Section 6.3: **Matrix Notation**

Matrices

- A *matrix* is a rectangular or square array of values arranged in rows and columns.
- An $m \times n$ matrix $\mathbf{A}$, has m rows and n columns, and has a general form of

$$\mathbf{A}_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

Examples of Matrices

$$\mathbf{A} = \begin{bmatrix} 3 & 5 & 8 \\ 1 & 2 & 2 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 3 & 6 \\ 7 & 2 \\ 2 & 9 \end{bmatrix}$$

$$C = \begin{vmatrix} -1 \\ 2 \\ 1 \\ 3 \end{vmatrix}$$

$$D = \begin{vmatrix} -1 & 2 & 1 & 4 \end{vmatrix}$$

Two matrices are equal if they have same <u>dimension</u>, same <u>elements</u>:

$$A = \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} \quad ; \quad B = \begin{vmatrix} 1 & x \\ -1 & y \end{vmatrix}$$

If $A = B$, then:

Addition of Matrices A and B:

A and B must have same dimensions:

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 8 \\ 9 & -1 & 5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 3 & -7 \end{bmatrix}$$

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} 2+1 & 4+5 & 8+3 \\ 9+2 & (-1)+3 & 5+(-7) \end{bmatrix} = \begin{bmatrix} 3 & 9 & 11 \\ 11 & 2 & -2 \end{bmatrix}$$

$$\mathbf{A}-\mathbf{B} = \begin{bmatrix} 2-1 & 4-5 & 8-3 \\ 9-2 & -1-3 & 5-(-7) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 5 \\ 7 & -4 & 12 \end{bmatrix}$$

If k is a real number, then the scalar product $k.A$ is abstained by multiplying each element of A by k

If k = -2 and $A = \begin{vmatrix} -1 & -4 & 5 \\ 1 & 3 & -3 \\ 2 & 4 & -2 \end{vmatrix}$

Then $k.A = -2.A =$

Example: for the following matrices, find $2A-3B$

$$A=\begin{vmatrix} -1 & -4 & 5 \\ 1 & 3 & -3 \\ 2 & 4 & -2 \end{vmatrix} \qquad B=\begin{vmatrix} 4 & -1 & 2 \\ -1 & -2 & 3 \\ -3 & -4 & 1 \end{vmatrix}$$

$2A-3B=$

The Transpose of a Matrix:

Each row becomes column, and each column becomes row.

$$A = \begin{vmatrix} -1 & 4 \\ 1 & 3 \\ 2 & 4 \end{vmatrix} \qquad A^t = \begin{vmatrix} -1 & 1 & 2 \\ -4 & 3 & 4 \end{vmatrix}$$

Matrix Multiplication

To Multiply matrix A by matrix B:

- Multiply each Row in matrix A by each Column in matrix B
- Multiply corresponding entries and then add the resulting products

$$A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \qquad B = \begin{vmatrix} -1 & 1 & 2 \\ 3 & -2 & 3 \end{vmatrix}$$

2 x 2   2 x 3

$$A.B = \begin{vmatrix} (1)(-1) + (2)(3) & (1)(1) + (2)(-2) & (1)(2) + (2)(3) \\ (3)(-1) + (4)(3) & (3)(1) + (4)(-2) & (3)(2) + (4)(3) \end{vmatrix} = \begin{vmatrix} 5 & -3 & 8 \\ 9 & -5 & 18 \end{vmatrix}$$

We had:

3 columns

Result: 2 rows by 3 columns

2 rows $A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$, $B = \begin{vmatrix} -1 & 1 & 2 \\ 3 & -2 & 3 \end{vmatrix}$ and $A.B = \begin{vmatrix} 5 & -3 & 8 \\ 9 & -5 & 18 \end{vmatrix}$

2 x 2 **2 x 3** **2 x 3**

How about $B.A$:

$B = \begin{vmatrix} -1 & 1 & 2 \\ 3 & -2 & 3 \end{vmatrix}$, $A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

2 x 3 **2 x 2**

For the following matrices, using the multiplication of Row by Column :

$$A = \begin{vmatrix} 1 & 1 & 5 \\ 2 & 1 & -1 \end{vmatrix} \quad , \quad B = \begin{vmatrix} 2 \\ 1 \\ 0 \end{vmatrix} \quad , \quad C = \begin{vmatrix} 2 & -1 \\ 1 & 2 \\ 0 & 4 \end{vmatrix}$$

a) Which of the following multiplication is possible ?
b) If it is possible, find the dimension of the resulting matrix

A.B ***A.C*** ***B.C*** ***C.A***

For the following matrices, using the multiplication of Row by Column :

$$A = \begin{vmatrix} 1 & 1 & 5 \\ 2 & 1 & -1 \end{vmatrix} \quad , \quad B = \begin{vmatrix} 2 \\ 1 \\ 0 \end{vmatrix} \quad , \quad C = \begin{vmatrix} 2 & -1 \\ 1 & 2 \\ 0 & 4 \end{vmatrix}$$

a) Which of the following multiplication is possible ?
b) If it is possible, find the dimension of the resulting matrix

$$\mathbf{A.B} = \begin{vmatrix} 1 & 1 & 5 \\ 2 & 1 & -1 \end{vmatrix} \cdot \begin{vmatrix} 2 \\ 1 \\ 0 \end{vmatrix}$$

