## Section 9.1: Transition Matrices

- **In Section 4.4, Bernoulli Trails:**
  The probability of each outcome is independent of the outcome of any previous experiments and the probability stays the same.

**Ex. 1**: Computer chips are manufactured with 5% defective. Fifteen are drawn at random from an assembly line by an inspector, what is the probability that he will find 3 defective chips?

**Ex. 2**: Flipping a fair coin 30 times, the probability stays the same and does not depend on the previous result.

- **In Section 9.1, Markov Chain**:
  What happens next is governed by what happened immediately before. (see the next Examples)

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**Ex. 3**: An independent landscape contractor works in a weekly basis.
Each week he works (W), there is a probability of 80% that will be called again to work the following week.
Each week he is not working (N), there is a probability of only 60% that he will be called again to work.

Draw the tree for all possibilities of 2 weeks from now and show all probabilities.
*Note: You need to draw 2 different trees, one if he is working now and the other if he is not. We cannot start from one point covering the initial states with one branch for W and the other for N since it is not given to us and we cannot assume it 50% each.*

Use the tree to find

a) The probability that if he is working now, then he will be working in 2 weeks.
b) The probability that if he is not working now, then he will be working in 2 weeks.

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**Ex. 4:** Use the information of example 3 again and find:

a) The **Transition Matrix**. Show all probabilities and make sure the sum per row = 1

b) The **Transition Diagram.** Show all probabilities and make sure:
the sum of probabilities leaving a nod + itself = 1

c) The probability that if he is working now, then he will be working in 2 weeks
d) The probability that if he is not working now, then he will be working in 2 weeks
e) The probability that if he is working now, then he will be working in 4 weeks

**Ex. 5:** A study by an overseas travel agency reveals that among the airlines: American, Delta and United, traveling habits are as follows:

- If a customer has just traveled on American, there is a 50% chance he will choose American again on his next trip, but if he switches, he is just likely to switch to Delta or United.
- If a customer has just traveled on Delta, there is a 60% chance he will choose Delta again on his next trip, but if he switches, he is three times as likely to switch to American as to United.
- If a customer has just traveled on United, there is a 70% chance he will choose United again on his next trip, but if he switches, he is twice as likely to switch to Delta as to American.

a) Find the probability transition matrix

b) Find the transition diagram

c) Find the matrix that describes the customers habits two trips from now, then find the probability that a current Delta ticket holder will not travel on Delta the next time after

d) Find the matrix that describes the customers habits three trips from now, then find the probability that a current American ticket holder will switch to United three trips from now.

**Ex. 6:** A market analyst is interested in whether consumers prefer Dell or Gateway computers. Two market surveys taken one year apart reveals the following:

- 10% of Dell owners had switched to Gateway and the rest continued with Dell.
- 35% of Gateway owners had switched to Dell and the rest continued with Gateway.

At the time of the first market survey, 40% of consumers had Dell computers and 60% had Gateway.

a) What percentage will by their next computer from Dell?

b) What percentage will buy their second computer from Dell?

c) Suppose that each consumer buy a new computer each year, what will be the market distribution after 4 years?

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|---|
| $\boldsymbol{P_n = P_0 T^n}$ ($P_0$: the initial state vector, $T$: the transition matrix) |

**Ex. 7:** Suppose that taxis pick up and deliver passengers in a city which is divided into three zones: *A*, *B* and *C*. Records kept by the drivers show that:

- Of the passengers picked up in zone *A*, 50% are taken to a destination in zone *A*, 40% to zone *B*, and 10% to zone C.
- Of the passengers picked up in zone *B*, 40% go to zone *A*, 30% to zone *B*, and 30% to zone *C*.
- Of the passengers picked up in zone *C*, 20% go to zone *A*, 60% to zone *B*, and 20% to zone *C*.

