POLYGONAL BILLIARDS: SOME OPEN PROBLEMS

P. HUBERT AND S. TROUBETZKOY

A billiard ball, i.e. a point mass, moves inside a polygon Q with unit speed along a straight line until it reaches the boundary ∂Q of the polygon, then instantaneously changes direction according to the mirror law: "the angle of incidence is equal to the angle of reflection," and continues along the new line. The orbit of a point which reaches a vertex is not defined further.

Question 1. Is there at least one periodic billiard orbit in every polygon?

This question is open even for non acute triangles. It is known that if Q is a rational polygon, i.e. all the angles between sides are rational multiples of π, then there are periodic orbits [M1, GSV, B] and in fact they are dense in the phase space of Q [BGKT]. There are constructive [GSV] and non constructive [GuTr, Tr2] methods of producing periodic orbits in certain irrational billiards.

Question 2. For which polygons is the billiard flow topologically transitive? For which is it ergodic with respect to the invariant phase volume? Is it ever mixing? weak mixing?

It is known that the typical polygon (in the topological sense) is both topologically transitive and ergodic [KZ, KMS]. There are explicit examples of ergodic polygons [Vo]. It is not known if polygonal billiards can be weak mixing or mixing. In a conversation J. Moser attributed the question of ergodicity to E. Artin.

We call an (oriented) orbit segment which starts and ends at a vertex of Q a generalized diagonal. Let $N(t)$ be the the number of generalized diagonal whose geometric length is less than or equal to t.

Question 3. What is the growth rate of $N(t)$?

For rational polygons $N(t)$ has quadratic upper and lower bounds [M2, M3]. The limit $\lim_{t\to\infty} N(t)/t^2$ exists for a special class of polygons called Veech polygons, which includes for example all regular polygons [V1, V2].

Label the sides of Q by symbols from a finite alphabet $\mathcal{A}$ whose cardinality is equal to the number of sides of Q and code the orbit by the sequence of sides it hits. Consider $L(n)$ be the set of all words of length n which arise via this coding.

Question 4. What is the growth rate of $\#L(n)$?

The only general results known about the complexity function is that it grows slower than any exponential [K], at least quadratically [Tr1] and has cubic upper and lower bounds for convex rational polygons [CHTr]. For billiards in a square the complexity function has been explicitly calculated, albeit for a slightly different coding (the alphabet consists of two symbols, one for vertical sides and one for horizontal sides) [Mi, BP].

If Q is a rational polygon then the phase space is foliated by invariant surfaces. One can use the theory of quadratic differentials and Teichmüller theory as tools to prove results about the restriction of the billiard flow to an invariant surface.

Several good reviews exist [Gu1, Gu2, MT, T]. They include more open problems.

References

[BP] J. Berstel and M. Pocchiola *A geometric proof of the enumeration formula for Sturmian words* Jour. Alg. Comp. 3 (1993) 349–355.

[B] M. Boshernitzan *Billiards and rational periodic directions in polygons* Amer. Math. Monthly June-July (1992) 522-529.

[BGKT] M. Boshernitzan, G. Galperin, T. Krüger and S. Troubetzkoy *Periodic billiard orbits are dense in rational polygons* Trans. AMS 350 (1998) 3523–3535.

[CHTr] J. Cassaigne, P. Hubert and S. Troubetzkoy *Complexity and growth for polygonal billiards* preprint Inst. Math. Luminy (2000).

[GSV] G.A. Galperin, A.M. Stepin and Ya.B. Vorobets *Periodic billiard trajectories in polygons: generating mechanisms* Russian Math. Surv. 47:3 (1992) 5-80.

[Gu1] E. Gutkin *Billiards in polygons* Physica D 19 (1986) 311–333.

[Gu2] E. Gutkin *Billiard in polygons: survey of recent results* J. Stat. Phys. 174 (1995) 43–56.

[GuTr] E. Gutkin, and S. Troubetzkoy *Directional flows and strong recurrence for polygonal billiards* in Proceedings of the International Congress of Dynamical Systems, Montevideo, Uruguay, F. Ledrappier et. al. eds., Pitman Research Notes 362 (1996) 21–45.

[K] A. Katok *The growth rate for the number of singular and periodic orbits for a polygonal billiard* Comm. Math. Phys. 111 (1987) 151–160.

[KZ] A. Katok and A. Zemlyakov *Topological transitivity of billiards in polygons* Math. Notes 18 (1975) 760–764.

[KMS] S. Kerckhoff, H. Masur, and J. Smillie *Ergodicity of billiard flows and quadratic differentials* Annals Math. 124 (1986) 293-311.

[M1] H. Masur *Closed trajectories for quadratic differentials with an application to billiards* Duke Math. J. 53 (1986) 307-313.

[M2] H. Masur *Lower bounds for the number of saddle connections and closed trajectories of a quadratic differential* In Holomorphic functions and moduli, vol. 1, D. Drasin ed., Springer–Verlag 1988.

[M3] H. Masur *The growth rate of a quadratic differential* Ergod. Th. Dyn. Sys. 10 (1990) 151-176.

[MT] H. Masur and S. Tabachnikov *Rational billiards and flat structures* preprint Max Planck Institut (1999).

[Mi] F. Mignosi *On the number of factors of sturmian words* Theor. Comp. Sci. 82 (1991) 71–84.

[T] S. Tabachnikov *Billiards* "Panoramas et Synthèses", Soc. Math. France (1995).

[Tr1] S. Troubetzkoy *Complexity lower bounds for polygonal billiards* Chaos 8 (1998) 242–244.

[Tr2] S. Troubetzkoy *Recurrence and periodic billiard orbits in polygon* preprint (1998).

[V1] W. Veech *Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards* Invent. Math. **97** (1989) 553–583.

[V2] W. Veech *The billiard in a regular polygon* Geom. Func. Anal. **2** (1992) 341–379.

[Vo] Ya. Vorobets *Ergodicity of billiards in polygons* Mat. Sb. 188 (1997) 65–112.

Institut de mathématiques de Luminy, CNRS Luminy, Case 907, F-13288 Marseille Cedex 9, France

E-mail address: hubert 'at' iml.univ-mrs.fr

Centre physique théorique et Institut de mathématiques de Luminy, CNRS Luminy, Case 907, F-13288 Marseille Cedex 9, France

E-mail address: troubetz 'at' iml.univ-mrs.fr

URL: http://iml.univ-mrs.fr/~troubetz/