OPEN PROBLEMS IN DYNAMICS AND RELATED FIELDS

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Contents

This paper contains a collection of open problems from the the workshop "Emerging applications of measure rigidity" held at the American Institute of Mathematics in June, 2004. During the workshop researchers in dynamical systems, number theory, arithmetic geometry, and mathematical physics had a unique opportunity to discuss new links between these already richly connected subjects. We hope that this collection will give a snapshot of the active and rapidly developing field of modern dynamics and its applications.

The presented open problems were collected from the participants of the workshop. I also tried to include current status of the problems as well as related references, and I am greatly in debt to the participants for providing this information. I apologize for all omissions and inaccuracies which are solely due to my lack of knowledge.

I would like to thank to the participants of the workshop for contributing the problems and for numerous comments/suggestions, and the American Institute of Mathematics and the National Science Foundation for the financial support of the workshop. The author was partially supported by NSF DMS-0400631 and by AIM.

1. Local rigidity

We refer to [79] for a recent comprehensive survey on local rigidity.

1.1. An action α of a finitely generated discrete group A on a manifold M is called $C^{k,r,l}$-*locally rigid* if any C^k-perturbation $\tilde{\alpha}$ which is sufficiently small in C^r-topology is C^l-conjugate to α, i.e., there exists a C^l-close to identity diffeomorphism $\mathcal{H}$ of M which conjugates $\tilde{\alpha}$ to α:

$$\mathcal{H} \circ \alpha(g) = \tilde{\alpha}(g) \circ \mathcal{H} \quad \text{for all } g \in A.$$

The $C^{\infty,1,\infty}$-local rigidity is often referred to as C^∞-*local rigidity.* The case of $C^{1,1,0}$ is known as C^1-*structural stability.* In the definitions for continuous groups such as $\mathbb{R}^k$ one has to allow a "time change", i.e., an automorphism ρ of the acting group close to *id* such that

$$\mathcal{H} \circ \alpha(\rho(g)) = \tilde{\alpha}(g) \circ \mathcal{H} \quad \text{for all } g \in A.$$

It is well-known that an Anosov diffeomorphism is C^1-*structurally stable*, but the conjugation map is not differentiable in general. On the other hand, *Anosov*[1] actions by higher rank abelian groups exhibit much more rigid behavior (see [123] for the first result of this type). It was shown in [128] that most of known *algebraic*[2] Anosov $\mathbb{Z}^k$- and $\mathbb{R}^k$- actions, $k \geq 2$, are locally C^∞-rigid provided that they do not reduce to rank one actions via some elementary constructions. We call such actions *"irreducible"*. See, for example, [128] for some natural conditions that guarantee that an action is "irreducible".

Recently, local rigidity was proved in [44, 45] for partially hyperbolic higher rank abelian actions by toral automorphisms using the KAM method. The method of [44, 45] allows to construct C^∞-conjugacy only for C^l-perturbations of the original action for some large l. Another interesting example of a partially hyperbolic action is given in the following conjecture, which was communicated by R. Spatzier:

Conjecture 1. *Let G be a connected semisimple Lie group, Γ an irreducible lattice in G, and A a closed subgroup of a split Cartan subgroup of G with* $\dim A > 1$. *Then any C^1-small perturbation of the action of A on G/Γ is C^∞-conjugate to the action of A defined by a continuous homomorphism from A to the centralizer of A in G.*

We also state one of important partial cases of Conjecture 1:

Conjecture 2. *If in Conjecture 1 the group A is not contained in a wall of a Weyl chamber of the split Cartan subgroup D, then any C^1-small perturbation*

[1] An action of a group G is called *Anosov* if there is an element $g \in G$ that acts normally hyperbolically with respect to the orbit foliation of G.

[2] That is, the actions on infrahomogeneous spaces of Lie groups induced by either automorphisms or translations.

of the action of A on G/Γ is C^∞-conjugate to the action of A defined by a continuous homomorphism from A to D.

Conjecture 2 was proved in [128] when A is the full split Cartan subgroup. Recently, D. Damjanović and A. Katok [47, 48] developed a totally different from [44, 45] approach to Conjectures 1 and 2, which is based on noncommutativity of various stable/unstable foliations (see [46]). This gives a proof of Conjectures 1 and 2 for C^2-small perturbations in the case when $G = \mathrm{SL}(n, \mathbb{R})$ and A is any subgroup which contains a lattice in a two-plane in general position in the maximal split Cartan group. Extension to other classical simple Lie groups is in progress [49].

1.2. Local rigidity for semisimple Lie groups of higher rank and their lattices (motivated by the program of R. Zimmer [267]) has been an active area of research too. First results in this direction were obtained for Anosov actions (see [111, 124, 125, 128]) and for actions with weaker hyperbolicity assumptions (see [174] and references therein). Recently, local rigidity results were established without any hyperbolicity assumptions (see [81, 82, 78, 80]).

2. Global rigidity

2.1. The only known examples of Anosov diffeomorphisms are automorphisms of infranilmanifolds. Moreover, every Anosov diffeomorphism on an infranilmanifold is topologically conjugate to a hyperbolic automorphism (see [84, 170]). This motivates the following "€100,000" folklore conjecture, which is already implicit in [84] (see also [173]):

Conjecture 3. *Every Anosov diffeomorphism is topologically conjugate to a hyperbolic automorphism on an infranilmanifold.*

Conjugacy with a toral automorphism has been proven for codimension-one (i.e., the stable or unstable foliation is of codimension one) Anosov diffeomorphisms [84, 197] and for Anosov diffeomorphisms on infranilmanifolds [170]. See also [83, 14, 93, 118] for other partial results on Conjecture 3. In general, it is not even known whether an Anosov diffeomorphism is topologically transitive. We also mention that there are infinitely many moduli of smooth conjugacy, and there are examples of Anosov diffeomorphisms on manifolds that are homeomorphic but not diffeomorphic to a torus (see [69]).

