# EVOLUTION OF THE SHARKOVSKY THEOREM

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Abstract. We briefly describe some results that evolved from the Sharkovsky Theorem.

## 1. Introduction

Among the many achievements of Oleksandr Mykolayovych Sharkovsky, the best known one is what has been called The Sharkovsky Theorem. Its subject is quite simple, periods of periodic orbits of continuous maps of an interval to itself. However, the result is surprisingly strong.

As it happens with theorems of that strength and beauty, various generalizations and refinements started to appear. They grew into a discipline that is called Combinatorial Dynamics.

The aim of this paper is to give a brief description of the Sharkovsky Theorem and some results that evolved from it. We do not depart too far from the original theorem, as this would require writing several books[1].

In Section 2 we describe the Sharkovsky Theorem. In Section 3 we discuss a similar result for continuous maps of the circle to itself. In Section 4 we consider continuous maps of trees into themselves. Section 5 is devoted to some general ideas. In Sections 6 and 7 we use those ideas to produce refinements of the Sharkovsky Theorem for interval maps.

## 2. The Sharkovsky Theorem

To state the Sharkovsky Theorem, we first describe the Sharkovsky Ordering. It is a linear ordering on the set of natural numbers $\mathbb{N}$ (we do not consider 0 natural), together with the symbol $2^\infty$. We have to include $2^\infty$ in order to ensure the existence of the leftmost ("strongest") element of every subset in this ordering. The ordering is

$$\begin{aligned}3 \succ_s 5 \succ_s 7 \succ_s 9 \succ_s \dots \succ_s 2\cdot 3 \succ_s 2\cdot 5 \succ_s 2\cdot 7 \succ_s 2\cdot 9 \succ_s \dots \succ_s 4\cdot 3 \succ_s 4\cdot 5 \succ_s 4\cdot 7 \succ_s 4\cdot 9 \succ_s \dots\\ 2^n\cdot 3 \succ_s 2^n\cdot 5 \succ_s 2^n\cdot 7 \succ_s 2^n\cdot 9 \succ_s \dots \succ_s 2^\infty \succ_s \dots \succ_s 2^n \succ_s \dots \succ_s 16 \succ_s 8 \succ_s 4 \succ_s 2 \succ_s 1.\end{aligned}$$

For every $k \in \mathbb{N} \cup \{2^\infty\}$ we denote by $\mathrm{Sh}(k)$ the set of all natural numbers $n$ such that $k \succ_s n$, and $k$ itself, if $k \in \mathbb{N}$.

Now we are ready to state the Sharkovsky Theorem. While originally Sharkovsky was considering maps of the real line, rather than a compact interval, the only difference is that for the real line case there may be no periodic orbits, and all the proofs are practically the same.

When speaking of the set of periods of periodic orbits (of a map), we will use a simpler term "set of periods (of a map)."

**Theorem 2.1.** *Let $I$ be a compact interval. For every continuous map $f : I \to I$ there exists $s \in \mathbb{N} \cup 2^\infty$ such that the set of periods of $f$ is equal to* $\mathrm{Sh}(s)$. *Conversely, for every $s \in \mathbb{N} \cup 2^\infty$ there exists a continuous map $f : I \to I$ such that its set of periods is equal to* $\mathrm{Sh}(s)$.

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[1]One such book, [3], has been published in 1993, with the second edition in 2000. A brief survey-type book dealing with the same topic [20] was apparently one of the last contributions of O. M. Sharkovsky.

The first part of this theorem was proved by Sharkovsky in [31] (its translation to English can be found in [32]). The arguments given there also prove the second part, except the case of $s = 2^\infty$. This missing case was proved by Sharkovsky in [33].

Note that some authors treat Theorem 2.1 as two separate theorems.

## 3. Circle

Usually, when a completely new theorem is proved, a lot of natural questions appear. Here are some of the simplest ones. Does the theorem hold only for the class of objects mentioned in it, or maybe it can be extended onto a larger class of objects? And if it cannot be extended, can similar theorems be proved for similar classes of objects?

In the concrete case of the Sharkovsky Theorem, the simplest question of the above type concerns continuous circle maps. One can immediately see that the Sharkovsky Theorem does not hold in this case. If we take the rotation of the circle by the $1/n$ of the full rotation, all orbits are periodic of period $n$. This also shows that one cannot just replace the Sharkovsky ordering by some other ordering and hope that similar results will hold. Thus, in order to work on extending the Sharkovsky Theorem, one should first understand its true meaning, its true essence. What is the main thing that this theorem does? It provides a simple description of all possible sets of periods of periodic orbits for continuous interval maps. Thus, we should try to do the same for continuous circle maps.

It turns out that it is possible to do it, but we should divide the set of all continuous circle maps into subsets, according to the degree of the map. The simplest definition of the degree of a continuous circle map $f : \mathbb{S} \to \mathbb{S}$ is to look at the map $f_* : \mathbb{Z} \to \mathbb{Z}$ induced by $f$ in the first homology group of the circle $\mathbb{S}$. This map is the multiplication by a certain integer $d$, and $d$ is the degree of $f$.

Another approach, more useful for us, is to treat $\mathbb{S}$ as the quotient $\mathbb{R}/\mathbb{Z}$ and look at a lifting $F : \mathbb{R} \to \mathbb{R}$ of $f$. Then for every $x \in \mathbb{R}$ we have $F(x+1) = F(x) + d$, where $d$ is the degree of $f$.

The easiest case is $|d| \geq 2$. For those continuous circle maps it is not hard to see that there should be periodic orbits of all periods. However, there is a small surprise: there is one exception. For instance, if $d = -2$ and the lifting is $F(x) = -2x$, there is no periodic orbit of period 2. To see this, note that the fixed points of $f^2$ are projections to $\mathbb{S}$ of points $x \in \mathbb{R}$, for which $F^2(x) = x + p$ for some integer $p$. However, $F^2(x) = 4x$, so if $F^2(x) = x + p$, then $x = p/3$. Then $F(x) = -2p/3 = x - p$, so the projection of $x$ to $\mathbb{S}$ is a fixed point of $f$.

