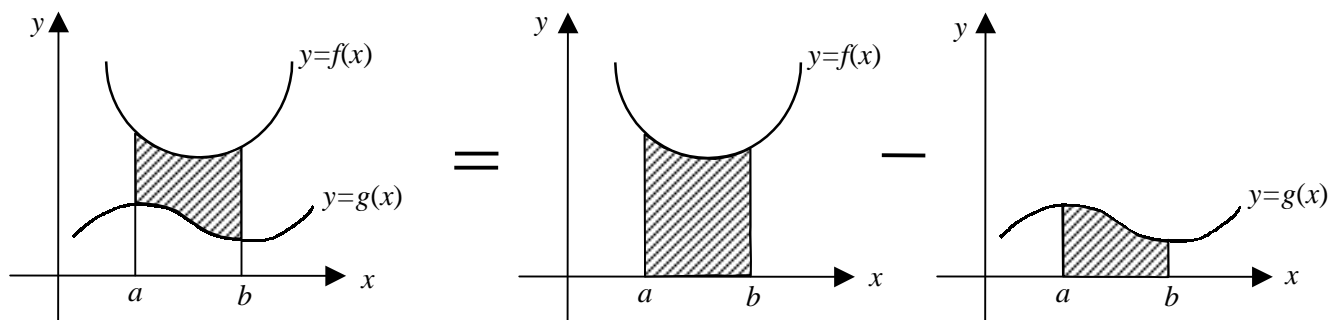


Important: In this section we will learn just how to set up the integrals to find area between curves. The final answer for each example in this handout is given for information only. You will be required to solve some problems with final answers when we cover section 7.3.

A) Area Between Curves in $[a, b]$



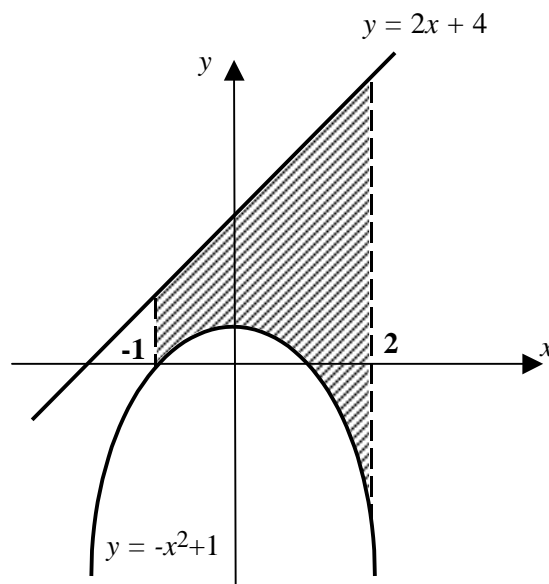
$$\begin{aligned}
 \text{Total Area between } f(x) \text{ and } g(x) &= \int_a^b f(x)dx - \int_a^b g(x)dx \\
 \text{in } [a, b] &= \int_a^b [f(x) - g(x)]dx \\
 \Rightarrow \text{Area} &= \int_a^b [\text{Top Function} - \text{Bottom Function}]dx
 \end{aligned}$$

Example 1: Find the area of the region that is enclosed by $y = -x^2 + 1$, $y = 2x + 4$, $x = -1$ and $x = 2$.

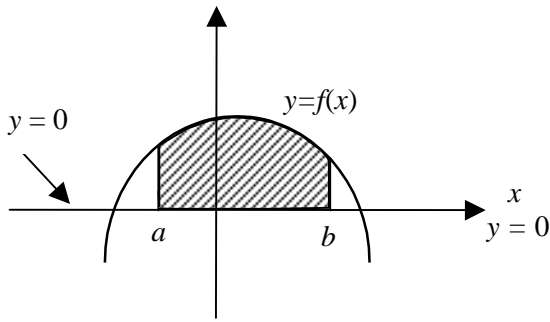
Solution:

- Graph the functions $y = -x^2 + 1$, and $y = 2x + 4$.
Shade in the region in the interval $[-1, 2]$
- The function $y = 2x + 4$ is on the top in the entire interval of $[-1, 2]$

$$\begin{aligned}
 \text{3) Area: } A &= \int_a^b [\text{Top} - \text{Bottom}]dx \\
 &= \int_{-1}^2 [(2x + 4) - (-x^2 + 1)]dx \\
 &= \int_{-1}^2 (x^2 + 2x + 3)dx \\
 &= 15
 \end{aligned}$$

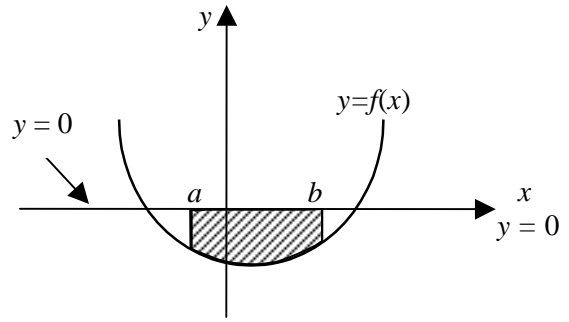


B) Area Between a Curve and the x-Axis in [a, b] (x-axis is same as $y = 0$)



Top function is $y = f(x)$, Bottom is $y = 0$

$$\begin{aligned} \text{Area} &= \int_a^b [\text{Top Function} - \text{Bottom Function}] dx \\ &= \int_a^b [f(x) - 0] dx \\ &= \int_a^b f(x) dx \end{aligned}$$



Top function is $y = 0$, Bottom is $y = f(x)$

$$\begin{aligned} \text{Area} &= \int_a^b [\text{Top Function} - \text{Bottom Function}] dx \\ &= \int_a^b [0 - f(x)] dx \\ &= \int_a^b -f(x) dx \quad \text{or} \quad \text{Area} = -\int_a^b f(x) dx \end{aligned}$$

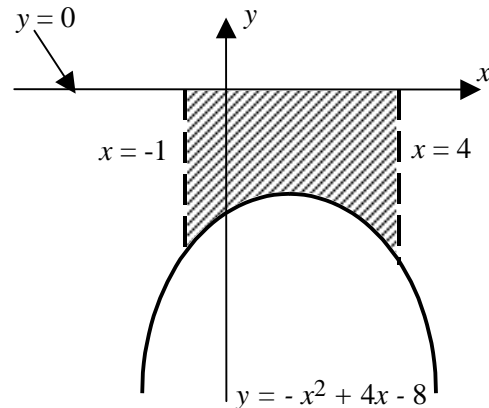
Example 2: Find the area of the region that is enclosed by the x-axis, the graph $y = -x^2 + 4x - 8$ over the interval $[-1, 4]$.

Solution:

- 1) Graph the functions $y = -x^2 + 4x - 8$, and $y = 0$ (the x-axis). Shade in the region in the interval $[-1, 4]$
- 2) The function $y = 0$ is on the top in the entire interval of $[-1, 4]$

- 3) Area: $A = \int_a^b [\text{Top} - \text{Bottom}] dx$

$$\begin{aligned} &= \int_{-1}^4 [(0) - (-x^2 + 4x - 8)] dx \\ &= \int_{-1}^4 (x^2 - 4x + 8) dx = 31.67 \end{aligned}$$



C) Area Between Intersecting Curves

When $[a, b]$ is not given, then we must determine the appropriate values from the graph. The area bounded between two functions $y = f(x)$ and $y = g(x)$ is from the intersection point on the left to the intersection point on the right. To find these points of intersection, we solve the equation: $f(x) = g(x)$ for x .