In contrast, there exist Anosov flows that are not topologically transitive (see [85]), and it is not clear how to state a conjecture regarding classification of general Anosov flows. Such a conjecture is available for codimension-one Anosov flows (see [251, 94]).

2.2. One can also state an analog of Conjecture 3 for Anosov $\mathbb{Z}^k$- and $\mathbb{R}^k$-actions, $k \geq 2$. In this case, it is usually possible to show that if a continuous conjugation map exists, it is also smooth (see, for example, [111, 123, 128]). R. Spatzier communicated the following conjecture.

Conjecture 4. *Every "irreducible" Anosov $\mathbb{Z}^k$- and $\mathbb{R}^k$- action, $k \geq 2$, is C^∞-conjugate to an algebraic action.*

Some partial results on Conjecture 4 were obtained in [123, 187] and, recently, by F. Rodriguez-Hertz [214] and B. Kalinin, R. Spatzier [120]. One may also hope to classify "irreducible" partially hyperbolic $\mathbb{Z}^k$- and $\mathbb{R}^k$- action, $k \geq 2$, and higher rank actions of commuting expanding maps.

2.3. There are also analogous conjectures for actions of connected semisimple Lie groups of higher rank and their lattices satisfying some hyperbolicity assumptions (see [111, 174]).

Conjecture 5. *Every action of a connected semisimple Lie group of higher rank (i.e., all simple factors have real rank at least 2) or its lattice, which that has an element which acts non-trivially uniformly partially hyperbolicly, is C^∞-conjugate to an algebraic action.*

Partial results on Conjecture 5 were obtained in [111, 124, 125, 209, 96, 174]. Note that without partial hyperbolicity assumption, one may only hope to classify the actions when restricted to an open dense subset. See [124, 77] for examples of nonstandard lattice actions. In general, there are conjectures originated from [267] on classification of actions satisfying some transitivity assumptions or preserving a rigid geometric structure in the sense of Gromov (see [152, 173] for up-to-date statements).

2.4. Ratner's measure rigidity theorem has applications to the study of general properties of continuous volume preserving actions of higher rank semisimple Lie groups and their lattices on compact manifolds. In particular, Ratner's theorem plays a key role in the construction of arithmetic quotients of such actions (see [164, 165] for connected groups, [74, 75] for lattices, and [76] for a survey). This raises the following question:

Question 6 (D. Fisher)**.** *Do the new results on measure rigidity for actions of higher rank abelian groups give rise to obstructions to smooth or continuous irreducible actions of a higher rank abelian group on a compact manifold?*

Some basic obstructions for smooth volume preserving actions of higher rank abelian groups are already known, see particularly works of H. Hu [110] and A. Katok, J.-P. Thouvenot [130]. Unfortunately, both of those restrictions apply to smooth actions with respect to any Borel invariant measure. The question is whether one can use results on measure rigidity to obtain more information and to restrict possible properties of smooth actions of abelian groups.

The results on arithmetic quotients for actions of semisimple groups and their lattices have no straightforward analogues here, since the proofs of those results use not only Ratner's theorem but applications of the cocycle superrigidity theorems to cocycles which are necessarily only measurable. Though some cocycle superrigidity theorems are known for particular classes of actions of higher rank abelian groups, none of these can possibly apply to measurable cocycles because of the Dye theorem [54] and its generalizations [203, 42].

3. Measure rigidity

3.1. One of the main topics of the workshop was the measure rigidity of higher rank abelian actions, which started with the following conjecture of H. Furstenberg:

Conjecture 7 (H. Furstenberg [87]). *Let m and n be multiplicatively independent positive integers. Then the only probability Borel measures on $\mathbb{R}/\mathbb{Z}$ which are ergodic under the action of the semigroup generated by multiplications by m and n are the Lebesgue measure and measures with finite support.*

H. Furstenberg proved a topological analog of this result in [87]. There has been a series of papers [169, 218, 116, 73, 109, 204] proving Conjecture 7 under some positive entropy assumptions.

A. Katok and R. Spatizer [127, 129] were first to obtain analogous results for more general homogeneous spaces, and this led to a generalization of Conjecture 7:

Conjecture 8 (A. Katok, R. Spatzier [127]). *Let M be a manifold which is a biquotient of a connected Lie group G. Given an algebraic Anosov action of $\mathbb{R}^k$ or $\mathbb{Z}^k$, $k \geq 2$, on M and an ergodic probability Borel measure μ on M, one of the following holds:*

(i) *The measure μ is a finite sum of measures constructed from Haar measures supported on closed subgroups of G.*
(ii) *The support of μ fibers over a manifold N in an equivariant way such that the action on N reduces to a rank one action.*

We refer to a survey [160] for a more detailed discussion and to [173] where stronger conjectures and questions are stated. Examples for case (ii) in Conjecture 8 were given by M. Rees (see also [58]).

Positive results on Conjecture 8 and its analogs have led to important advances in number theory — Littlewood conjecture (see [60] and Conjecture 38) and in mathematical physics — quantum unique ergodicity (see [159] and Conjecture 50).

The method of [127, 129] was clarified and extended in [117, 119]. Subsequently two new approaches, which allow to overcome limitations of this method, were developed: the high entropy case [58, 59], which uses noncommutativity of stable/unstable foliations and the product structure of conditional measures, and the low entropy case [159], which is based on recurrence of foliations. These two approaches were combined in [61] and [60] to prove many cases of Conjecture 8 assuming only that at least one element has positive entropy. Note that all of these techniques are based on the study of conditional measures, and they provide essentially no information without some positive entropy assumption.

It was observed by E. Lindenstrauss that a positive entropy assumption is not needed in the adelic setting. Let

$$\mathbb{A} \subset \prod_{v\text{-place}} \mathbb{Q}_v$$

denote the ring of adeles and D the diagonal subgroup in $\mathrm{SL}(2)$. E. Lindenstrauss showed that the only probability $D(\mathbb{A})$-invariant measure on $\mathrm{SL}(2,\mathbb{A})/\mathrm{SL}(2,\mathbb{Q})$ is the Haar measure (see [161]).