The following theorem was proved independently by Efremova [23] and Block, Guckenheimer, Misiurewicz and Young [9].

**Theorem 3.1.** *Let $f$ be a circle map of degree $d$ such that $|d| \geq 2$. Then the set of periods of $f$ is $\mathbb{N}$, except for $d = -2$ when it is either $\mathbb{N}$ or $\mathbb{N} \setminus \{2\}$, and both possibilities can occur.*

If the degree of $f$ is 0, then $F$ is periodic, so $F(\mathbb{R})$ is an interval. Then it is not a surprise that the situation is very similar as for interval maps. It is not intuitively obvious that if the degree of $f$ is $-1$ then also the situation is similar to that for the interval maps (and the proof in this case is more complicated).

**Theorem 3.2.** *The Sharkovsky Theorem holds for continuous circle maps of degree 0 and for continuous circle maps of degree $-1$.*

In the case of degree 0, partial results were obtained by Efremova [23], and the full result was obtained (independently) by Block, Guckenheimer, Misiurewicz and Young [9]. In the case of degree $-1$, the full result was obtained independently by Efremova [23] and Block [10].

In such a way we are left with the case of degree of $f$ equal to 1. This is definitely the most intricate case. To understand what is going on, we have to introduce the notion of the *rotation interval.*

Let us fix a lifting $F$ of $f$. For any other lifting $G$ of $f$ there is an integer $j$ such that $G = F + j$, so it will be clear that our final results do not depend on the choice of a particular lifting. If $x \in \mathbb{R}$, call the average displacement

$$\rho(x) = \limsup_{n\to\infty} \frac{F^n(x) - x}{n}$$

the *rotation number* of $x$. This notion was introduced by Poincaré [29] for homeomorphisms of the circle of degree 1. He showed that in that case the limit actually exists and is independent of $x$. Newhouse, Palis and Takens [28] extended this notion to continuous circle maps of degree 1. They showed that the set of rotation numbers of all points is an interval. Ito [25] proved that the rotation interval is a closed interval. A shorter proof of this fact was given by Chenciner, Gambaudo and Tresser [22]. We will denote the rotation interval of $F$ by $R(F)$.

It is clear that if the projection of $x$ to the circle is periodic of period $q$, then $F^q(x) = x + p$ for some integer $p$. Then $\rho(x) = p/q$. It turns out that if $p/q$ belongs to the interior of $R(F)$ and $p, q$ are coprime, then for every $m \in \mathbb{N}$ there exists $x \in \mathbb{R}$ such that the projection of $x$ to the circle is periodic of period $mq$ and $\rho(x) = p/q$. If $p/q$ is an endpoint of $R(F)$ and $p, q$ are coprime, then there exists $s \in \mathbb{N} \cup \{2^\infty\}$ such that for every $m \in \mathrm{Sh}(s)$ there exists $x \in \mathbb{R}$ such that the projection of $x$ to the circle is periodic of period $mq$ and $\rho(x) = p/q$.

From those properties, one can deduce the characterization of all possible sets of periods for continuous maps of the circle of degree 1. For $c \le d$ we denote

$$M(c, d) = \{n \in \mathbb{N} : c < k/n < d \quad \text{for some integer } k\}.$$

Given $r \in \mathbb{R}$ and $s \in \mathbb{N} \cup \{2^\infty\}$, we set

$$S(r, s) = \begin{cases} \emptyset & \text{if } r \notin \mathbb{Q}, \\ \{nq : q \in \mathrm{Sh}(s)\} & \text{if } r = k/n \text{ with } k \text{ and } n \text{ coprime.} \end{cases}$$

The following theorem was proved by Misiurewicz [26].

**Theorem 3.3.** *Let $F$ be a lifting of a continuous circle map $f$ of degree 1, and let $R(F) = [c, d]$. Then there exist numbers $s_c, s_d \in \mathbb{N} \cup \{2^\infty\}$, such that the set of periods of $f$ is equal to $S(c, s_c) \cup M(c, d) \cup S(d, s_d)$.*

Of course, if we do not want to mention liftings or rotation intervals, we can restate this theorem as follows.

**Theorem 3.4.** *Let $f$ be a continuous circle map of degree 1. Then there exist numbers $c, d \in \mathbb{R}$, with $c \le d$, and numbers $s_c, s_d \in \mathbb{N} \cup \{2^\infty\}$, such that the set of periods of $f$ is equal to $S(c, s_c) \cup M(c, d) \cup S(d, s_d)$.*

Similarly as for the Sharkovsky Theorem, the above characterization of the sets of periods of periodic orbits for continuous circle maps is the best possible. That is, if we have any set of the form mentioned in Theorem 3.4, then there is a continuous circle map of degree 1 with that set of periods.

At the end of this section we want to state the results of [26] in full generality (although they were nor presented in this way there). The reader will see how they imply Theorems 3.3 and Theorem 3.4, but also, quite importantly, this will allow us to introduce ideas that motivated the results in Section 6. Namely, define the *rotation pair* $(p, q)$ of a periodic orbit of a degree one circle map $f$, where $q$ is the period, and $p/q$ the rotation number of the orbit (here we fix the lifting $F$ of $f$). Basically, the number $p$ shows how many times the periodic point rotates around the circle before it comes back to its initial location. The full results of [26] describe all possible sets of rotation pairs of periodic orbits of degree one circle maps. For $c \le d$ we set

$$M'(c,d) = \{(p,q) : c < p/q < d, \text{ where } p \text{ and } q > 0 \text{ are integers}\}.$$

Given $r \in \mathbb{R}$ and $s \in \mathbb{N} \cup \{2^\infty\}$, we set

$$S'(r,s) = \begin{cases} \emptyset & \text{if } r \notin \mathbb{Q}, \\ \{(kq, nq) : q \in \mathrm{Sh}(s)\} & \text{if } r = k/n \text{ with } k \text{ and } n \text{ coprime.} \end{cases}$$

The following theorem is the main result of [26]; recall that when we talk about rotation numbers and pairs of a degree one circle maps we fix a lifting of the map.