For the following matrices, using the multiplication of Row by Column :

$$A = \begin{vmatrix} 1 & 1 & 5 \\ 2 & 1 & -1 \end{vmatrix} \quad , \quad B = \begin{vmatrix} 2 \\ 1 \\ 0 \end{vmatrix} \quad , \quad C = \begin{vmatrix} 2 & -1 \\ 1 & 2 \\ 0 & 4 \end{vmatrix}$$

a) Which of the following multiplication is possible ?
b) If it is possible, find the dimension of the resulting matrix

$$A.C = \begin{vmatrix} 1 & 1 & 5 \\ 2 & 1 & -1 \end{vmatrix} \cdot \begin{vmatrix} 2 & -1 \\ 1 & 2 \\ 0 & 4 \end{vmatrix}$$

For the following matrices, using the multiplication of Row by Column :

$$A = \begin{vmatrix} 1 & 1 & 5 \\ 2 & 1 & -1 \end{vmatrix} \quad , \quad B = \begin{vmatrix} 2 \\ 1 \\ 0 \end{vmatrix} \quad , \quad C = \begin{vmatrix} 2 & -1 \\ 1 & 2 \\ 0 & 4 \end{vmatrix}$$

a) Which of the following multiplication is possible ?
b) If it is possible, find the dimension of the resulting matrix

$$\mathbf{B.C} = \begin{vmatrix} 2 \\ 1 \\ 0 \end{vmatrix} . \begin{vmatrix} 2 & -1 \\ 1 & 2 \\ 0 & 4 \end{vmatrix}$$

For the following matrices, using the multiplication of Row by Column :

$$A = \begin{vmatrix} 1 & 1 & 5 \\ 2 & 1 & -1 \end{vmatrix} \quad , \quad B = \begin{vmatrix} 2 \\ 1 \\ 0 \end{vmatrix} \quad , \quad C = \begin{vmatrix} 2 & -1 \\ 1 & 2 \\ 0 & 4 \end{vmatrix}$$

a) Which of the following multiplication is possible ?
b) If it is possible, find the dimension of the resulting matrix

$$C.A = \begin{vmatrix} 2 & -1 \\ 1 & 2 \\ 0 & 4 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 5 \\ 2 & 1 & -1 \end{vmatrix}$$

For the following matrices, Find $A.B$ and $B.A$ if possible:

$$A = \begin{vmatrix} 1 & 1 & 5 \end{vmatrix} \quad , \qquad B = \begin{vmatrix} 2 \\ 1 \\ 0 \end{vmatrix}$$

$$A.B = \begin{vmatrix} 1 & 1 & 5 \end{vmatrix} \cdot \begin{vmatrix} 2 \\ 1 \\ 0 \end{vmatrix}$$

$$B.A = \begin{vmatrix} 2 \\ 1 \\ 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 5 \end{vmatrix}$$

Section 6.3 Cont.: Inverse Matrix

To find the inverse matrix of $A = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$ using the All Integer Method:

- Step 1: Re-write it with the Identity Matrix I next to it on the right side:
  (The Identity Matrix I: the square matrix where all Diagonal elements = 1, the rest are zeros)

$$\left|\begin{array}{cc:cc} 2^* & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right|$$

- Step 2: Do the pivot steps (2 pivots for two rows), and the last step should be:

$$\left|\begin{array}{cc:cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array}\right|$$

- Step 3: The Identity Matrix I is now on the left side, and the Inverse Matrix A^{-1} is on the right side:

$$A^{-1} = \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}$$

- You can check your answer by multiplying the original matrix A and the inverse A^{-1}. The answer must be an Identity Matrix I.

Note: Not every matrix has an inverse, for example: $A = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix}$ does not have an inverse (the second pivot is zero).

Solve the same example and show steps

$$A = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

Examples: Find the inverse matrix for each of the following and check your answer by multiplying the original matrix by its inverse, the resulting matrix must be an Identity Matrix:

1) $\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix}$

2) $\begin{vmatrix} 1 & 0 \\ 3 & -2 \end{vmatrix}$

1)

1	1	1	0
2	4	0	1

2)

1	0	1	0
3	-2	0	1

3) $\begin{vmatrix} 4 & -1 \\ 3 & -1 \end{vmatrix}$

4) $\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}$

Note: Not every matrix has an inverse, for example: $A = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix}$ does not have an inverse (the second pivot is zero).

Find the inverse of the following 3x3 matrix:

$$A = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 1 & 1 \end{vmatrix}$$

Original Matrix A			*Identity matrix I*		
2	1	1	1	0	0
1	2	-1	0	1	0
1	1	1	0	0	1
2	1	1	1	0	0
0	3	-3	-1	2	0
0	1	1	-1	0	2
3	0	3	2	-1	0
0	3	-3	-1	2	0
0	0	3	-1	-1	3
3	0	0	3	0	-3
0	3	0	-2	1	3
0	0	3	-1	-1	3
1	0	0	1	0	-1
0	1	0	- 2/3	1/3	1
0	0	1	- 1/3	- 1/3	1
	I			A^{-1}	