Question 9 (E. Lindenstrauss)**.** *Let $\mathbb{A}'$ be defined as the ring of adeles, but the product is taken over a subset of places of $\mathbb{Q}$ and Γ an "irreducible" lattice in $\mathrm{SL}(2,\mathbb{A}')$. What are the finite ergodic $D(\mathbb{A}')$-invariant measures on $\mathrm{SL}(2,\mathbb{A}')/\Gamma$?*

3.2. Let G be a Lie group, Γ a discrete subgroup, and H a subgroup of G generated by one-parameter unipotent subgroups. One of the prototypical examples of measure rigidity is the classification of finite ergodic H-invariant measures on G/Γ (see [210], and [190] for an accessible exposition).

Problem 10 (L. Silberman)**.** *Extend the results on measure rigidity of unipotent flows to adelic setting.*

It seems natural to expect (and is known in some cases) that the set of finite ergodic invariant measures for other dynamical systems with parabolic behavior has a manageable structure, which is possible to described in algebraic terms.

Suppose that H is a connected semisimple Lie subgroup of a Lie group G, and let P be a parabolic subgroup of H. One of manifestations of the measure rigidity of unipotent flows is the fact that every finite P-invariant measure on G/Γ is H-invariant (see [192]).

Question 11 (E. Lindenstrauss)**.** *Suppose that H acts on a space X preserving some geometric structure. Under what conditions on X, every finite P-invariant measure is H-invariant? In other words, which H-actions are stiff (see* [88]*)?*

For example, one may consider an $\mathrm{SL}(2,\mathbb{R})$-action on the moduli space of quadratic differentials over complex structures on a compact surface. There are a lot of similarities between this action and $\mathrm{SL}(2,\mathbb{R})$-actions on homogeneous spaces (see [63, 255, 186]). Some partial results on topological and measure rigidity for the latter actions were obtained in [188] and [66].

3.3. Let Γ be a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$. If Γ has infinite covolume, then the only finite ergodic invariant measures for the horocyclic flow $u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ on $\mathrm{SL}(2,\mathbb{R})/\Gamma$ are the the ones supported on periodic orbits (see [212]). It turns out that there is a large family of infinite ergodic invariant Radon measures. Such measures can be constructed from the minimal positive Γ-invariant eigenfunction of the Laplacian (see, for example, [6]). Recently, F. Ledrappier and O. Sarig [155, 156] proved that if Γ is a normal subgroup of a lattice in $\mathrm{SL}(2,\mathbb{R})$, then every u_t-ergodic Radon measure on $\mathrm{SL}(2,\mathbb{R})/\Gamma$ is of this form up to a constant (see also [7, 219] for previous classification results).

Question 12 (F. Ledrappier, O. Sarig)**.** *Let G be a noncompact semisimple Lie group of rank one, Γ a discrete subgroup of G, and U a horospherical subgroup of G. What are the U-ergodic Radon measures on G/Γ? In particular, are they either carried by closed U-orbits or given by the harmonic function construction?*

A u_t-invariant measure μ is called *squashable* if the centralizer of u_t contains an invertible nonsingular transformation that does not preserve μ.

Question 13 (F. Ledrappier, O. Sarig)**.** *Let Γ be a normal conilpotent subgroup of a uniform lattice in* $\mathrm{SL}(2,\mathbb{R})$. *Is the Haar measure nonsquashable?*

For coabelian subgroups, the answer to this question is positive [154]. In fact, it was shown that the Haar measure has a generalized law of large numbers. For general discrete subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$, it is not known whether the Haar measure on $\mathrm{SL}(2,\mathbb{R})/\Gamma$ is nonsquashable, or whether there exist other nonsquashable u_t-ergodic Radon measures.

3.4. One may also expect measure rigidity for algebraic actions of "large" groups.

Conjecture 14 (A. Furman)**.** *Consider one of the following actions of a group Γ:*

(1) Γ *is a "large" (e.g., Zariski dense) subgroup of the group of automorphism of a nilmanifold of finite volume.*
(2) Γ *is a "large" subgroup of a Lie group acting by translations on G/Λ where Λ is a lattice in G.*

Then the only ergodic Γ-invariant probability measures are the measures supported on finite Γ-orbits and the Haar measure.

There are results on the topological analog of this conjecture (see [17, 240, 195, 196, 98]). Also there are results about measurable centralizers, quotients, and joinings for actions on some infinite volume homogeneous spaces [86].

3.5. Let M be a compact Riemannian manifold, $\phi_t : M \to M$ is an Anosov flow that defines the decomposition $TM = E^0 \oplus E^s \oplus E^u$ of the tangent bundle. We suppose that there exists a continuous invariant splitting $E^s = E^s_+ + E^s_-$ such that for some $C > 0$ and $\mu_+ > \mu_- > \lambda$,

$$\|D\phi_t v\| \le Ce^{-\mu_+ t}\|v\|, \quad v \in E^s_+,\ t \ge 0;$$
$$\|D\phi_t v\| \ge C^{-1}e^{-\mu_- t}\|v\|, \quad v \in E^u_-,\ t \ge 0.$$

A basic example of such splitting is the geodesic flow of $\mathbb{CH}^2$. The distribution E^s_+ integrates to the *fast stable* foliation W^s_+.

The following is a nonlinear analog of the Ragunathan's question about classifications of measures invariant under unipotent flows:

Question 15 (F. Ledrappier)**.** *Describe the invariant ergodic measures for the fast stable foliation W^s_+.*

One may ask the same question for Anosov diffeomorphisms as well.