**Theorem 3.5.** *Let $F$ be a lifting of a continuous circle map $f$ of degree 1, and let $R(F) = [c,d]$. Then there exist numbers $s_c, s_d \in \mathbb{N} \cup \{2^\infty\}$, such that the set of rotation pairs of periodic orbits of $f$ is equal to $S'(c, s_c) \cup M'(c,d) \cup S'(d, s_d)$. In particular, all rotation numbers of periodic orbits of $f$ are simply all rational numbers that belong to $[c,d]$.*

It is easy to see that Theorem 3.5 implies Theorems 3.3 and 3.4. Moreover, Theorem 3.5 describes all rotation numbers of periodic orbits of degree one circle maps. This allows one to draw a parallel between this theorem and the Sharkovsky Theorem. Indeed, in the Sharkovsky Theorem, if we know that our interval map has a periodic orbit of a given period, we can deduce that it has periodic orbits of all periods prescribed by this theorem. Here, if we know that the degree 1 circle map has periodic orbits of rotation numbers $p/q$ and $s/t$ then it has a periodic orbit of any rotation number $m/n$ just as long as $m/n$ is located between $p/q$ and $s/t$.

In fact, Theorem 3.5 allows one to deal with rotation pairs in the same fashion. To this end let us expand the Sharkovsky Ordering over rotation pairs. Namely, if $(a,b)$ and $(c,d)$ are two rotation pairs such that $a/b = c/d = u/v$, where $u$ and $v$ are coprime, and $b/v \succ_s d/v$ then we write $(a,b) \succ_s (c,d)$. Then if a degree one circle map $f$ has periodic orbits of rotation pairs $(p,q)$ and $(s,t)$, then it has to have a periodic orbit of rotation pair $(m,n)$ in any of the following cases:

- $p/q < m/n < s/t$ or $s/t < m/n < p/q$,
- $(p,q) \succ_s (m,n)$ or $(s,t) \succ_s (m,n)$.

Observe that if $p/q = s/t$ then they can be compared in the sense of $\succ_s$-ordering, and one of them (say, $(p,q)$) is $\succ_s$-stronger than the other one. Then the set of rotation pairs forced by $(p,q)$ and $(s,t)$ is simply the set of rotation pairs that are $\succ_s$-weaker than $(p,q)$.

## 4. Trees

Another natural class of spaces onto which one can try to generalize the Sharkovsky Theorem is the class of trees. The simplest trees, resembling the interval, are *$n$-ods*. One can define the *canonical $n$-od* as the set of all complex numbers $z$ for which $z^n$ belongs to the interval $[0,1]$, and then call every space homeomorphic to the canonical $n$-od simply an *$n$-od*.

Note that both 1-od and 2-od are intervals. It turns out that for our purposes it makes more sense to view an interval as a 2-od, so we will assume that $n \ge 2$.

Of course the simplest non-interval case is $n = 3$. The 3-od (*triod*) resembles the letter Y, so let us denote it by $Y$. This was the first tree for which all possible sets of periods of periodic orbits were identified for a large class of continuous maps (namely, those for which the branching point is fixed). This was done by Alsedà, Llibre and Misiurewicz in [2].

It turned out that for this class of maps in order to describe possible sets of periods we need not one, but three linear orderings. In addition to the Sharkovsky ordering, let us define *green* and *red* orderings (the names came from the colors of markers used on the whiteboard). If Sh is the

Sharkovsky ordering, then $n \cdot \mathrm{Sh}$ is the same ordering with every element $k \in \mathbb{N} \cup \{2^\infty\}$ replaced by $nk$ (so in particular we have the symbol $n \cdot 2^\infty$). Then the green ordering is

$$5,\ 8,\ 4,\ 11,\ 14,\ 7,\ 17,\ 20,\ 10,\ 23,\ 26,\ 13,\ \dots,\ 3\cdot\mathrm{Sh},\ 1,$$

and the red ordering is

$$7,\ 10,\ 5,\ 13,\ 16,\ 8,\ 19,\ 22,\ 11,\ 25,\ 28,\ 14,\ \dots,\ 3\cdot\mathrm{Sh},\ 1.$$

Instead of giving the formal definitions of those orderings, let us describe how they are produced. We write the arithmetic progression starting with some number (5 for the green ordering and 7 for the red one) and with difference 3, then after each even number we insert its half. Finally, we append at the end $3 \cdot \mathrm{Sh}$ and 1. Note that in the green ordering 2 is missing, and in the red one 2 and 4 are missing.

Alternatively, the green ordering can be described as

$$6\cdot 1-1,\ 6\cdot 1+2,\ 3\cdot 1+1,\ \dots, 6n-1,\ 6n+2,\ 3n+1,\ \dots,\ 3\cdot\mathrm{Sh},\ 1,$$

with $n$ running from $n = 1$ to infinity in the initial countable segment of the green ordering (i.e. until the beginning of $3 \cdot \mathrm{Sh}$ segment). Similarly, the red ordering can be described as

$$6\cdot 1+1,\ 6\cdot 1+4,\ 3\cdot 1+2,\ \dots, 6n+1,\ 6n+4,\ 3n+2,\ \dots,\ 3\cdot\mathrm{Sh},\ 1,$$

with $n$ running from $n = 1$ to infinity in the initial countable segment of the red ordering.

Similarly as for the Sharkovsky ordering, for every $k \in \mathbb{N}_g = \mathbb{N} \cup \{3 \cdot 2^\infty\} \setminus \{2\}$ we denote by $\mathrm{Green}(k)$ the set of all natural numbers to the right of $k$ (including $k$, if $k \in \mathbb{N}$) in the green ordering, and for every $k \in \mathbb{N}_r = \mathbb{N} \cup \{3 \cdot 2^\infty\} \setminus \{2, 4\}$ we denote by $\mathrm{Red}(k)$ the set of all natural numbers to the right of $k$ (including $k$, if $k \in \mathbb{N}$) in the red ordering.

