model on Cayley trees (with external field).

family of rational map Family of maps:

$$f_{\lambda,t}(z) = \lambda \left(\frac{z+t}{1+tz} \right)^2 .$$

Marked point:

$$a(\lambda) = \lambda.$$

Known: Fix $0 < t < 1$ (ferromagnetic region), $\mathrm{supp}(\mu_t)$ is either the whole circle, or an arc of the circle.

Pointwise Dimension vs Critical Exponent

Theorem (C-He-Ji-Roeder (2018))

Fix $0 < t < 1$, there is a Lebesgue full measure subset of $\mathrm{supp}(\mu_t)$ such that

$$d_{\mu_t}(z) = \frac{\log 2}{\chi_{z,t}},$$

where $\chi_{z,t}$ is the Lyapunov exponent for the a.c. invariant measure for $f_{z,t}$. Moreover, for such z the free energy has radial critical exponent equal to $\frac{\log 2}{\chi_{z,t}}$.

Proof relating pointwise dimension and critical exponent relies on the fact that $\mathrm{supp}(\mu_t)$ is either the whole circle, or an arc of the circle.

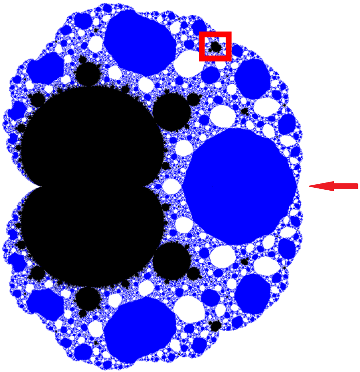
Our Motivation

Montel's Theorem: Let $\lambda_0 \in \mathrm{supp}(\mu)$ (activity/bifurcation measure), then in any neighborhood of λ_0, there is another λ such that $a(\lambda)$ is pre-repelling under f_λ.

So, there are lots of these parameters.

Our Motivation

$\mathrm{supp}(\mu_t)$ for q-states Potts model on Diamond Hierarchical Lattice (parameter space)



Red arrow indicates $\lambda = 3$ (right most point), for which the marked point is mapped to a repelling fixed point after one iterate.

Main Result (repeated slide)

Suppose 0 is a parameter s.t. $a(0)$ is a repelling fixed point for $f_0(z)$ with multiplier η_0.

Theorem

Under a mild transversality assumption, the pointwise dimension of μ at 0 is

$$d_\mu(\lambda_0) = \frac{\log d}{\log|\eta_0|}.$$

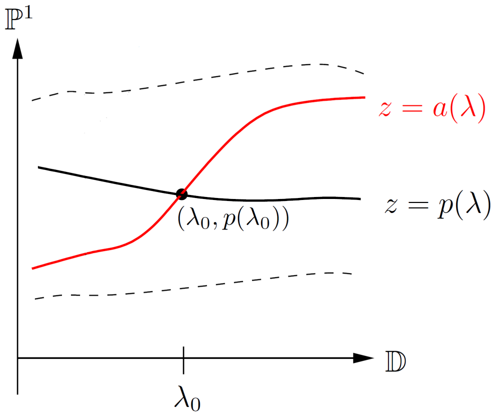
Assumption:

$$(\eta_0 - 1)^{-1} \frac{\partial}{\partial \lambda} f_0 a(0) + \frac{d}{d\lambda} a(0) \neq 0,$$

Sketch Proof

The repelling fixed point moves holomorphically.

Marked point a lies in the linearization domain.



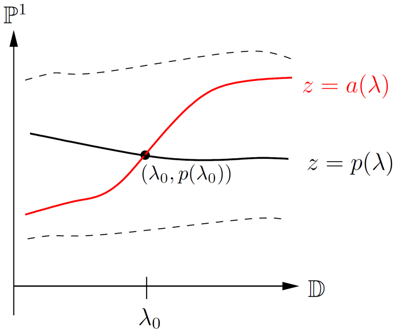
Sketch Proof

Notations:

$F(\lambda, z) = (\lambda, f_\lambda(z))$ - dynamics in product space.

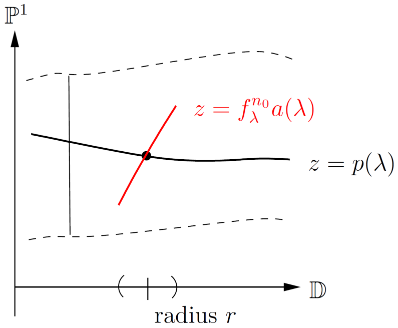
$\pi_2 : \mathbb{D} \times \mathbb{P}^1 \to \mathbb{P}^1$ is the projection.

$\widehat{a} : \mathbb{D} \to \mathbb{D} \times \mathbb{P}^1$ is just $\lambda \mapsto (\lambda, a(\lambda))$.



Sketch Proof

Under repelling, if we iterate the marked point long enough:



After n_0 iterates, the graph is 'vertical' in a restricted domain,

i.e. $(\pi_2 \circ F^{n_0} \circ \widehat{a})$ is univalent in $\mathbb{D}_r(0)$.