Suppose that at the beginning of the day 60% of the taxis are in zone *A*, 10% in zone *B*, and 30% in zone *C*.

a) What is the distribution of taxis in the various zones after all have had one rider?

b) What is the distribution of taxis in the various zones after all have had two riders?

c) What is the distribution of taxis in the various zones after all have had four riders

# <u>Section 9.2: Regular Markov Chains</u>

- **Irreducible Markov Chain:** When all its states communicate with each others. (*It is strongly recommended to draw the transition diagram*)

**Ex. 1**: Determine if the following is irreducible: $T = \begin{bmatrix} 0.25 & 0.75 \\ 0.65 & 0.35 \end{bmatrix}$

**Ex. 2**: Determine if the following is irreducible: $T = \begin{bmatrix} 0.2 & 0.4 & 0.4 \\ 0.3 & 0 & 0.7 \\ 0 & 0 & 1 \end{bmatrix}$

*Note: Anytime a state is communicating only with <u>itself</u> as in state 3, the matrix is not irreducible.*

**Ex. 3**: Determine if the following is irreducible: $T = \begin{bmatrix} 0.6 & 0 & 0.4 \\ 0.2 & 0 & 0.8 \\ 0 & 0.8 & 0.2 \end{bmatrix}$

**Ex. 4**: Determine if the following is irreducible: $T = \begin{bmatrix} 0.3 & 0 & 0.7 & 0 \\ 0 & 0.2 & 0 & 0.8 \\ 0.6 & 0 & 0.4 & 0 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}$

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- **Regular Markov Chain:** A transition matrix is regular when there is power of $T$ that contains all positive *no zeros* entries**.**
  a) If the transition matrix is not irreducible, then it is not regular
  b) If the transition matrix is irreducible and at least one entry of the main diagonal is nonzero, then it is regular
  c) If all entries on the main diagonal are zero, but $T^n$ (after multiplying by itself $n$ times) contain all positive entries, then it is regular

**Ex. 5**: Determine which of the following matrices is regular:

a) $T = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$ b) $T = \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0 \end{bmatrix}$ c) $T = \begin{bmatrix} 1 & 0 & 0 \\ 0.25 & 0.5 & 0.25 \\ 0 & 1 & 0 \end{bmatrix}$

a) yes, all entries are positive
b) yes because $T^2 = \begin{bmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{bmatrix}$ has only positive entries. You can also look at it as irreducible matrix with at least one element in the main diagonal not equal to zero.

c) No, because it is not irreducible. Also, if you multiply it by itself over and over it will still contain zeros

**Ex. 6**: Previously in section 9.1, we had the following example:

A market analyst is interested in whether consumers prefer Dell or Gateway computers. two market surveys taken one year apart reveals the following:

- 10% of Dell owners had switched to Gateway and the rest continued with Dell.
- 35% of Gateway owners had switched to Dell and the rest continued with Gateway.