**Theorem 4.1.** *For every continuous map $f : Y \to Y$ that fixes the branching point there exist $n_s \in \mathbb{N} \cup 2^\infty$, $n_g \in \mathbb{N}_g$ and $n_r \in \mathbb{N}_r$ such that the set of periods of $f$ is equal to $\mathrm{Sh}(n_s) \cup \mathrm{Green}(n_g) \cup \mathrm{Red}(n_r)$. Conversely, for every $n_s \in \mathbb{N} \cup 2^\infty$, $n_g \in \mathbb{N}_g$ and $n_r \in \mathbb{N}_r$ there exists a continuous map $f : Y \to Y$ that fixes the branching point, such that its set of periods is equal to $\mathrm{Sh}(n_s) \cup \mathrm{Green}(n_g) \cup \mathrm{Red}(n_r)$.*

In 1991 Baldwin [6] obtained a much more general result. He characterized all possible sets of periods of periodic orbits for maps of $n$-ods (for each $n$-od separately). However, he chose a different approach, departing in his theorem further from the Sharkovsky Theorem. Namely, instead of considering linear orderings, he considered partial orderings.

Let us describe those orderings. The $n$-th partial ordering is defined on

$$\mathbb{N}_n = \{k \in \mathbb{N} : k \geq n\} \cup \{n \cdot 2^\infty, 1\}$$

as follows. If $k, m > n$ and neither $k$ nor $m$ is divisible by $n$, then $k$ precedes $m$ if and only if $m = ik + jn$ for some $i, j \in \mathbb{N}$. This defines the initial part of the ordering. Then every $k$ not divisible by $n$ precedes every $m$ divisible by $n$ (including $n \cdot 2^\infty$). On the part divisible by $n$ the ordering is of course $n \cdot \mathrm{Sh}$. And then 1 stands at the very end.

It is easy to see that the second partial ordering is simply the Sharkovsky ordering. We illustrate the third partial ordering in Figure 1.

We will say that a set $S \subset \mathbb{N}_n \cap \mathbb{N}$ is a *tail* of the $n$-th partial ordering if whenever $k \in S$ and $k$ precedes $m$ then also $m$ belongs to $S$. While for the Sharkovsky ordering to define a tail we needed to specify only one element of $\mathbb{N} \cup 2^\infty$, here for a similar purpose we may need to specify up to $n-1$ elements of $\mathbb{N}_n$.

**Theorem 4.2.** *For every continuous map $f$ of an $n$-od to itself, the set of periods of $f$ is equal to a finite union of tails of $k$-th partial orderings with $2 \leq k \leq n$. Conversely, for every set $A$ which is a finite union of tails of $k$-th partial orderings with $2 \leq k \leq n$, there exists a continuous map of*

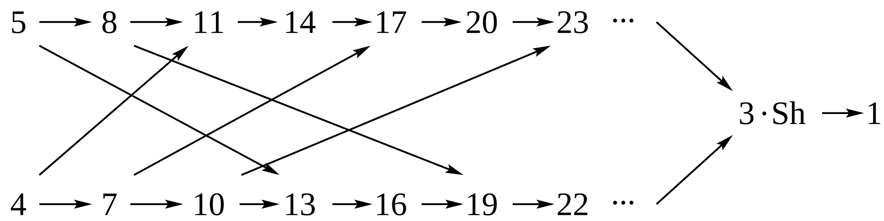


FIGURE 1. Third partial ordering.

*an $n$-od to itself, for which the set of periods is equal to $A$. Moreover, such a map can be chosen in such a way that the branching point is fixed.*

While the approaches with partial orderings and linear orderings are different, it turns out that one can replace Baldwin's partial orderings with linear ones. This was attempted by Alsedà and Moreno [5] in 1993. However, the linear orderings were not defined correctly there. This was corrected by Alsedà and Misiurewicz [4] in 2003.

**Theorem 4.3.** *For every $n$ there exist $n-1$ linear orderings, such that any set is a tail of the Baldwin's $n$-th partial ordering if and only if it is the union of tails of those linear orderings.*

While the approach with linear orderings is closer to the original Sharkovsky Theorem, those orderings are quite complicated and their connection with the dynamics of the map is weaker than for the Baldwin's partial orderings. Therefore, we will not include their definitions here. Nevertheless, we should mention that their form suggests some connection to the rotation intervals. This was observed not only in [4], but also by Blokh and Misiurewicz in [17] (for the green and red orderings).

The final result about sets of periods of tree maps was obtained in 2005 by Alsedà, Juher and Mumbrú [1]. For every tree, they gave the characterization of all possible sets of periods of maps of that tree to itself. Again we will not state the full result here, since it is quite complicated. The reader is encouraged to check the original paper [1]. A crude description of this set of periods is that up to a finite set, it is a union of a finite number of tails of Baldwin's partial orderings. The bounds on the cardinality of that finite set and the number of partial orderings are given in terms of the properties of the tree. Notice also, that the following version of this description is a particular case of a result obtained by Blokh in [11, 15].

**Theorem 4.4.** *The set of periods of a continuous map of a tree coincides, up to a finite set, with the union of finitely many sets $l_i\mathbb{Z} = \{l_i \cdot n : n \geq 1\}$ and finitely many sets $\{2^n k_j : n \geq 0\}$, where $\{l_i\}_{i=1}^m$ and $\{k_j\}_{j=1}^p$ are finite (perhaps empty) collections of positive integers.*

## 5. FORCING AMONG TYPES

In previous sections we discussed full or partial answers to the same question, namely: what are possible sets of periods of periodic orbits of maps from a certain class (these can be all continuous maps of an interval, all continuous maps of an $n$-od, all degree one circle maps, etc). Now we would like to consider two more ideas that yield other ways of understanding the Sharkovsky Theorem as well as its potential generalizations.

