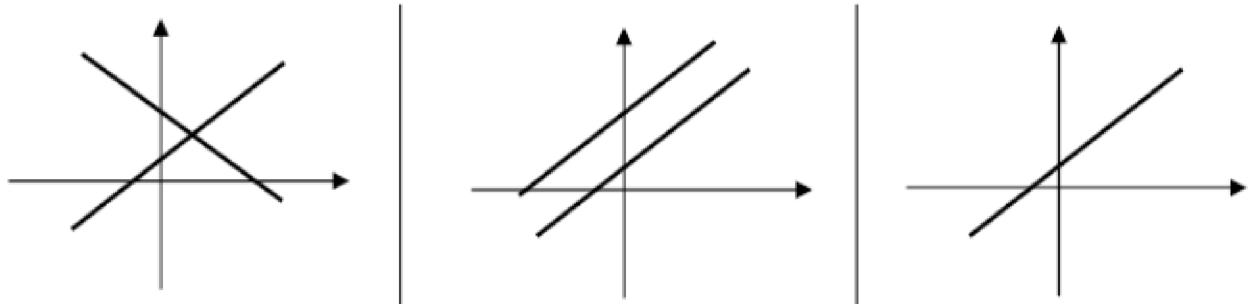
Chapter 6: Linear Equations and Matrix Algebra

<u>Section 6.1 and 6.2:</u> The All Integer Method

- **<u>Method 1:</u>** Solving Graphically



Consistent, Independent	Inconsistent, Independent	Consistent, Dependent
One solution	No solution	(y depends on x)
		Infinite number of solutions

- **<u>Method 2:</u>** Solving by **Substitution**
- **<u>Method 3:</u>** Solving by **Elimination**
- **<u>Method 4:</u>** Solving by **The All Integer Echelon Method**

The All Integer Method First and Last Steps:

Example:

Solve for x, y and z:

$$2x + y + 3z = 5$$
$$2y - z = 5$$
$$2x + y + 2z = 6$$

*The First step is to create the Setup (initial) table:

x	y	z	
2	1	3	5
0	2	-1	5
2	1	2	6

*The Last step will be:

x	y	z	
1	0	0	3
0	1	0	2
0	0	1	-1

*The Answer : $x = 3$, $y = 2$ and $z = -1$

The All Integer Method Steps:

Solve for x, y and z using the All-Integers Method:

$$2x + y + 3z = 5$$
$$2y - z = 5$$
$$2x + y + 2z = 6$$

x	y	z	
2*	1	3	5
0	2	-1	5
2	1	2	6

x	y	z	
2*	1	3	5
0			
0			

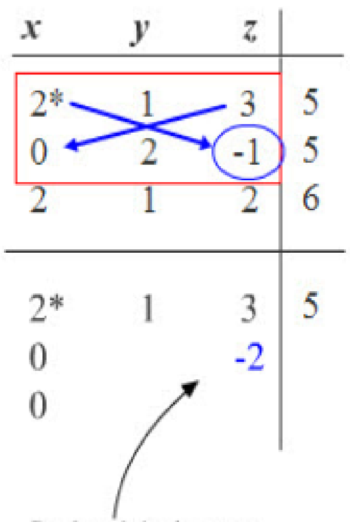
1) Setup the initial table and select the first Pivot Element (* *the first element in the table*).

2) Copy the pivot row and make all other elements in the pivot column = 0.

3) Replace the other elements using the "criss-cross" multiplication method.

"Criss-Cross" Operation Step by Step Using First Pivot

Current Pivot (*First Pivot*) = 2 ; **Previous** Pivot (*there is none, so assume it* = 1)



Replace it in the same location as its original.

Lets replace any element such as -1:

Create a rectangle where the pivot element and the element to be replaced are on facing corners.

Multiply the pivot element by the element to be replaced (2*)(-1)

Subtract the product of the two elements on the opposite diagonal. - (3)(0)

Divide the result by the previous pivot element

$$\frac{(2^*)(-1) - (3)(0)}{\textit{previous pivot}} = \frac{-2}{1} = -2$$

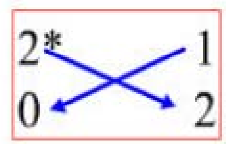
Note: The result must be an **Integer** (no decimal, no fraction) until the very last step (as we will see later).