Example 3: Find the area of the region that is enclosed by the graphs of the functions:

$$f(x) = 2x - 1 \quad \text{and} \quad g(x) = x^2 - 4$$

Solution:

- 1) Graph both functions
- 2) Find the points of intersection of the two curves by equating the two equations :

$$y = 2x - 1 \quad \text{and} \quad y = x^2 - 4:$$

$$x^2 - 4 = 2x - 1$$

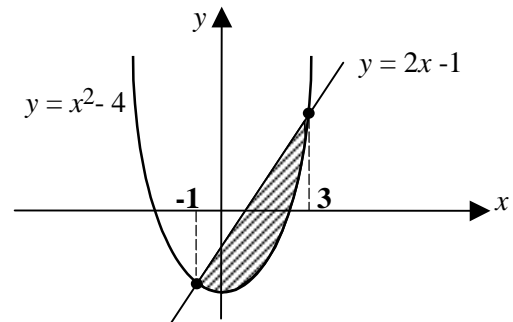
$$x^2 - 2x - 3 = 0$$

$$(x + 1)(x - 3) = 0$$

$$\text{Then } x = -1 \text{ or } x = 3$$

$$3) \text{ Area: } A = \int_a^b [\text{Top} - \text{Bottom}] dx$$

$$= \int_{-1}^3 [(2x - 1) - (x^2 - 4)] dx = \int_{-1}^3 (-x^2 + 2x + 3) dx = \frac{32}{3}$$



Example 4: Find the area of the region that is enclosed by $y = x^2 - 4x + 3$ and the x -axis.

Solution:

- 1) Graph the functions $y = x^2 - 4x + 3$, and $y = 0$ (the x -axis)
- 2) Find the points of intersection by equating the two equations $y = x^2 - 4x + 3$ and $y = 0$:

$$x^2 - 4x + 3 = 0$$

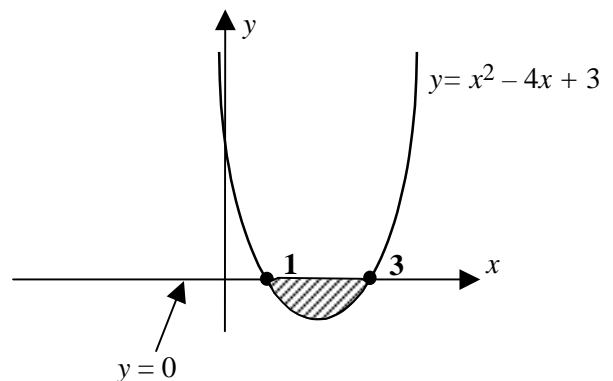
$$(x - 1)(x - 3) = 0$$

$$\text{Then } x = 1 \text{ or } x = 3$$

$$3) \text{ Area: } A = \int_a^b [\text{Top} - \text{Bottom}] dx$$

$$= \int_1^3 [(0) - (x^2 - 4x + 3)] dx$$

$$= \int_1^3 (-x^2 + 4x - 3) dx = 1.33$$



More Examples:

Example 5: Find the area of the region that is enclosed by the graphs of the functions:

$$y = x^2 \quad \text{and} \quad y = \sqrt{x}$$

Solution:

- 1) Graph both functions
- 2) Find the points of intersection of the two curves by equating the two equations :

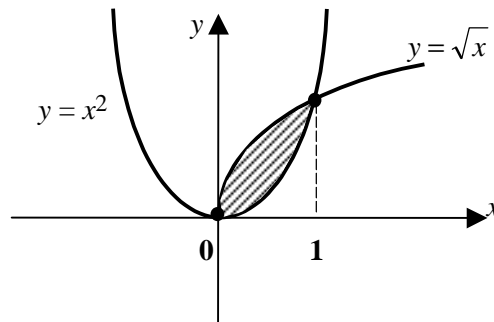
$$y = x^2 \quad \text{and} \quad y = \sqrt{x}$$

$$\text{Square both sides: } x^4 = x$$

$$x^4 - x = 0$$

$$x^3(x-1) = 0$$

$$\text{Then } x = 0 \text{ or } x = 1$$



$$\begin{aligned} \text{3) Area: } A &= \int_a^b [\text{Top} - \text{Bottom}] dx \\ &= \int_0^1 [\sqrt{x} - x^2] dx = 0.33 \end{aligned}$$

Example 6: Find the area of the region that is enclosed by the graphs of the functions:

$$y = -x^2 - 3x + 4 \quad \text{and} \quad y = x^2 - 3x - 4$$

Solution:

- 1) Graph both functions
- 2) Find the points of intersection of the two curves by equating the two equations :

$$x^2 - 3x - 4 = -x^2 - 3x + 4$$

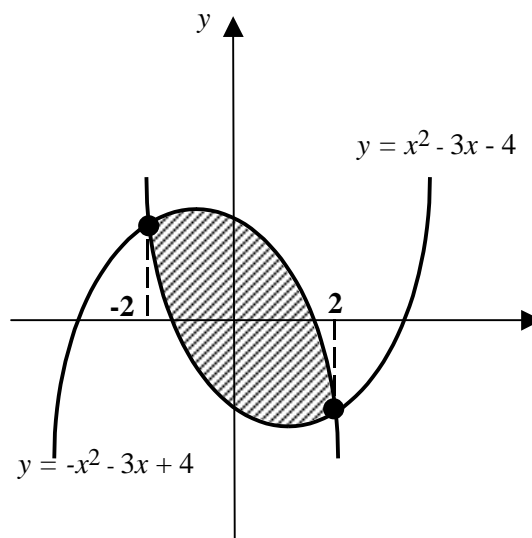
$$2x^2 - 8 = 0$$

$$2(x^2 - 4) = 0$$

$$2(x-2)(x+2) = 0$$

$$\text{Then } x = -2 \text{ or } x = 2$$

$$\begin{aligned} \text{3) Area: } A &= \int_a^b [\text{Top} - \text{Bottom}] dx \\ &= \int_{-2}^2 [(-x^2 - 3x + 4) - (x^2 - 3x - 4)] dx \\ &= \int_{-2}^2 (-2x^2 + 8) dx = 21.33 \end{aligned}$$



D) Area Between Curves With Multiple Points of Intersection (Crossing Curves)

If neither graph $f(x)$ or $g(x)$ lies above the other over the whole interval, then we break the area into two pieces. One on either side of the point at which the graphs cross and then compute each area separately. To do this, we need to know exactly where that crossing point is by solving the equation: $f(x) = g(x)$ for x .

Example 7: Find the area of the region that is enclosed by the x -axis, $y = x^2 - 4$ over the interval $[-1, 3]$.

Solution:

- 1) Graph the functions $y = x^2 - 4$, and $y = 0$ (the x -axis). Shade in the region in the interval $[-1, 3]$
- 2) Find the points of intersection by equating the two equations $y = x^2 - 4$ and $y = 0$:

$$\begin{aligned} x^2 - 4 &= 0 \\ (x - 2)(x + 2) &= 0 \end{aligned}$$

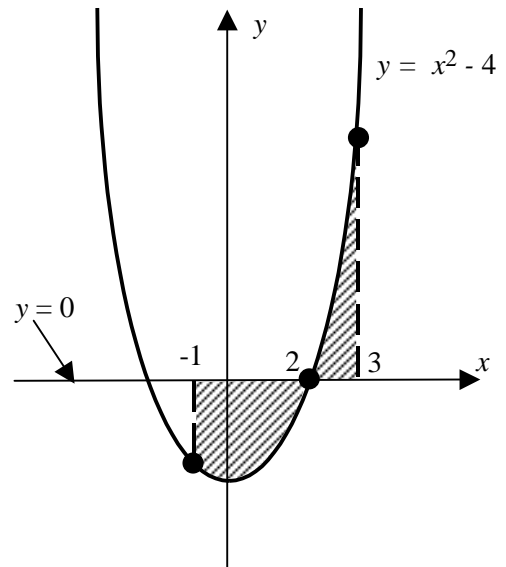
The graph crosses the x -axis at: $x = -2, x = 2$

- 3) The total area is the sum of the two areas: $[-1, 2]$ and $[2, 3]$

$$\begin{aligned} A &= \int_{-1}^2 [\text{Top} - \text{Bottom}] dx + \int_2^3 [\text{Top} - \text{Bottom}] dx \\ &= \int_{-1}^2 [(0) - (x^2 - 4)] dx + \int_2^3 [(x^2 - 4) - (0)] dx \\ &= \int_{-1}^2 (-x^2 + 4) dx + \int_2^3 (x^2 - 4) dx = 9 + 2.33 = 12.33 \end{aligned}$$

Note 1: If you write the area as $A = \int_{-1}^3 (x^2 - 4) dx$, then the answer will be $-9 + 2.33 = -6.67$.