First of all, observe that the Sharkovsky Theorem, and the previous sections of the present article, focused almost exclusively (with the exception of rotation pairs for degree one circle maps) on periods of periodic orbits as their characteristic (the "type"). However, the period is a rather coarse characteristic of a periodic orbit. Therefore, it is natural to look for other ways of describing periodic orbits and then consider the problem of characterizing possible sets of types of periodic orbits for a given class of maps, in our case for continuous maps of the interval. This is another way of generalizing the Sharkovsky Theorem, where instead of considering other topological spaces (as was done in previous sections) one considers other, more precise and detailed, ways of describing periodic orbits.

A natural choice here is to consider cyclic permutations of sets $\{1, \dots, n\}$ induced by $n$-periodic orbits on the interval. Yet, the problem of describing possible sets of cyclic permutations induced by periodic orbits of an interval map apparently does not have a transparent and elegant solution. The remaining sections of this article are devoted to the two "middle-of-the-road" ways of describing periodic orbits on the interval, neither as coarse as periods, nor as detailed as cyclic permutations. These options are easier to deal with if we adopt a slightly different point of view, based on the concept of *forcing*[2].

It can be summarized as follows. Suppose that finite collections of periodic orbits of certain types *force* the existence of periodic orbits of some other types. Then the description of full sets of types can be reduced to the description of the strongest, in the above sense, collections of types of periodic orbits admitted by the map.

Let us now make this idea more precise and illustrate it using interval maps and degree one circle maps. The discussion below is intimately related (basically, equivalent) to that in previous sections, but offers a different, and sometimes more productive, point of view on the problem.

The Sharkovsky Theorem states that if $m \succ_{s} n$ then the fact that an interval map has an orbit of period $m$ *forces* the presence of an orbit of period $n$ among the periodic orbits of the map. In short, $m$ *forces* $n$. Thus, if we consider all periods of periodic orbits of an interval map and discover that $m$ is the strongest among them in the sense of the $\succ_{s}$ -ordering, then the set of periods will coincide with the set of all numbers $n$ such that $m \succ_{s} n$, and $m$ itself. The exceptional case here is the situation when an interval map has periods that are all powers of 2 and nothing else; in this case there is no strongest period of a periodic orbit of $f$ in the $\succ_{s}$ -sense, and to treat this case we introduce the special symbol $2^\infty$. To complete the proof of Theorem 2.1 one needs to show that the potential sets of periods of periodic orbits described above are in fact actual sets of periods for specific interval maps.

As was noticed in Section 3, for the continuous circle maps of degree one, rotation pairs can be interpreted in the same fashion. Namely, if the situation described at the end of Section 3 holds, we will say that $(p, q)$ and $(s, t)$ together *rotationally force* $(m, n)$. Then, by [26], one can say that if a degree one circle map has periodic orbits of rotation pairs $(p, q)$ and $(s, t)$, and $(p, q)$ together with $(s, t)$ rotationally force $(m, n)$, then $f$ has a periodic orbit of rotation pair $(m, n)$.

Observe that one or both endpoints of the rotation interval of $F$ can be irrational, and this creates the situation similar to the one that made it necessary to introduce the symbol $2^\infty$ in the Sharkovsky Theorem. However, here we can just use this irrational number (or two of them) — note that numbers $c$ and $d$ in Theorem 3.5 are allowed to be irrational.

From this point of view results from Sections 2 and 3 have the following structure. First one proves that the existence of periodic orbits of certain types (one periodic orbit of some period, or two periodic orbits of certain rotation pairs) implies the existence of periodic orbits of some specific types of the same map. Then one uses this to describe all potential sets of periods of periodic orbits (for interval maps) or of rotation pairs (for degree one maps). This completes the step when one describes *potential* sets of periods (or rotation pairs) of periodic orbits of a map. At the next step one proves that all described *potential* sets of periods (or rotation pairs) of periodic orbits are in fact *actual* sets of periods (or rotation pairs) of periodic orbits of some map from the class of maps that is being studied.

[2] Not to be confused with the notion of forcing in the set theory.

To sum it all up, the forcing relation among periods of periodic orbits of interval maps, or rotational forcing among rotation pairs of periodic orbits of degree one circle maps, allows one to describe potential sets of periods of periodic orbits of interval maps, or potential sets of rotation pairs of periodic orbits of degree one circle maps.

This nicely fits into the general scheme (as presented by Misiurewicz [27]), which we now briefly describe. Consider the set of all periodic orbits of maps from some class (e.g., interval maps, or degree one circle maps) with various equivalence relations, and, accordingly, various equivalence classes. For each relation, we say that a collection of equivalence classes $A_1, \dots, A_n$ *forces* an equivalence class $B$ if every map with representatives in each $A_i$ (that is, periodic orbits belonging to $A_i$ for each $i$) has a representative of $B$. As explained above, ideally this would allow for a transparent description of possible families of classes represented for our maps.

In the rest of the paper we will describe two characteristics of periodic orbits of interval maps, namely *over-rotation pairs* and *renormalization towers*; they admit forcing relations among them and allow for concise descriptions of possible sets of over-rotation pairs and of renormalization towers of periodic orbits of continuous interval maps.

Let us make an important point here (it applies, e.g., to the Sharkovsky Theorem and to the results of Section 3). In all these cases knowing only one or two numbers allows one to describe an infinite set (of all periods, or of all rotation pairs, of periodic orbits). This shows the power of the Sharkovsky Theorem and motivates the entire field of (one-dimensional) combinatorial dynamics started by it.

## 6. Over-rotation numbers

At the end of Section 3 the notion of the *rotation pair* of a periodic orbit of a degree one circle map was introduced and used to interpret the results of [26]. This interpretation served as an inspiration for the papers of Blokh [14] and Blokh and Misiurewicz [16] (very similar results were obtained independently almost at the same time by Bobok and Kuchta [21]). In our presentation we mainly follow [16].

Results from Section 3 can be interpreted through the ergodic theory. Consider the situation when we have a map $f : X \to X$ and an *observable* $\varphi : X \to \mathbb{R}$ (or $\varphi : X \to \mathbb{R}^d$; then numbers will become vectors). We look at the sequence of the ergodic averages of $\varphi$ at a point $x \in X$. If they converge, then we call the limit the *rotation number* of $x$ (strictly speaking one should talk about $\varphi$-rotation numbers but we assume that a function $\varphi$ is fixed)[3]. Alternatively, one could call $\varphi$-rotation numbers *functional rotation numbers*, however for brevity we will still call them simply *rotation numbers* (this is how they were called, for instance, in [30]).