First Tableau, First Pivot

Current Pivot (*First Pivot*) = 2

Previous Pivot (*there is none, so assume it* = 1)

x	*y*	*z*	
2*	1	3	5
0	2	-1	5
2	1	2	6
2*	1	3	5
0	4	-2	
0			

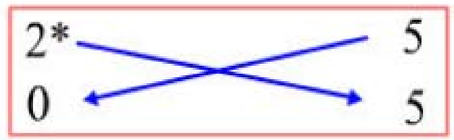


$$\frac{(2^*)(2) - (1)(0)}{\textit{previous pivot}} = \frac{4}{1} = 4$$

First Tableau, First Pivot

Current Pivot (*First Pivot*) = 2
Previous Pivot (*there is none, so assume it* = 1)

x	*y*	*z*	
2*	1	3	5
0	2	-1	5
2	1	2	6



2*	1	3	5
0	4	-2	10
0			

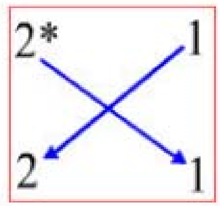
$$\frac{(2^*)(5) - (5)(0)}{\textit{previous pivot}} = \frac{10}{1} = 10$$

First Tableau, First Pivot

Current Pivot (*First Pivot*) = 2

Previous Pivot (*there is none, so assume it* = 1)

x	y	z	
2*	1	3	5
0	2	-1	5
2	1	2	6
2*	1	3	5
0	4	-2	10
0	0		

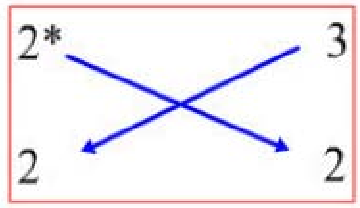


$$\frac{(2^*)(1) - (1)(2)}{\textit{previous pivot}} = \frac{0}{1} = 0$$

First Tableau, First Pivot

Current Pivot (*First Pivot*) = 2
Previous Pivot (*there is none, so assume it* = 1)

x	y	z	
2*	1	3	5
0	2	-1	5
2	1	2	6



2*	1	3	5
0	4	-2	10
0	0	-2	

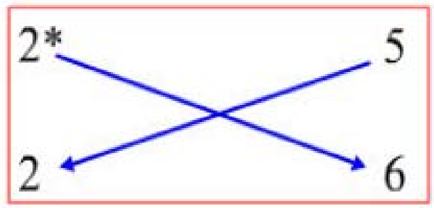
$$\frac{(2^*)(2) - (3)(2)}{\textit{previous pivot}} = \frac{-2}{1} = -2$$

First Tableau, First Pivot

Current Pivot (*First Pivot*) = 2

Previous Pivot (*there is none, so assume it* = 1)

x	*y*	*z*	
2*	1	3	5
0	2	-1	5
2	1	2	6



2*	1	3	5
0	4	-2	10
0	0	-2	2

$$\frac{(2^*)(6) - (5)(2)}{\textit{previous pivot}} = \frac{2}{1} = 2$$

First Tableau, First Pivot

Current Pivot (*First Pivot*) = 2
Previous Pivot (*there is none, so assume it* = 1)

x	y	z	
2*	1	3	5
0	2	-1	5
2	1	2	6

2*	1	3	5
0	4	-2	10
0	0	-2	2

First Tableau Finished

Second Tableau, Second Pivot

Select a new pivot element which is located diagonaly in the next row.

x	y	z	
2	1	3	5
0	4*	-2	10
0	0	-2	2

4	0		
0	4*	-2	10
0	0		

New (Second) Pivot = 4
Previous Pivot = 2

Repeat the steps used in the previous table:

Copy the pivot row and make all other elements in the pivot column = 0.

Note: For columns already pivoted, the old pivot will change to the new and current pivot and the 0's will stay.

Replace the other elements using the "criss-cross" multiplication method.