| **If a consumer has Dell Computer now:** | **If a consumer has Gateway Computer now:** |
|---|---|
| Now: $\begin{bmatrix} 1 & 0 \end{bmatrix}$ | Now: $\begin{bmatrix} 0 & 1 \end{bmatrix}$ |
| After 1 year: $\begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0.9 & 0.1 \\ 0.35 & 0.65 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.1 \end{bmatrix}$ | After 1 year: $\begin{bmatrix} 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0.9 & 0.1 \\ 0.35 & 0.65 \end{bmatrix} = \begin{bmatrix} 0.35 & 0.65 \end{bmatrix}$ |
| After 2 year: $\begin{bmatrix} 0.9 & 0.1 \end{bmatrix} \cdot \begin{bmatrix} 0.9 & 0.1 \\ 0.35 & 0.65 \end{bmatrix} = \begin{bmatrix} 0.85 & 0.16 \end{bmatrix}$ | After 2 year: $\begin{bmatrix} 0.35 & 0.65. \end{bmatrix} \cdot \begin{bmatrix} 0.9 & 0.1 \\ 0.35 & 0.65 \end{bmatrix} = \begin{bmatrix} 0.54 & 0.46 \end{bmatrix}$ |
| After 3 year: $\begin{bmatrix} 0.85 & 0.16 \end{bmatrix} \cdot \begin{bmatrix} 0.9 & 0.1 \\ 0.35 & 0.65 \end{bmatrix} = \begin{bmatrix} 0.81 & 0.19 \end{bmatrix}$ | After 3 year: $\begin{bmatrix} 0.54 & 0.46 \end{bmatrix} \cdot \begin{bmatrix} 0.9 & 0.1 \\ 0.35 & 0.65 \end{bmatrix} = \begin{bmatrix} 0.65 & 0.35 \end{bmatrix}$ |
| After 4 year: $\begin{bmatrix} 0.81 & 0.19 \end{bmatrix} \cdot \begin{bmatrix} 0.9 & 0.1 \\ 0.35 & 0.65 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix}$ | After 4 year: $\begin{bmatrix} 0.65 & 0.35 \end{bmatrix} \cdot \begin{bmatrix} 0.9 & 0.1 \\ 0.35 & 0.65 \end{bmatrix} = \begin{bmatrix} 0.71 & 0.29 \end{bmatrix}$ |
| After 5 year: $\begin{bmatrix} 0.8 & 0.2 \end{bmatrix} \cdot \begin{bmatrix} 0.9 & 0.1 \\ 0.35 & 0.65 \end{bmatrix} = \begin{bmatrix} 0.79 & 0.21 \end{bmatrix}$ | After 5 year: $\begin{bmatrix} 0.71 & 0.29 \end{bmatrix} \cdot \begin{bmatrix} 0.9 & 0.1 \\ 0.35 & 0.65 \end{bmatrix} = \begin{bmatrix} 0.74 & 0.26 \end{bmatrix}$ |
| After 6 year: $\begin{bmatrix} 0.79 & 0.21 \end{bmatrix} \cdot \begin{bmatrix} 0.9 & 0.1 \\ 0.35 & 0.65 \end{bmatrix} = \begin{bmatrix} 0.78 & 0.22 \end{bmatrix}$ | After 6 year: $\begin{bmatrix} 0.74 & 0.26 \end{bmatrix} \cdot \begin{bmatrix} 0.9 & 0.1 \\ 0.35 & 0.65 \end{bmatrix} = \begin{bmatrix} 0.76 & 0.24 \end{bmatrix}$ |
| After 7 year: $\begin{bmatrix} 0.78 & 0.22 \end{bmatrix} \cdot \begin{bmatrix} 0.9 & 0.1 \\ 0.35 & 0.65 \end{bmatrix} = \begin{bmatrix} 0.78 & 0.22 \end{bmatrix}$ | After 7 year: $\begin{bmatrix} 0.76 & 0.24 \end{bmatrix} \cdot \begin{bmatrix} 0.9 & 0.1 \\ 0.35 & 0.65 \end{bmatrix} = \begin{bmatrix} 0.77 & 0.23 \end{bmatrix}$ |
| After 8 year: $\begin{bmatrix} 0.78 & 0.22 \end{bmatrix} \cdot \begin{bmatrix} 0.9 & 0.1 \\ 0.35 & 0.65 \end{bmatrix} = \begin{bmatrix} 0.78 & 0.22 \end{bmatrix}$ | After 8 year: $\begin{bmatrix} 0.77 & 0.23 \end{bmatrix} \cdot \begin{bmatrix} 0.9 & 0.1 \\ 0.35 & 0.65 \end{bmatrix} = \begin{bmatrix} 0.78 & 0.22 \end{bmatrix}$ |
| After 9 year: $\begin{bmatrix} 0.78 & 0.22 \end{bmatrix} \cdot \begin{bmatrix} 0.9 & 0.1 \\ 0.35 & 0.65 \end{bmatrix} = \begin{bmatrix} 0.78 & 0.22 \end{bmatrix}$ | After 9 year: $\begin{bmatrix} 0.78 & 0.22 \end{bmatrix} \cdot \begin{bmatrix} 0.9 & 0.1 \\ 0.35 & 0.65 \end{bmatrix} = \begin{bmatrix} 0.78 & 0.22 \end{bmatrix}$ |

- After certain years, the probability stabilizes at 78% for Dell and 22% for Gateway. Notice that whether we start with Gateway or Dell, the result is the same and that is not accidental.

- The state vector of $P = \begin{bmatrix} 0.78 & 0.22 \end{bmatrix}$ is called the **Steady State Vector** where: $\boldsymbol{P.T = P}$
  *(multiplying the Steady State Vector by the Transition Matrix = the Steady State Vector.)*

- The above can only applied on **Regular** Markov chain

**Ex. 7**: The same example again:

A market analyst is interested in whether consumers prefer Dell or Gateway computers. two market surveys taken one year apart reveals the following:

- 10% of Dell owners had switched to Gateway and the rest continued with Dell.
- 35% of Gateway owners had switched to Dell and the rest continued with Gateway.

Find the distribution of the market after "a long period of time".