If $f^n(x) = x$ (i.e., if $x$ is periodic and $n$ is a multiple of its minimal period), then the rotation number of $x$ exists and is equal to $\frac{1}{n}\sum_{k=0}^{n-1}\varphi(f^k(x))$. If $h : X \to Y$ is a bijection, $g = h \circ f \circ h^{-1}$, $\psi = \varphi \circ h^{-1}$, and $y = h(x)$, then the rotation number of $y$ for the system $(Y, g, \psi)$ is equal to the rotation number of $x$ for $(X, f, \varphi)$. In this sense, the rotation numbers are conjugacy invariants. For a periodic point $x$ of (minimal) period $n$ one can also consider its *($\varphi$-)rotation pair*

$$\left(\sum_{k=0}^{n-1}\varphi(f^k(x)), n\right)$$

and study how rotation pairs of periodic orbits are related to each other. This approach was adopted by Blokh in [12, 13], where properties of rotation numbers and pairs were studied, in particular, for interval maps (but also, for example, for trees).

[3]Various special cases were considered by various authors, but it seems that for the first time the general case was described by Ziemian in [34].

Let us now interpret the results of Sections 2 and 3 using this approach. For interval maps one can take the trivial function $\varphi_0(x) \equiv 0$ as an observable. Then the $\varphi_0$-rotation pair of a periodic orbit of period $n$ is $(0, n)$. Clearly, the Sharkovsky Theorem gives the description of all $\varphi_0$-rotation pairs of periodic orbits of continuous interval maps.

For a circle map $f$ of degree one we took $\varphi_1(x) = F(x') - x'$, where $F$ is a (fixed) lifting of $f$ to the real line, and the projection of $x' \in \mathbb{R}$ to the circle is $x$. Then results of Section 3 describe sets of $\varphi_1$-rotation pairs of those maps. However, a simple and natural observable $\varphi_1(x) = g(x) - x$ that was quite productive for degree one circle maps, in the case of interval maps gives rise to the $\varphi_1$-rotation pairs equal to the $\varphi_0$-rotation pairs and, hence, does not provide any new information. It turns out that an alternative observable is more useful, and we will now describe it.

Suppose that $f$ is a continuous interval map. Then we take the function $\Phi_f(x)$, which is equal to $1/2$ if $(f^2(x) - f(x))(f(x) - x) \leq 0$ and 0 otherwise as our observable. A visualization of $\Phi_f$ is as follows. Take a point $x$ such that (eventual) images of $x$ are not fixed. Take a vector from $x$ to $f(x)$ and consider its evolution as we apply $f$ to it. Each time when the vector changes its direction, think of its rotation in the positive direction by $\frac{1}{2}$ (we normalize the full angle to 1). The function $\Phi_f$ catches the rotation experienced by this vector along the orbit. If $x$ is periodic of period $q > 1$ then in the end the vector comes back, so that the overall cumulative rotation of the vector is by an integer $p$. Then the *over-rotation pair* of $x$ is $\mathrm{orp}(x) = (p, q)$ and the *over-rotation number* of $x$ is $\rho(x) = \frac{p}{q}$. Since at each step the vector rotates by at most $\frac{1}{2}$, then $\rho(x) = \frac{p}{q} \leq \frac{1}{2}$. An even more "physical" picture can be obtained by connecting $x$ and $f(x)$ with a rubber string, and then following the movement of the string assuming that it always rotates in the positive direction (i.e., counterclockwise). During this rotation one end of the string goes over the other one, and this motivates the term *over*-rotation.

In our considerations we exclude interval maps that have no periodic points that are not fixed, since their dynamics is trivial. By the Sharkovsky Theorem the maps we consider always have a periodic orbit of period 2 and, hence, of over-rotation number 1/2. Together with the observation that the rotation number of any periodic orbit is at most 1/2, it shows that if we look at the set of all over-rotation numbers of periodic orbits of $f$, the right endpoint of this set is always 1/2.

For continuous circle maps $f$ of degree one, the observable $\varphi$ that we were considering had the *completeness property*: if $(p, q)$ and $(k, m)$ rotationally force $(r, s)$ and $f$ has periodic orbits with rotation pairs $(p, q)$ and $(k, m)$ then it has also a periodic orbit with the rotation pair $(r, s)$. This was the basic property behind Theorem 3.5. The main result of [16] is that for a continuous interval map $f$ the observable $\Phi_f$ has the completeness property.

Denote by $\mathrm{ORP}(f)$ the set of all over-rotation pairs of periodic orbits of a continuous interval map $f$. Also, given a pair of integers $(p, q)$ with $p/q \leq 1/2$ set $\Psi(p, q) = (3, 6)$ if $p/q < 1/2$ and $\Psi(p, q) = (1, 2)$ if $p/q = 1/2$.

**Theorem 6.1.** *Suppose that $f$ is a continuous interval map and $x$ is a periodic point of $f$ with* $\mathrm{orp}(x) = (p, q)$. *Then for any pair of integers $(s, t)$ such that $(p, q)$ and $\Psi(p, q)$ rotationally force $(s, t)$ there exists a periodic point $y$ of $f$ with* $\mathrm{orp}(y) = (s, t)$.

Given a pair of positive integers $(p, q)$ with $p/q \leq 1/2$ let $\mathfrak{I}(p, q)$ be the set of all pairs of positive integers that are rotationally forced by $(p, q)$ and $\Psi(p, q)$. Moreover, for any irrational number $\tau$, $0 < \tau < 1/2$, or for $\tau = 0$ let $\mathfrak{I}(\tau)$ be the set of all pairs of positive integers $(m, n)$ with $\tau < m/n \leq 1/2$. Theorem 6.1 implies that potential sets of over-rotation pairs of periodic orbits of an interval map $f$ are all sets $\mathfrak{I}(\eta)$ where $\eta$ is either a pair of integers $p, q$ with $p/q \leq 1/2$, or an irrational number between 0 and 1/2, or 0. Then we can restate Theorem 6.1 as follows.