Second Tableau, Second Pivot

New (Second) Pivot = 4
Previous Pivot = 2

x	y	z	
2	1	3	5
0	4*	-2	10
0	0	-2	2

4	0	7	5
0	4*	-2	10
0	0	-4	4

Second Tableau Finished

Third Tableau, Third Pivot

Select a new pivot element which is located diagonaly in the next row.

x	y	z	
4	0	7	5
0	4	-2	10
0	0	-4*	4
-4	0	0	
0	-4	0	
0	0	-4*	4

New (Third) Pivot = -4
Previous Pivot = 4

Repeat the steps used in the previous table:

Copy the pivot row and make all other elements in the pivot column = 0.

Note: For columns already pivoted, the old pivot will change to the new and current pivot and the 0's will stay.

Replace the other elements using the "criss-cross" multiplication method.

Third Tableau, Third Pivot

New (Third) Pivot = -4
Previous Pivot = 4

x	y	z	
4	0	7	5
0	4	-2	10
0	0	-4*	4

-4	0	0	-12
0	-4	0	-8
0	0	-4*	4

Third Tableau Finished

Last Tableau

Select a new pivot element which is located diagonaly in the next row.

x	y	z	
-4	0	0	-12
0	-4	0	-8
0	0	-4	4
1	0	0	3
0	1	0	2
0	0	1	-1

New Pivot = No more rows
No more pivot
Previous or last Pivot = -4

Divide all elements by the last pivot wich is = -4.

This is the last step and the ***only step*** *where you can get fractions or decimals as answers.*

$x = 3$

$y = 2$

$z = -1$

Summary of All Tableaus and Pivots

x	y	z		
2	1	3	5	
0	2	-1	5	Setup
2	1	2	6	
2*	1	3	5	
0	4	-2	10	First
0	0	-2	2	
4	0	7	5	
0	4*	-2	10	Second
0	0	-4	4	
-4	0	0	-12	
0	-4	0	-8	Third
0	0	-4*	4	

Last Tableau, Result:

x	y	z	
1	0	0	3
0	1	0	2
0	0	2	-1

$x = 3$

$y = 2$

$z = -1$

The All Integer Method
Example 2

Solve for x, y and z using the all integers method:

$$3x + 2z = 9$$
$$x - y - 3z = -3$$
$$-x + 2y + 4z = 5$$

x	y	z	
3	0	2	9
1	-1	-3	-3
-1	2	4	5

Setup Table

Solve for x, y and z using the all integers method:

$$3x + 2z = 9$$
$$x - y - 3z = -3$$
$$-x + 2y + 4z = 5$$

x	y	z		
3	0	2	9	Setup Table
1	-1	-3	-3	
-1	2	4	5	
3*	0	2	9	First Tableau
0	-3	-11	-18	First Pivot = 3
0	6	14	24	
-3	0	-2	-9	Second Tableau
0	-3*	-11	-18	Second Pivot = -3
0	0	8	12	
8	0	0	16	Third Tableau
0	8	0	4	Third Pivot = 8
0	0	8*	12	

x	y	z	
1	0	0	2
0	1	0	1/2
0	0	1	3/2

$x = 2$

$y = 1/2$

$z = 3/2$

The All Integer Method

Example 3

Solve for x and y using the all integers method:

$$2x - y = 4$$
$$x + y = 5$$

Solve for x and y using the all integers method:

$$2x - y = 4$$
$$x + y = 5$$

x	y		
2	-1	4	Setup Table
1	1	5	
2*	-1	4	First Tableau
0	3	6	First Pivot = 2
3	0	9	Second Tableau
0	3*	6	Second Pivot = 3
1	0	3	
0	1	2	

$x = 3$

$y = 2$

The All Integer Method , Special Cases:

Case 1, No Solution

Solve for x, y and z using the all integers method:

$$\begin{aligned} 2x - 6y + 4z &= 1 \\ 4x - 10y + 10z &= 3 \\ x - 2y + 3z &= 2 \end{aligned}$$

Solve for x, y and z using the all integers method:

$$2x - 6y + 4z = 1$$
$$4x - 10y + 10z = 3$$
$$x - 2y + 3z = 2$$

x	y	z		
2	-6	4	1	Setup Table
4	-10	10	3	
1	-2	3	2	
2*	-6	4	1	First Tableau First Pivot = 2
0	4	4	2	
0	2	2	3	
2	-6	4	1	Second Tableau Second Pivot = 4
0	4*	4	2	
0	2	2	3	
4	0	20	8	
0	4	4	2	
0	0	0	4	

No Solution.
Any time we have a row with all 0's to the left, and a nonzero to the right, the system is inconsistent or no solution. $0 = 4$ is not possible.