4. Equidistribution

4.1. Let G be a Lie group, Γ a lattice in G, and $U = \{u(t)\} \subset G$ a one-parameter Ad-unipotent subgroup. Suppose that for $x \in G/\Gamma$, Ux is dense in G/Γ.

Question 16 (G. Margulis [173])**.** *Prove equidistribution of the sequence $\{u(t_n)x\}$ in G/Γ, where t_n is one of the following:*

(1) *$t_n = [n^\alpha]$[3] for $\alpha > 1$,*
(2) *$t_n = [P(n)]$, where $P(x)$ is a polynomial,*
(3) *t_n is the n-th prime number.*

A. Venkatesh [250] gave a proof of (1) for horocyclic flow when α is close to 1.

It is known that Česaro averages along sequences as in Question 16 converge almost everywhere for functions in L^p, $p > 1$ (see [24, 25, 26, 27, 258]). Note that there is a subtle difference between sequences $t_n = [n^\alpha]$ and $t_n = n^\alpha$ for $\alpha \in \mathbb{Q} - \mathbb{Z}$. In fact, there is no general pointwise ergodic theorem possible for the latter sequence (see [19]).

[3] Here $[x]$ denotes the integer part of x.

4.2. Let V be a connected Ad-unipotent subgroup of the Lie group G such that Vx is dense in G/Γ for some $x \in G/\Gamma$.

Question 17 (G. Margulis [173]). *Show that for a "good" sequence of subsets $A_n \subset V$ and every $f \in C_c(G/\Gamma)$,*

$$\lim_{n\to\infty} \frac{1}{\mathrm{Vol}(A_n)} \int_{A_n} f(vx)\, dv = \int_{G/\Gamma} f\, d\mu$$

with effective error term, where dv is a Haar measure on V, and μ is the probability Haar measure on G/Γ.

Such equidistribution results were proved by several authors (see [211, 175, 231]), but the methods of the proofs are not effective. In the case when V is a horospherical subgroup of G (see Section 4.3 below), one can deduce an equidistribution result with explicit error term from decay of matrix coefficient on $L^2(G/\Gamma)$ (see [142]).

4.3. Let L be Lie group, G a closed subgroup of L, and Λ a lattice in L. For a semisimple element $a \in G$, the *expanding horospherical subgroup* U of G associated to a is defined by

$$U = \{g \in G :\ a^{-n} g a^n \to e \text{ as } n \to \infty\}.$$

Suppose that for $x_0 \in L/\Lambda$, the orbit Gx_0 is dense in L/Λ.

Let μ be a measure on Ux_0 which is the image of a probability measure on U, absolutely continuous with respect to the Haar measure on U, under the map $u \mapsto ux_0$, $u \in U$. Then it is known that $a^n\mu \to \lambda$ as $n \to \infty$ where λ is the probability Haar measure on L/Λ (see [232]).

One may consider the following refinement of the above result. Take any analytic curve $\gamma : [0,1] \to U$, and let ν be the image of the Lebesgue measure on $[0,1]$ under the map $t \mapsto \gamma(t)x_0$.

Question 18 (N. Shah). *Under what condition on γ, we have that $a^n\nu \to \lambda$ as $n \to \infty$?*

Recently Question 18 was solved by N. Shah for $L = \mathrm{SO}(m,1)$ and $G = \mathrm{SO}(n,1)$, $m > n$. He showed that $a^n\nu \to \lambda$ as $n \to \infty$ provided $\gamma([0,1])$ does not lie on an proper affine subspace or an $(n-2)$-dimensional sphere in U.

Conjecture 19 (N. Shah). *The same result holds for all Lie groups L containing $G = \mathrm{SO}(n,1)$.*

The above kind of questions are related to the following more general problem. Consider a representation of a semisimple Lie group G on real vector space V equipped with a norm $\|\cdot\|$. Take a point $p \in V$, and consider the set

$$R_T = \{g \in G :\ \|gp\| < T\}$$

for $T > 0$. Suppose that the stabilizer of p is finite, so R_T is compact for each T. Let Γ be a lattice in G, and let μ_T denote the image of the normalized Haar measure on R_T projected to G/Γ.

Question 20 (N. Shah). *What is the limiting distribution of the measure μ_T as $T \to \infty$?*

In some examples such results are known (see [68, 97]), but more general answers can be very important for understanding distribution of Γ-orbits on homogeneous spaces G/H where either $\Gamma \cap H$ is a lattice in H or ΓH is dense in G.

4.4. For irrational α, the sequence $\{\alpha n^2 \pmod 1 : n \geq 1\}$ is equidistributed in $[0,1]$. In fact, one expects that if α is badly approximable by rationals, then statistical properties of this sequence are the same as the sequence of independent uniformly distributed random variables. For $[a,b] \subset [0,1]$, we define pair correlation:

$$R_2([a,b],N,\alpha) = \frac{1}{N}\#\left\{1 \leq i \neq j \leq N : \alpha i^2 - \alpha j^2 \in \frac{1}{N}[a,b] \pmod 1\right\}.$$

Conjecture 21 (Z. Rudnick, P. Sarnak). *If $\alpha \in \mathbb{R}$ is badly approximable by rationals (see* [217] *for exact conditions), then*

$$R_2([a,b],N,\alpha) \to b-a \quad \textit{as} \quad N \to \infty. \tag{1}$$

Although it was shown that (1) holds on the set of α of full measure (see [216]) and on a residual set of α in the sense of Baire category (see [217]), one does not know any explicit α for which it is true. It is expected that (1) holds for algebraic integers, and it is not hard to show that there are well approximable irrational α for which (1) fails.