**Theorem 6.2.** *Let $f$ be a continuous interval map. Then there exists $\eta$, which is either a pair of positive integers $p, q$ with $p/q \leq 1/2$, or an irrational number between* 0 *and* $1/2$, *or* 0, *such that* $\mathrm{ORP}(f) = \mathfrak{I}(\eta)$.

In fact, by [16], these potential sets of over-rotation pairs are the actual sets of over-rotation pairs of interval maps.

**Theorem 6.3.** *Let $\eta$ be either a pair of positive integers $p, q$ with $p/q \le 1/2$, or an irrational number between $0$ and $1/2$, or $0$. Then there exists a continuous interval map $f$ such that $\mathfrak{I}(\eta) = \mathrm{ORP}(f)$.*

While in the above theorems we use the notion of forcing of one pair by two pairs, the fact that the right endpoint of the rotation interval is $1/2$ allows us to switch to forcing one pair by one pair. Namely, considering only pairs of positive integers $(i, j)$ with $i/j \le 1/2$, we can write $(p, q) \succ_{\mathcal{T}} (s, t)$ if either $p/q < s/t$ or $p/q = s/t = m/n$ with $m, n$ coprime, and $q/n \succ_{\mathcal{S}} t/n$. This is a linear order on the set of all possible over-rotation pairs for continuous interval maps. Then subsets that actually occur are exactly the tails in this order (the sets $A$ such that if $(p, q) \in A$ and $(p, q) \succ_{\mathcal{T}} (s, t)$, then $(s, t) \in A$). Thus, the over-rotation pair $(p, q)$ *forces* the over-rotation pair $(s, t)$.

Finally, observe that just like in the circle case, in the above case there exists a simpler version of the results dealing with over-rotation numbers, rather than over-rotation pairs. Namely, if a continuous interval map $f$ has a periodic orbit of over-rotation number $p/q$ then for every rational number $m/n$ located between $p/q$ and $1/2$ there exists a periodic orbit of over-rotation number $m/n$. One can say that the rotation number $p/q$ *forces* every rotation number $m/n$ with $p/q < m/n \le 1/2$. In particular, this shows that the closure of the set of all over-rotation numbers of periodic orbits of a continuous interval map is a closed interval with $1/2$ as the right endpoint (this interval is called the *over-rotation* interval of $f$). This is analogous to the results concerning the degree one circle maps and their rotation intervals (see Section 3).

## 7. Renormalization towers

In this section we suggest another way of describing periodic orbits of continuous interval maps. Namely, given a periodic orbit, we associate with it its *renormalization tower*. It reflects the structure of a periodic orbit more faithfully than the period and, arguably, than the over-rotation pair. Still, it allows for a transparent description of the associated forcing relation, and, therefore, of potential sets of renormalization towers for continuous interval maps. Observe that even though the concepts that we introduce will be defined only for cyclic permutations, they can be similarly defined for the corresponding periodic orbits; the same applies to the corresponding notation.

**Definition 7.1.** A cyclic permutation $\pi : \{1, \dots, N\} \to \{1, \dots, N\}$ has a *block structure* if there is a partition of $\{1, \dots, N\}$ into $k \notin \{1, N\}$ segments of consecutive integers (*blocks*) of the same length, permuted by $\pi$. That is, $\pi$ maps a block to a block, but the order within blocks does not have to be preserved. We will call $k$ the *period* of this block structure.

The concept of a block structure makes sense thanks to the fact that there is a natural order among numbers $\{1, \dots, N\}$. It is easy to see that if two block structures of the same permutation have periods $p < q$ then $q$ is a multiple of $p$. Thus, a permutation with a block structure can be studied in two steps: study the factor-permutation obtained if each block is collapsed to a point, while the order among blocks is kept, and then study the restriction of the appropriate power of the permutation to blocks. Evidently, this step-by-step approach is easiest to implement if one uses the most basic moves. Therefore, it is natural to consider all possible block structures and navigate among them moving toward larger and larger periods of block structures while making the smallest possible steps.

**Definition 7.2.** Let $p_1 < \dots < p_s$ be periods of *all possible* block structures on $\pi$. Call the finite string $(p_1/1, p_2/p_1, \dots, p_s/p_{s-1}, N/p_s)$ the *renormalization tower of* $\pi$ and denote it by $\mathrm{RT}(\pi)$.

Call the cardinality $s+1$ of the renormalization tower of $\pi$ the *height* of $\pi$. For consistency, if $\pi$ has no block structure, we define its renormalization tower as $(N)$. For simplicity we will pretend that the singleton of a fixed point is not a periodic orbit.

The concept of the renormalization tower extends that of the period of a cyclic permutation and reflects its structure in a more detailed way than the period itself. Even though in general the renormalization tower of a permutation is a string of more than one numbers, its height does not necessarily increase to infinity with the period of a permutation. In fact, the height of the tower of a permutation of *any* period, however big, but with no block structure, is 1. On the other hand, renormalization towers do not reflect the actual combinatorics of a cyclic permutation, only the periods of its block structures. In this sense the renormalization towers are *numerical* rather than *combinatorial*. One can say that the renormalization tower as a characteristic of a permutation that takes a place somewhere between the period of the permutation and the permutation itself.

**Definition 7.3.** Let $\mathcal{M}$ and $\mathcal{N}$ be two finite strings of integers greater than 1. If every continuous interval map that has a periodic orbit with renormalization tower $\mathcal{M}$ has a periodic orbit with renormalization tower $\mathcal{N}$, then we say that the renormalization tower $\mathcal{M}$ *forces* the renormalization tower $\mathcal{N}$.