But, it is ok to have a zero on the right and nonzeros on the left (when variables are = 0). For example:

0	3	0	0

or: $3y = 0$

The All Integer Method , Special Cases:
Case 2, Infinite Number of Solutions

Solve for x, y and z using the all integers method:

$$\begin{aligned} 2x - 3y - 9z &= -5 \\ x + 3z &= 2 \\ -3x + y - 4z &= -3 \end{aligned}$$

Solve for x, y and z using the all integers method:

$$2x - 3y - 9z = -5$$
$$x + 3z = 2$$
$$-3x + y - 4z = -3$$

x	y	z		
2	-3	-9	-5	Setup Table
1	0	3	2	
-3	1	-4	-3	
2*	-3	-9	-5	First Tableau, First Pivot = 2
0	3	15	9	
0	-7	-35	-21	
3	0	9	6	Second Tableau, Second Pivot = 3
0	3*	15	9	Eliminate the line of all 0's. Move to the next pivot if possible.
0	0	0	0	
1	0	3	2	No more pivotes, then divide all elements by last pivot.
0	1	5	3	Translate the solution. It is **Linearly Dependent**

$x + 3z = 2$
$y + 5z = 3$

or

$x = 2 - 3z$
$y = 3 - 5z$
$z =$ any number

Section 6.3: **Matrix Notation**

Matrices

- A *matrix* is a rectangular or square array of values arranged in rows and columns.
- An $m \times n$ matrix $\mathbf{A}$, has m rows and n columns, and has a general form of

$$\mathbf{A}_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

Examples of Matrices

$$\mathbf{A} = \begin{bmatrix} 3 & 5 & 8 \\ 1 & 2 & 2 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 3 & 6 \\ 7 & 2 \\ 2 & 9 \end{bmatrix}$$

$$C = \begin{vmatrix} -1 \\ 2 \\ 1 \\ 3 \end{vmatrix}$$

$$D = \begin{vmatrix} -1 & 2 & 1 & 4 \end{vmatrix}$$

Two matrices are equal if they have same dimension, same elements:

$$A = \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} \quad ; \quad B = \begin{vmatrix} 1 & x \\ -1 & y \end{vmatrix}$$

If $A = B$, then:

Addition of Matrices A and B:

A and B must have same <u>dimensions</u>:

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 8 \\ 9 & -1 & 5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 3 & -7 \end{bmatrix}$$

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} 2+1 & 4+5 & 8+3 \\ 9+2 & (-1)+3 & 5+(-7) \end{bmatrix} = \begin{bmatrix} 3 & 9 & 11 \\ 11 & 2 & -2 \end{bmatrix}$$

$$\mathbf{A}-\mathbf{B} = \begin{bmatrix} 2-1 & 4-5 & 8-3 \\ 9-2 & -1-3 & 5-(-7) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 5 \\ 7 & -4 & 12 \end{bmatrix}$$

If k is a real number, then the scalar product $k.A$ is abstained by multiplying each element of A by k

If k = -2 and $A = \begin{vmatrix} -1 & -4 & 5 \\ 1 & 3 & -3 \\ 2 & 4 & -2 \end{vmatrix}$

Then $k.A$ = -2.A =

Example: for the following matrices, find $2A - 3B$

$$A = \begin{vmatrix} -1 & -4 & 5 \\ 1 & 3 & -3 \\ 2 & 4 & -2 \end{vmatrix} \qquad B = \begin{vmatrix} 4 & -1 & 2 \\ -1 & -2 & 3 \\ -3 & -4 & 1 \end{vmatrix}$$