Note 2: You should get the correct answer for the **Total Area** as long as each integral produces a **positive** area. This is one way to check your work.



Example 8: Find the area of the region that is enclosed by the x -axis, $y = 1 - x^2$, $x = -1$ and $x = 2$.

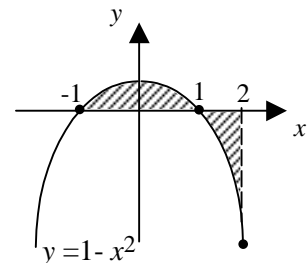
Solution:

- 1) Graph $y = 1 - x^2$ and shade the region between $x = -1$ and $x = 2$
- 2) Find the points of intersection:

$$1 - x^2 = 0 \text{ or } (x - 1)(x + 1) = 0$$

The graph crosses the x -axis at $x = -1$ and $x = 1$

$$3) \quad A = \int_{-1}^1 (1 - x^2) dx + \int_1^2 (x^2 - 1) dx = 1.33 + 1.33 = 2.66$$



Example 9: Find the area of the region that is enclosed by the graphs of the functions:

$$y = -x, \quad y = 2 - x^2, \text{ over the interval } [-1, 3].$$

Solution:

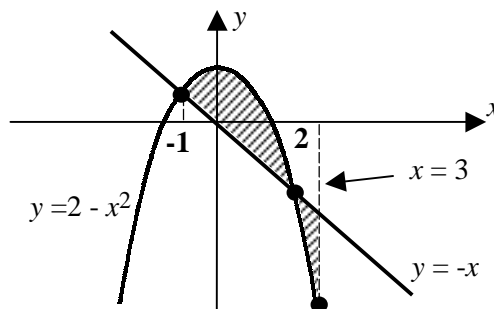
- 1) Graph the functions $y = -x$, and $y = 2 - x^2$.
- 2) Find the points of intersection of the two curves by equating the two equations :

$$\begin{aligned} -x &= 2 - x^2 \\ x^2 - x - 2 &= 0 \\ (x - 2)(x + 1) &= 0 \end{aligned}$$

The two graphs cross when $x = -1$ and $x = 2$.

- 3) The total area is the sum of the two areas: $[-1, 2]$ and $[2, 3]$

$$\begin{aligned} A &= \int_{-1}^2 [(2 - x^2) - (-x)] dx + \int_2^3 [(-x) - (2 - x^2)] dx \\ &= \int_{-1}^2 (2 - x^2 + x) dx + \int_2^3 (x^2 - x - 2) dx \\ &= 4.5 + 1.83 = 6.33 \end{aligned}$$



Example 10: Find the area of the region that is enclosed by the graphs of the functions:

$$y = x, \quad y = x^3$$

Solution:

- 1) Graph the functions $y = x^3$, and $y = x$.
- 2) Find the points of intersection of the two curves by equating the two equations :

$$\begin{aligned} x^3 &= x \\ x^3 - x &= 0 \\ x(x^2 - 1) &= 0 \\ x(x - 1)(x + 1) &= 0 \end{aligned}$$

The two cross at: $x = 0, x = -1$ or $x = 1$

- 3) The total area is the sum of the two areas: $[-1, 0]$ and $[0, 1]$

$$\begin{aligned} A &= \int_{-1}^0 [Top - Bottom] dx + \int_0^1 [Top - Bottom] dx \\ &= \int_{-1}^0 [(x^3) - (x)] dx + \int_0^1 [(x) - (x^3)] dx \\ &= 0.25 + 0.25 = 0.5 \end{aligned}$$

(notice that the answer will be zero if one area is negative)