It was discovered in [181] that Conjecture 21 is related to an equidistribution problem on a hyperbolic surface $X = \Gamma\backslash\mathbb{H}^2$, Γ a lattice. We assume that

$$\Gamma \cap \{z \mapsto z + a : a \in \mathbb{R}\} = \{z \mapsto z + a : a \in \mathbb{Z}\}.$$

Then the curve $\{x + iy : x \in [0,1]\}$ corresponds to a closed horocycle $\{u_y(t) : 0 \leq t \leq 1\}$ of length y^{-1} in in the unit tangent bundle $\mathrm{T}^1(X)$, and it is well-known that it becomes equidistributed in $\mathrm{T}^1(X)$ as $y \to 0^+$. Also, for irrational α, the sequence $\{u_y(\alpha n) : n \geq 1\}$ is equidistributed in the horocycle. This motivates the following conjecture:

Conjecture 22 (J. Marklof, A. Strömbergsson). *Let f be a continuous function on $T^1(X)$ with given growth condition at the cusps (see* [181]*). Then for $\alpha \in \mathbb{R}$ which is badly approximable by rationals and $0 < c_1 < c_2$,*

$$\frac{1}{M}\sum_{m=1}^{M} f(u_y(\alpha m)) \to \int_{\mathrm{T}^1(X)} f\, d\lambda$$

uniformly as $M \to \infty$ and $c_1 M^{-2} \le y \le c_2 M^{-2}$, where λ denotes the Liouville measure.

It was observed in [181] that Conjecture 22 implies Conjecture 21. Conjecture 22 was proved in [181] under the condition that $c_1 M^{-\nu} \le y \le c_2 M^{-\nu}$ for some $\nu < 2$. Furthermore, the statement of Conjecture 22 holds for almost all α with respect to Lebesgue measure [181] for any positive ν, in particular for $\nu = 2$. Hence this gives a new proof of the main result in [216].

4.5. Let X be a Riemannian locally symmetric space of noncompact type and of finite volume. A *flat* in X is a totally geodesic submanifold of sectional curvature zero. Note that $X = \Gamma\backslash G/K$, where G is a connected semisimple real algebraic group, K is a maximal compact subgroups of G, and Γ is a lattices in G, and flats are $\Gamma g A K$, $g \in G$, for a Cartan subgroup A of G. It was shown (see [199]) that the number of compact flats with bounded volume is finite. Note that this number is related to the number of totally real number fields of fixed degree with bounded regulator (see [199]).

Question 23 (H. Oh)**.** *Determine the asymptotics of the number of compact flats with volume less than T as $T \to \infty$.*

This asymptotics and the rate of convergence has been determined for rank one spaces (see [171, 103, 91, 92, 261, 148, 207, 146]); however, the question about optimal rate of convergence is still open (see [114, 167, 166, 33]). When X is compact, using techniques developed in [238], one can determine the asymptotics of the sum

$$\sum_{\mathcal{F}\text{ - regular, systol}(\mathcal{F})<T} \mathrm{Vol}(\mathcal{F}).$$

Here a flat is called *regular* if its shortest closed geodesics goes in the regular direction and $\mathrm{systol}(\mathcal{F})$ denotes the length of this geodesics. See also [50] for another analog of the prime geodesic theorem for higher rank compact symmetric spaces.

Another open question concerns the distribution of compact maximal flats on the unit tangent bundle $\mathrm{T}^1(X)$. Since the identity component G of the isometry group does not act transitively on X in the higher rank case, it is more convenient to consider the compact orbits of a Cartan subgroup A on the homogeneous space $\Gamma\backslash G$, which we also call flats. Denote by $\mu_{\mathcal{F}}$ the Lebesgue measure on a flat $\mathcal{F} \subset \Gamma\backslash G$ and by $\bar{\mu}_{\mathcal{F}}$ the normalized Lebesgue measure on $\mathcal{F}$. Let

$$\nu_T = \frac{\sum_{\mathcal{F}:\,\mathrm{Vol}(\mathcal{F})<T} \mu_{\mathcal{F}}}{\sum_{\mathcal{F}:\,\mathrm{Vol}(\mathcal{F})<T} \mathrm{Vol}(\mathcal{F})} \quad \text{and} \quad \bar{\nu}_T = \frac{\sum_{\mathcal{F}:\,\mathrm{Vol}(\mathcal{F})<T} \bar{\mu}_{\mathcal{F}}}{\#\{\mathcal{F}:\ \mathrm{Vol}(\mathcal{F})<T\}}.$$

Question 24 (H. Oh)**.** *Do the measures ν_T and $\bar{\nu}_T$ converge to the normalized Haar measure on $\Gamma\backslash G$?*

Motivated by the work of Y. Linnik [162], M. Einsidler, E. Lindenstrauss, P. Michel, and A. Venkatesh [62] recently proved an equidistribution result related to Question 24 for some higher rank homogeneous spaces. They showed that a positive proportion of the normalized sum of the measure supported on compact flats (indexed by the discriminant) converges to Haar measure. Nondivergence is deduced from subconvexity bounds for L-functions.

According to [147], the normalized Haar measure on $\Gamma\backslash G$ is the unique ergodic measure of maximal entropy for the geodesic flow. Thus, to resolve Question 24, it suffices to estimate the entropy of the weak* limit points of ν_T and $\bar{\nu}_T$ and show that they do not escape to infinity.

For the rank one groups, Question 24 has been answered positively (see [28, 262, 206, 224, 138]). In higher rank case, the equidistribution of ε-separated closed geodesics from different homotopy classes was established in [147]. As in [147], one can also prove analog of the Conjecture 24 for compact $\Gamma\backslash G$ and the measure ν_T as above with summation taken over regular flats $\mathcal{F}$ such that $\mathrm{systol}(F) < T$. Another equidistribution result was obtained in a recent work of Y. Benoist and H. Oh [15], where the averages along Hecke orbits of maximal compact flats were considered.