$2A - 3B =$

The Transpose of a Matrix:

Each row becomes column, and each column becomes row.

$$A = \begin{vmatrix} -1 & 4 \\ 1 & 3 \\ 2 & 4 \end{vmatrix} \qquad A^t = \begin{vmatrix} -1 & 1 & 2 \\ -4 & 3 & 4 \end{vmatrix}$$

Matrix Multiplication

To Multiply matrix A by matrix B:

- Multiply each Row in matrix A by each Column in matrix B
- Multiply corresponding entries and then add the resulting products

$$A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \qquad B = \begin{vmatrix} -1 & 1 & 2 \\ 3 & -2 & 3 \end{vmatrix}$$

2 x 2 2 x 3

$$A.B = \begin{vmatrix} (1)(-1) + (2)(3) & (1)(1) + (2)(-2) & (1)(2) + (2)(3) \\ (3)(-1) + (4)(3) & (3)(1) + (4)(-2) & (3)(2) + (4)(3) \end{vmatrix} = \begin{vmatrix} 5 & -3 & 8 \\ 9 & -5 & 18 \end{vmatrix}$$

We had:

3 columns

Result:
2 rows by 3 columns

2 rows $A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$, $B = \begin{vmatrix} -1 & 1 & 2 \\ 3 & -2 & 3 \end{vmatrix}$ and $A.B = \begin{vmatrix} 5 & -3 & 8 \\ 9 & -5 & 18 \end{vmatrix}$

2 x 2 **2 x 3** **2 x 3**

How about $B.A$:

$B = \begin{vmatrix} -1 & 1 & 2 \\ 3 & -2 & 3 \end{vmatrix}$, $A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

2 x 3 **2 x 2**

For the following matrices, using the multiplication of Row by Column :

$$A = \begin{vmatrix} 1 & 1 & 5 \\ 2 & 1 & -1 \end{vmatrix} \quad , \quad B = \begin{vmatrix} 2 \\ 1 \\ 0 \end{vmatrix} \quad , \quad C = \begin{vmatrix} 2 & -1 \\ 1 & 2 \\ 0 & 4 \end{vmatrix}$$

a) Which of the following multiplication is possible ?
b) If it is possible, find the dimension of the resulting matrix

A.B ***A.C*** ***B.C*** ***C.A***

For the following matrices, using the multiplication of Row by Column :

$$A = \begin{vmatrix} 1 & 1 & 5 \\ 2 & 1 & -1 \end{vmatrix} \quad , \quad B = \begin{vmatrix} 2 \\ 1 \\ 0 \end{vmatrix} \quad , \quad C = \begin{vmatrix} 2 & -1 \\ 1 & 2 \\ 0 & 4 \end{vmatrix}$$

a) Which of the following multiplication is possible ?
b) If it is possible, find the dimension of the resulting matrix

$$\boldsymbol{A.B} = \begin{vmatrix} 1 & 1 & 5 \\ 2 & 1 & -1 \end{vmatrix} \cdot \begin{vmatrix} 2 \\ 1 \\ 0 \end{vmatrix}$$

For the following matrices, using the multiplication of Row by Column :

$$A = \begin{vmatrix} 1 & 1 & 5 \\ 2 & 1 & -1 \end{vmatrix} \quad , \quad B = \begin{vmatrix} 2 \\ 1 \\ 0 \end{vmatrix} \quad , \quad C = \begin{vmatrix} 2 & -1 \\ 1 & 2 \\ 0 & 4 \end{vmatrix}$$

a) Which of the following multiplication is possible ?
b) If it is possible, find the dimension of the resulting matrix

$$A.C = \begin{vmatrix} 1 & 1 & 5 \\ 2 & 1 & -1 \end{vmatrix} \cdot \begin{vmatrix} 2 & -1 \\ 1 & 2 \\ 0 & 4 \end{vmatrix}$$