Distortion Control

Theorem (Koebe distortion theorem)

If ϕ is univalent in a domain D and $w_0 \in D \subset \mathbb{C}$, then

$$\begin{aligned}\frac{1}{4}|\phi'(w_0)|\,dist(w_0, \partial D)&\\ &\leq dist(\phi(w_0),\partial(\phi(D)))\\ &\qquad \leq 4|\phi'(w_0)|\,dist(w_0, \partial D).\end{aligned}$$

Distortion Control

For each n, let $\epsilon_n = \frac{r}{|\eta_0|^n}$. Consider $\mathbb{D}_{\epsilon_n}$.

The derivative satisfies

$$\frac{d}{d\lambda}\left(\pi_2 \circ F^{n_0+n} \circ \widehat{a}\right)(0) \sim \eta^{n_0+n}$$

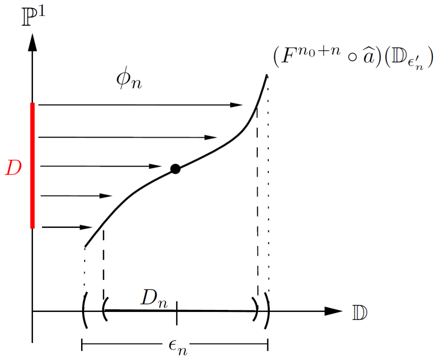
By Koebe distortion:

There are constants $C_1, C_2 > 0$, independent of n,

$$C_1 < \operatorname{dist}\left(a(0), \partial(F_{n_0+n}(\mathbb{D}_{\epsilon_n}))\right) < C_2.$$

Sketch Proof

For all n, the projected disks in $\mathbb{P}^1$ are of comparable sizes, make them the same size by shrinking the domain $\mathbb{D}_{\epsilon_n}$ (check it's not too much).



Let $\phi_n : D \to (F^{n_0+n} \circ \widehat{a})(D_n)$ be the inverse of projection π_2.

Fiberwise Green current revisit

The disks in the product space converge to the vertical disk $\{0\} \times D$,

i.e. ϕ_n converges uniformly to $i_D : D \to \{0\} \times \mathbb{P}^1$, $i_D(z) = (0, z)$.

Green current $\widehat{T}$ sliced at:

$\{0\} \times D$ is just the MME (for f_0): $\mu_0(D)$.

$(F^{n_0+n} \circ \widehat{a})(D_n)$ gives a measure ν_n on D.

Uniform convergence of ϕ_n to i_D implies $\nu_n \to \mu_0|_D$ in weak-$*$ topology,

so there is $C_3 > 0$ independent of n such that

$$\frac{1}{C^3} < d^{n_0+n}\mu(D_n) < C_3.$$

(F-invariance of $\widehat{T}$ gives $\nu_n = d^{n_0+n}\mu$)

Sketch Proof

Since we didn't shrink $\mathbb{D}_{\epsilon_n}$ too much to get D_n, we have

$$\frac{1}{C^3} < d^n \mu(\mathbb{D}_{\epsilon_n}) < C_3.$$

Then,

$$\log(1/C^3) - n \log d < \log \mu(\mathbb{D}_{\epsilon_n}) < \log C_3 - n \log d,$$

Divide by $\log \epsilon_n = \log(r/|\eta_0|^n)$,

$$\frac{\log C_3 - n \log d}{\log r - n \log |\eta_0|} < \frac{\log \mu(\mathbb{D}_{\epsilon_n})}{\log \epsilon_n} < \frac{\log(1/C^3) - n \log d}{\log r - n \log |\eta_0|},$$

Taking the limit as $n \to \infty$, done.

Next Steps:

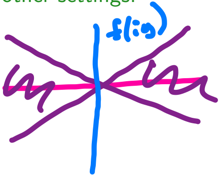
Apply this result to calculate critical exponents of free energy.
(in progress..)

Roeder's Proposed Proposition

Critical exponents related to pointwise dimension of limiting measure 2.

Proposition from previous page is good for Lee-Yang measure since supported on $\mathbb{T}$. Not as good for Fisher measure and other settings.

$$f_\mu(z) := \int_{\mathbb{C}} \log|\zeta - z| d\mu(\zeta),$$



where μ is supported in a cone centered on real axis.

Proposed Proposition: Suppose

$$\kappa \equiv \dim_\mu(0) := \lim_{\delta \to 0} \frac{\log \mu(\mathbb{D}_\delta(0))}{\log 2\delta} > 0.$$

Then, there exists a real-analytic function $f_{\mathrm{reg}}(y)$ such that

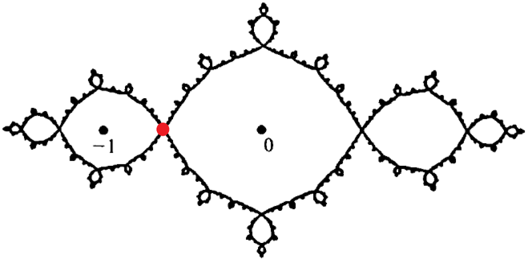
$$\lim_{y \to 0} \frac{\log|f_\mu(iy) - f_{\mathrm{reg}}(y)|}{\log|y|} = \kappa.$$

In other words, f_μ has critical exponent κ when crossing $\mathbb{R}$ vertically.

Not as Straightforward

(Courtesy of Rivera-Letelier)

$f(z) = z^2 - 1$, Julia set is the Basilica.



Red dot is a repelling fixed point, multiplier τ.

Pointwise dimension of MME at red dot should be $\log 2 / \log \tau$.

Green function G (measures escape rate) is identically 0 across an interval centered at red dot.

Thank you