Solution:

The answer is in finding the **Steady State Vector *P*** where***:*** ***P.T = P***

$$P = \begin{bmatrix} p_1 & p_2 \end{bmatrix} \quad ; \quad T = \begin{bmatrix} 0.9 & 0.1 \\ 0.35 & 0.65 \end{bmatrix}$$

***P.T = P*** then : $\begin{bmatrix} p_1 & p_2 \end{bmatrix} \cdot \begin{bmatrix} 0.9 & 0.1 \\ 0.35 & 0.65 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 \end{bmatrix}$

Or: $0.9p_1 + 0.35p_2 = p_1$

$0.1p_1 + 0.65p_2 = p_2$

Simplify the above equations by moving all variable to one side:

$$-0.1p_1 + 0.35p_2 = 0$$
$$0.1p_1 - 0.35p_2 = 0$$

The two equations are dependent and have infinite number of solutions. We must add another equation in order to get the answer: $p_1 + p_2 = 1$

Now, use the Echelon's Method to solve:

$$-0.1p_1 + 0.35p_2 = 0$$
$$0.1p_1 - 0.35p_2 = 0$$
$$p_1 + p_2 = 1$$

It makes it easier if you multiply the first and the second equation by 100 to remove the decimal:

| $p_1$ | $p_2$ | | |
|---|---|---|---|
| -10* | 35 | 0 | |
| 10 | -35 | 0 | |
| 1 | 1 | 1 | |
| -10 | 35 | 0 | |
| 0 | 0 | 0 | Remove the line with all zeros |
| 0 | -45 | -10 | |
| -10 | 35 | 0 | |
| 0 | -45* | -10 | |
| -45 | 0 | -35 | |
| 0 | -45 | -10 | |
| 1 | 0 | 0.78 | |
| 0 | 1 | 0.22 | |

The answer is $p_1$ = 78% and $p_2$ = 22% which is the same answer we got in example 6 when we did it the long way.

**Ex. 8:** Suppose that General Motors (GM), Ford (F), and Chrysler (C) each introduce a new SUV vehicle.

- General Motors keeps 85% of its customers but loses 10% to Ford and 5% to Chrysler.
- Ford keeps 80% of its customers but loses 10% to General motors and 10% to Chrysler.
- Chrysler keeps 60% of its customers but loses 25% to General Motors and 15% to Ford..

Find the distribution of the market in the long run.

Solution: Lets assume the probabilities to be *x* for GM, *y* for F and *z* for C just to make it easier to solve

$$P = \begin{bmatrix} x & y & z \end{bmatrix} \quad ; \quad T = \begin{bmatrix} 0.85 & 0.1 & 0.05 \\ 0.1 & 0.8 & 0.1 \\ 0.25 & 0.15 & 0.6 \end{bmatrix}$$

***P.T = P*** then : $\begin{bmatrix} x & y & z \end{bmatrix} \cdot \begin{bmatrix} 0.85 & 0.1 & 0.05 \\ 0.1 & 0.8 & 0.1 \\ 0.25 & 0.15 & 0.6 \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix}$

Or:

$$0.85x + 0.1y + 0.25z = x$$
$$0.1x + 0.8y + 0.15z = y$$
$$0.05x + 0.1y + 0.6z = z$$

Simplify the above equations by moving all variable to one side:

$$-0.15x + 0.1y + 0.25z = 0$$
$$0.1x - 0.2y + 0.15z = 0$$
$$0.05x + 0.1y - 0.4z = 0$$

and: $x + y + z = 1$

It makes it easier if you multiply the first 3 equations by 100 to remove the decimal:

| *x* | *y* | *z* | | |
|---|---|---|---|---|
| -15* | 10 | 25 | 0 | |
| 10 | -20 | 15 | 0 | |
| 5 | 10 | -40 | 0 | |
| 1 | 1 | 1 | 1 | |
| -15 | 10 | 25 | 0 | |
| 0 | 200* | -475 | 0 | |
| 0 | -200 | 475 | 0 | |
| 0 | -25 | -40 | -15 | |
| 200 | 0 | -650 | 0 | |
| 0 | 200 | -475 | 0 | |
| 0 | 0 | 0 | 0 | Remove the line with all zeros |
| 0 | 0 | 1325 | 200 | |
| 200 | 0 | -650 | 0 | |
| 0 | 200 | -475 | 0 | |
| 0 | 0 | 1325* | 200 | |
| 1325 | 0 | 0 | 650 | |
| 0 | 1325 | 0 | 475 | |
| 0 | 0 | 1325 | 200 | |
| 1 | 0 | 0 | 0.49 | GM = 49% |
| 0 | 1 | 0 | 0.36 | Ford = 36% |
| 0 | 0 | 1 | 0.15 | Chrysler = 15% |