For the following matrices, using the multiplication of Row by Column :

$$A = \begin{vmatrix} 1 & 1 & 5 \\ 2 & 1 & -1 \end{vmatrix} \quad , \quad B = \begin{vmatrix} 2 \\ 1 \\ 0 \end{vmatrix} \quad , \quad C = \begin{vmatrix} 2 & -1 \\ 1 & 2 \\ 0 & 4 \end{vmatrix}$$

a) Which of the following multiplication is possible ?
b) If it is possible, find the dimension of the resulting matrix

$$\textbf{B.C} = \begin{vmatrix} 2 \\ 1 \\ 0 \end{vmatrix} \cdot \begin{vmatrix} 2 & -1 \\ 1 & 2 \\ 0 & 4 \end{vmatrix}$$

For the following matrices, using the multiplication of Row by Column :

$$A = \begin{vmatrix} 1 & 1 & 5 \\ 2 & 1 & -1 \end{vmatrix} \quad , \quad B = \begin{vmatrix} 2 \\ 1 \\ 0 \end{vmatrix} \quad , \quad C = \begin{vmatrix} 2 & -1 \\ 1 & 2 \\ 0 & 4 \end{vmatrix}$$

a) Which of the following multiplication is possible ?
b) If it is possible, find the dimension of the resulting matrix

$$C.A = \begin{vmatrix} 2 & -1 \\ 1 & 2 \\ 0 & 4 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 5 \\ 2 & 1 & -1 \end{vmatrix}$$

For the following matrices, Find $A.B$ and $B.A$ if possible:

$$A = \begin{vmatrix} 1 & 1 & 5 \end{vmatrix} \quad , \qquad B = \begin{vmatrix} 2 \\ 1 \\ 0 \end{vmatrix}$$

$$A.B = \begin{vmatrix} 1 & 1 & 5 \end{vmatrix} \cdot \begin{vmatrix} 2 \\ 1 \\ 0 \end{vmatrix}$$

$$B.A = \begin{vmatrix} 2 \\ 1 \\ 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 5 \end{vmatrix}$$

Section 6.3 Cont.: Inverse Matrix

To find the inverse matrix of $A = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$ using the All Integer Method:

- Step 1: Re-write it with the Identity Matrix I next to it on the right side:
  (The Identity Matrix I: the square matrix where all Diagonal elements = 1, the rest are zeros)

$$\left|\begin{array}{cc|cc} 2^* & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right|$$

- Step 2: Do the pivot steps (2 pivots for two rows), and the last step should be:

$$\left|\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array}\right|$$

- Step 3: The Identity Matrix I is now on the left side, and the Inverse Matrix A^{-1} is on the right side:

$$A^{-1} = \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}$$

- You can check your answer by multiplying the original matrix A and the inverse A^{-1}. The answer must be an Identity Matrix I.

Note: Not every matrix has an inverse, for example: $A = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix}$ does not have an inverse (the second pivot is zero).

Solve the same example and show steps

$$A = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

Examples: Find the inverse matrix for each of the following and check your answer by multiplying the original matrix by its inverse, the resulting matrix must be an Identity Matrix:

1) $\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix}$

2) $\begin{vmatrix} 1 & 0 \\ 3 & -2 \end{vmatrix}$

1)

1	1	1	0
2	4	0	1

2)

1	0	1	0
3	-2	0	1

3) $\begin{vmatrix} 4 & -1 \\ 3 & -1 \end{vmatrix}$

4) $\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}$

Note: Not every matrix has an inverse, for example: $A = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix}$ does not have an inverse (the second pivot is zero).

Find the inverse of the following 3x3 matrix:

$$A = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 1 & 1 \end{vmatrix}$$

Original Matrix A			*Identity matrix I*		
2	1	1	1	0	0
1	2	-1	0	1	0
1	1	1	0	0	1
2	1	1	1	0	0
0	3	-3	-1	2	0
0	1	1	-1	0	2
3	0	3	2	-1	0
0	3	-3	-1	2	0
0	0	3	-1	-1	3
3	0	0	3	0	-3
0	3	0	-2	1	3
0	0	3	-1	-1	3
1	0	0	1	0	-1
0	1	0	- 2/3	1/3	1
0	0	1	- 1/3	- 1/3	1
	I			A^{-1}	