5. Divergent trajectories

Let G be a semisimple real algebraic group, Γ a noncocompact arithmetic lattice in G, and D a closed subgroup of a maximal $\mathbb{R}$-split torus A. An orbit Dx of D in G/Γ is called *divergent* is the map $d \mapsto dx$, $d \in D$, is proper. One can construct a divergent orbits using the following observation. Suppose that D is the union of open subsemigroups $D_1, \dots, D_l$ such that for every i there exists a representation $\rho_i : G \to \mathrm{GL}(V_i)$, defined over $\mathbb{Q}$, and $v_i \in V_i$, such that $\rho_i(dx)v_i \to 0$ as $d \in D_i$ goes to ∞. Then Dx is divergent. Such divergent orbits are called *obvious.*

Conjecture 25 (Barak Weiss)**.**
1. *If $\dim D > \mathrm{rank}_{\mathbb{Q}} G$, then there are no divergent orbits of D.*
2. *If $\dim D = \mathrm{rank}_{\mathbb{Q}} G$, then the only divergent trajectories are obvious ones.*
3. *If $\dim D < \mathrm{rank}_{\mathbb{Q}} G$, then there are non-obvious divergent trajectories.*

Conjecture 25 was formulated in [256] where several special cases of it were checked, and it was shown in particular that Conjecture 25 holds when $\dim D = 1$ (see also [43]). The case $D = A$ was settled in [243]. Recently, Conjecture 25(1) was proved in [35] when $\mathrm{rank}_{\mathbb{Q}} G = 2$ and in [257] in complete generality. We also mention that Conjecture 25(3) was checked in [256] for $G = \mathrm{SL}(4,\mathbb{R})$ for all diagonal subgroups D except

$$D = \{(s, s^{-1}, t, t^{-1}) : s, t > 0\}.$$

Next, we discuss a similar problem when D is a cone in A. There are examples of cones D that admit non-obvious divergent trajectories (e.g., a Weyl chamber) as well as an example of cones that admit only obvious divergent trajectories (see [256]). The latter example was constructed for $G = \mathrm{SL}(3,\mathbb{R})$ and the argument used essentially that $\dim D = 2$.

Question 26 (Barak Weiss)**.** *Construct examples of cones D in A with* $\dim D \geq 3$ *and no non-obvious divergent trajectories.*

Let A_T denote the ball of radius T in A and λ a Haar measure on A.

Question 27 (Barak Weiss)**.** *Suppose that* $\dim A \geq 2$ *and for some $x \in G/\Gamma$ and every one-parameter subgroup D of A, the orbits Dx is not divergent in G/Γ. Is it true that there exists a compact set $K \subset G/\Gamma$ such that*

$$\limsup_{T\to\infty} \frac{1}{\lambda(A_T)} \lambda(\{a \in A_T : ax \in K\}) > 0?$$

6. Symbolic coding

Symbolic dynamics plays important role in the study of Anosov flows (see, for example, [29]). In the case of surfaces of constant negative curvature, a symbolic representation of the geodesic flow in terms of a Markov chain can be given quite explicitly. Such constructions go back to M. Morse, E. Artin, and G. Hedlund. More recently, these constructions were generalized and improved by several authors (see [32, 228, 229, 230], [1, 2], [132, 99, 133, 134, 135]). The *geometric code* of a geodesic is a biinfinite sequence of symbols that obtained by fixing a fundamental domain and recording which sides the geodesic hits along its pass. The *arithmetic code* of a geodesic is obtained by expanding the coordinates of the endpoints of the geodesic into a continued fraction expansion.

Question 28 (S. Katok)**.** *Construct analogs of the geometric coding and the arithmetic codings for the Weyl chamber flow (i.e., the action the diagonal group) on* $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$.

It was pointed out by Barak Weiss that an interesting symbolic coding for the Weyl chamber flow was used in [236], where a (wrong) proof of the Littlewood conjecture was given (see also [234, 235, 9]). The above question is related to the problem of constructing effective multidimensional continuous fraction algorithms (see [153, 137]).

One should mention that symbolic representations in higher dimensions are usually quite involved and not explicit. For example, any Markov partition of a hyperbolic toral automorphism on the 3-dimensional torus must consist of fractal sets (see [31]). F. Ledrappier and S. Mozes suggested to look for a convenient symbolic representation of the Weyl chamber flow using fractal tilings. This approach was successfully applied to construct explicit symbolic representations for some automorphisms and shifts on higher-dimensional tori (see [213, 13, 112, 113]) and for the Cartan action on $\mathrm{GL}(2,\mathbb{Q}_p)\times\mathrm{GL}(2,\mathbb{Q}_q)/\Gamma$, Γ a irreducible lattice (see [191, 193]).

7. Polygonal billiards

Question 29 (A. Katok)**.** *Construct periodic orbits for triangular billiards.*

Every acute triangle has one obvious periodic orbit, but it is not known whether a general acute triangle has other periodic orbits. It is also not known whether a general obtuse triangle has at least one periodic orbit. Periodic orbits were constructed for some special classes of triangles (see [90, 108, 37, 101]). Recently, a computer aided proof, which uses the program McBilliards, was found that shows that every triangle with all angles less than 100 degrees has a periodic orbit (see [227]).

The situation is much better for rational triangles and polygons (i.e., if the angles are rational multiples of π). Unfolding the billiard table, one can construct a compact Riemannian surface with a flat structure so that billiard trajectories correspond to geodesics on this surface (see [186] for a survey). Using this technique, it was shown that the number $N(T)$ of periodic orbits of length at most T is bounded from above and below by quadratic polynomials in T (see [182, 183]). Moreover, for some billiard table this number has quadratic asymptotics (see [248, 249, 67, 66]), but it seems unknown whether the quadratic asymptotics holds for rational polygons in general. Note that the convergence $N(T)\to\infty$ cannot be uniform even on a compact set of triangles. In fact, it was shown by R. E. Schwatz [226] that for any given any $\varepsilon>0$ there exists a triangle, within ε of the 30–60–90 triangle, which has no periodic paths of length less than $1/\varepsilon$.