It turns out that the order corresponding to the forcing relation among renormalization towers is linear and can be fully and transparently described (a priori it is not at all clear if this should be the case). This was done by Blokh and Misiurewicz in [19]. Namely, in their previous paper [18] the following order among natural numbers was defined (the order in [18] does not include numbers 2 and 1, but it is easy to see that the results of [18] hold with the order described below which includes 2 and 1):

$$4 \gg 6 \gg 3 \gg \cdots \gg 4n \gg 4n+2 \gg 2n+1 \gg \cdots \gg 2 \gg 1 \tag{1}$$

We will call this order the $\gg$-*order*[4] and understand it in the strict sense (i.e., $k \gg k$ is not true for integers $k$).

The $\gg$-order covers all positive integers: all positive integers except for 1 and 2 are ordered in a rather transparent fashion by placing triples $4n \gg 4n+2 \gg 2n+1$ in the increasing order, and at the end putting the missing so far numbers 2 and 1. Notice the similarity between the $\gg$-order and the green and red orders defined in Section 4.

**Definition 7.4.** A set $A$ of numbers is said to be a $\gg$-*tail* if for any number $m \in A$ and any number $n$ with $m \gg n$ the set $A$ must contain $n$.

Clearly, the structure of a $\gg$-tail can be described explicitly. Namely, given a $\gg$-tail $A$, choose the $\gg$-greatest number $n$ of all numbers in $A$. It is easy to see that the $\gg$-greatest element of $A$ always exists. Hence $A$ is the set $N(n)$ of all numbers $k$ such that $k = n$ or $n \gg k$.

**Definition 7.5.** A finite string of integers larger than 1 is said to be a *tower*.

Evidently, each renormalization tower of a permutation is a tower. It is not hard to see that, conversely, for every tower $\mathcal{M}$ there exists a permutation $\pi$ such that $\mathrm{RT}(\pi) = \mathcal{M}$.

Now we want to introduce the order among towers that will play a crucial role in describing the forcing relation among towers. In fact, this order is quite simply the *lexicographic extension* of $\gg$ onto the family of towers. Let

$$\mathcal{N} = (n_1, n_2 \ldots, n_k), \quad \mathcal{M} = (m_1, m_2 \ldots, m_l)$$

be two towers. Append each of them by infinite strings of 1s and denote these *canonical extensions* by $\mathcal{N}'$ and $\mathcal{M}'$. Observe that in any canonical extension constructed above the number 1 does not

[4]The readers can use any name for the symbol $\gg$; we say "gg".

show in the beginning for some time, but once it shows, the rest of the numbers in the tower must be only 1s. Let $s$ be the first place at which $\mathcal{N}'$ and $\mathcal{M}'$ are different. Then we write $\mathcal{N} \gg \mathcal{M}$ if $n_s \gg m_s$. We keep the same notation $\gg$ for the lexicographic extension of $\gg$, as no confusion arises. Evidently, the family of all towers with the $\gg$-order is linearly ordered, as any two distinct towers are comparable in the sense of the $\gg$-order.

Before we state the main theorem dealing with renormalization towers, we need to introduce additional notions.

**Definition 7.6.** An *infinite tower* is an infinite string $\mathcal{N}$ of positive integers $(m_0, m_1, \dots)$ such that either (a) $m_i > 1$ for every $i$, or (b) there exists minimal $j$ such that $m_i > 1$ for each $i < j$ and $m_i = 1$ for every $i \geq j$. In case (b) we identify the infinite tower $\mathcal{M} = (m_0, \dots)$ with finite tower of all initial numbers $(m_0, m_1, \dots, m_{j-1})$ of $\mathcal{M}$ not equal to 1. The relation $\gg$ extends onto all infinite towers lexicographically.

In particular, when we write $\mathcal{M} \gg \mathcal{N}$ for an infinite tower $\mathcal{M}$ and a finite tower $\mathcal{N}$, we mean that $\mathcal{M} \gg \mathcal{N}'$ in the sense of Definition 7.6 (here the infinite tower $\mathcal{N}'$ is the canonical extension of the tower $\mathcal{N}$ defined above).

**Definition 7.7.** Given an infinite tower $\mathcal{M}$, the set $\mathrm{Tow}(\mathcal{M})$ is defined as the set of all finite towers $\mathcal{N}$ such that $\mathcal{M} \gg \mathcal{N}$, together with the finite tower $\widetilde{\mathcal{M}}$ in case $\mathcal{M}$ is the canonical extension of $\widetilde{\mathcal{M}}$.

Observe that infinite towers of type (a) from Definition 7.6, viewed as elements of the ordered space of all towers with the $\gg$-order, are similar to $2^\infty$ in the Sharkovsky ordering. Moreover, we introduce infinite towers for the same reason the symbol $2^\infty$ was introduced in the Sharkovsky ordering, namely to be able to explicitly describe $\gg$*-tails*, i.e. naturally defined tails of $\gg$-ordering.

**Definition 7.8.** A set $A$ of towers is said to be a $\gg$*-tail* if for any tower $\mathcal{M} \in A$ and any tower $\mathcal{N}$ with $\mathcal{M} \gg \mathcal{N}$ the set $A$ must contain $\mathcal{N}$.

The next lemma relates infinite towers and $\gg$-tails of towers.

**Lemma 7.9.** *Every $\gg$-tail of towers is of the form* $\mathrm{Tow}(\mathcal{N})$ *for some infinite tower $\mathcal{N}$. Conversely, for every infinite tower $\mathcal{N}$ the set* $\mathrm{Tow}(\mathcal{N})$ *is a $\gg$-tail of towers.*

Now we can state the main result of [19].

**Theorem 7.10.** *If $\mathcal{N} \gg \mathcal{M}$ and a continuous interval map $f$ has a periodic orbit with tower $\mathcal{N}$ then it has a periodic orbit with tower $\mathcal{M}$, and so there exists an infinite tower $\mathcal{K}$ such that the set of all (finite) towers of periodic orbits of $f$ is* $\mathrm{Tow}(\mathcal{K})$. *Conversely, if $\mathcal{K}$ is an infinite tower, then there exists a continuous interval map $g$ such that the set of all (finite) towers of periodic orbits of $g$ coincides with* $\mathrm{Tow}(\mathcal{K})$.

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