Question 30 (A. Katok)**.** *Find a triangle whose angle are Diophantine (mod 2π) with ergodic (with respect to the Liouville measure) billiard flow.*

Ergodic triangular billiards are generic in the sense of Baire category [136]. Y. Vorobets [252] gave an explicit, albeit extremely fast approximation, condition for the angles which is sufficient for ergodicity. In the case when one of the angles of a triangle is rational, there is a useful unfolding procedure (see [244]) that may lead to a proof of ergodicity.

Question 31 (A. Katok)**.** *Does there exist a weakly mixing polygonal billiard? Specifically, are Vorobets' billiards* [252] *weakly mixing?*

See [100] for a related result for rational billiards.

It was proved that the set of ergodic (topologically transitive) triangular billiard table is residual in the sense of Baire category (see [136] and [131] respectively).

Question 32 (A. Katok)**.** *Does the set of ergodic billiard tables have positive measure?*

These problems are better understood for the case of rational billiards. It is easy to see that rational billiards cannot be ergodic. In this case, the phase space decomposes into invariant subsets $\mathcal{P}_\theta$ that correspond to directions θ of the flow. It was shown in [136] that the billiard flow is uniquely ergodic on $\mathcal{P}_\theta$ for the set of directions θ of full measure. There is an estimate of the Hausdorff dimension of this set, which may be positive (see [185, 184, 36]). It is also known that the restriction of the billiard flow on all but countably many of the subsets $\mathcal{P}_\theta$ is minimal [186, Section 1.6].

Since a polygonal billiard is a parabolic dynamical system, one expects that invariant measures and invariant closed sets should be scarce.

Question 33 (A. Katok)**.** *Classify ergodic invariant probability measure and closed invariant subsets for polygonal billiards.*

Note that Questions 29 and 30 are special cases of Question 33.

8. Arithmeticity

Let G be the direct product of k copies of $\mathrm{SL}(2,\mathbb{R})$, $k \geq 2$, and U^+ and U^- the upper and lower unipotent subgroups G respectively. Let Γ^+ and Γ^- be lattices in U^+ and U^-. We assume that these lattices are "irreducible" in the sense that the projection maps from G to its components are injective on Γ^+ and Γ^-.

The following conjecture was communicated by H. Oh:

Conjecture 34 (G. Margulis, A. Selberg)**.** *If the group $\langle \Gamma^+, \Gamma^- \rangle$ is discrete, then it an arithmetic lattice in G.*

It was observed in [200] that Conjecture 34 for $k \geq 3$ follows from Conjecture 39. In particular, one can show that the Hausdorff dimension of the set of irreducible lattices $\Gamma^+ \subset U^+$ for which $\langle \Gamma^+, \Gamma^- \rangle$ is discrete for some irreducible lattice $\Gamma^- \subset U^-$ is exactly k (see [201]).

9. Diophantine analysis

9.1. A vector $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ is called *v-approximable* (for $v > 0$) if there are infinitely many $\mathbf{q} = (q_1, \dots, q_n) \in \mathbb{Z}^n$ and $p \in \mathbb{Z}$ such that

$$|y_1 q_1 + \cdots + y_n q_n - p| < \|\mathbf{q}\|^{-v}.$$

Here $\|\cdot\|$ denotes the max-norm on $\mathbb{R}^n$. If a vector $\mathbf{y} \in \mathbb{R}^n$ is $(n+\varepsilon)$-approximable for some $\varepsilon > 0$, it is called *very well approximable* (VWA). An easy argument (using Borel-Cantelli lemma) implies that the set of VWA vectors has Lebesgue measure zero in $\mathbb{R}^n$. Therefore, it is natural to expect that a generic point on a "nondegenerate" submanifold of $\mathbb{R}^n$ is not VWA. This is the Sprindžuk conjecture proved in [143] (see also [18]).

Similarly, an $m \times n$ real matrix A is called VWA if for some $\varepsilon > 0$ there are infinitely many $\mathbf{q} \in \mathbb{Z}^n$ and $\mathbf{p} \in \mathbb{Z}^m$ such that

$$\|A\mathbf{q} - \mathbf{p}\|^m < \|\mathbf{q}\|^{-n-\varepsilon}.$$

It is easy to see that the set of VWA matrices has measure zero in $\mathbb{R}^{m\times n}$, and one hopes that an analog of the Sprindžuk conjecture holds in this set-up as well.

The definition of nondegenerate submanifold of $\mathbb{R}^n$ in [143], which is well-suited for the case of vectors, is a manifold with smooth coordinate charts $\mathbf{f} : U(\subset \mathbb{R}^k) \to \mathbb{R}^n$ such that the spaces spanned by the partial derivatives $\mathbf{f}$ at points of U have dimension n. It is not quite clear what is the right definition of "nondegenerate" submanifold for the case of matrices.

Question 35 (D. Kleinbock, G. Margulis (see [143], Sec. 6.2))**.** *Find reasonable and checkable conditions for a smooth map $\mathbf{f} : U(\subset \mathbb{R}^k) \to \mathrm{M}_{m\times n}(\mathbb{R})$ which generalizes nondegeneracy of vector-valued maps and implies that almost every point of $\mathbf{f}(U)$ is not VWA.*

Such conditions were obtained in some cases in [150, 151, 139].

9.2. A far-reaching generalization of the Sprindžuk conjecture was suggested in [140]. Let μ be a measure on $\mathbb{R}^k$ and and $\mathbf{f} : \mathrm{supp}(\mu) \to \mathbb{R}^n$ that satisfy some reasonable conditions. What are the Diophantine properties of the generic points in $\mathbb{R}^n$ with respect to the measure $\mathbf{f}_*\mu$? Several results in this direction were obtained in [254, 141, 145] for locally finite measure. It would be interesting to consider the case of Hausdorff measures:

Question 36 (D. Kleinbock)**.** *Give estimates on Hausdorff dimension of v-approximable vectors in a nondegenerate submanifold of $\mathbb{R}^n$ using dynamics.*

See [52] for a discussion of what is currently known about the Hausdorff dimension and for a related result.

9.3. For $\alpha \in \mathbb{R}$, let $\langle\alpha\rangle = \mathrm{dist}(\alpha, \mathbb{Z})$. It is not hard to show that the set of $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$\liminf_{q\to\infty} q(\log q)^{2+\varepsilon} \langle q\alpha\rangle \langle q\beta\rangle > 0$$

for every $\varepsilon > 0$ has full Lebesgue measure. In fact, it was shown in [239] that

$$\lim_{q\to\infty} q(\log q)^{2+\varepsilon} \langle q\alpha\rangle \langle q\beta\rangle = \infty$$

on a set of $(\alpha, \beta) \in \mathbb{R}^2$ of full measure. On the other hand, the following question remains open:

Question 37 (A. Pollington). *Are there $\alpha, \beta \in \mathbb{R}$ such that for every $\varepsilon > 0$,*

$$\liminf_{q \to \infty} q(\log q)^{2-\varepsilon} \langle q\alpha \rangle \langle q\beta \rangle > 0?$$

It follows from [89] that

$$\liminf_{q \to \infty} q(\log q)^{2} \langle q\alpha \rangle \langle q\beta \rangle = 0$$

for almost all (α, β). Thus, the set of (α, β) in Question 37 has measure zero. Question 37 is related to the well-known conjecture of Littlewood:

Conjecture 38 (Littlewood). *For any $\alpha, \beta \in \mathbb{R}$,*

$$\liminf_{q \to \infty} q \langle q\alpha \rangle \langle q\beta \rangle = 0.$$

The best result on Conjecture 38 is [60], which shows that the set of exceptions (α, β) for the Littlewood conjecture is a countable union of sets of box dimension zero. The proof in [60] uses dynamics on the homogeneous space $\mathrm{SL}(3, \mathbb{R})/\mathrm{SL}(3, \mathbb{Z})$, and the crucial step is to establish measure rigidity discussed in Section 3.1.

It was observed some time ago that Conjecture 38 is implied by the following conjecture:

Conjecture 39 (G. Margulis [173]). *Let A be the group of all diagonal matrices in $\mathrm{SL}(3, \mathbb{R})$. Then every bounded A-orbit in $\mathrm{SL}(3, \mathbb{R})/\mathrm{SL}(3, \mathbb{Z})$ is closed.*

Conjecture 39 is a very special case of the general conjecture describing closed invariant subsets for actions of Cartan subgroups on general homogeneous spaces (see [173]).

G. Margulis suggested the following conjecture, which might be easier to handle than Conjecture 39:

Conjecture 40 (G. Margulis). *For every compact set K of $\mathrm{SL}(3, \mathbb{R})/\mathrm{SL}(3, \mathbb{Z})$, there are only finitely many closed A-orbits contained in K.*

This conjecture can be reformulated in terms of the Markov spectrum of forms

$$F(x) = \prod_{i=1}^{3} \left(\sum_{j=1}^{3} a_{ij} x_j \right), \quad a_{ij} \in \mathbb{R}.$$

Let

$$\Delta(F) = \det(a_{ij}) \text{ and } m(F) = \inf \left\{ \left| \frac{F(x)}{\Delta(F)} \right| : x \in \mathbb{Z}^3 - 0 \right\}.$$

Then Question 40 is equivalent to the following question:

Question 41 (G. Margulis). *Show that for every $\varepsilon > 0$, the set $[\varepsilon, \infty) \cap \{m(F)\}$ is finite.*

9.4. For $0 \leq s \leq 1$, define

$$\mathcal{C}_s = \left\{ (\alpha, \beta) \in \mathbb{R}^2 : \inf_{q \geq 1} \max\{q^s \langle q\alpha \rangle, q^{1-s} \langle q\beta \rangle\} > 0 \right\}.$$

In particular, $\mathcal{C}_{1/2}$ is the set of badly approximable vectors. Since Conjecture 38 holds for all $(\alpha, \beta) \notin \mathcal{C}_s$, one may naively hope to prove it by showing that intersection of the sets $\mathcal{C}_s$, $0 \leq s \leq 1$, is empty. In this regard, we mention the following conjecture:

Conjecture 42 (W. Schmidt [225]). *For any $s, t \in [0, 1]$, we have $\mathcal{C}_s \cap \mathcal{C}_t \neq \emptyset$.*

Note that W. Schmidt stated this conjecture in [225] only for $s = 1/3$ and $t = 2/3$.

It is known that each of the sets $\mathcal{C}_s$ has zero measure and full Hausdorff dimension. It was shown that the set $\mathcal{C}_s \cap \mathcal{C}_0 \cap \mathcal{C}_1$ has full Hausdorff dimension as well (see [208]). Conjecture 42 is related to the following conjecture:

Conjecture 43. *Let A be the group of all diagonal matrices in $\mathrm{SL}(3, \mathbb{R})$, and $A_1, A_2 \subset A$ are rays in A. Then there exists $x \in \mathrm{SL}(3, \mathbb{R})/\mathrm{SL}(3, \mathbb{Z})$ such that $A_1 x$ and $A_2 x$ are bounded, but Ax is not bounded in $\mathrm{SL}(3, \mathbb{R})/\mathrm{SL}(3, \mathbb{Z})$.*

Note that for rays A_1 and A_2 which lie in the cone

$$\{\mathrm{diag}(e^u, e^v, e^{-u-v}) : u, v \geq 0\} \subset A,$$

Conjecture 43 follows from Conjectures 42. On the other hand, it was pointed out by D. Kleinbock that in the case when A_1 and A_2 lie in the opposite Weyl chambers, Conjecture 43 can be proved using the argument from [142], and moreover, the set of x which satisfy Conjecture 43 has full Hausdorff dimension.

9.5. Let Q be a nondegenerate positive definite quadratic form of dimension $d \geq 3$.

Conjecture 44 (Davenport-Lewis). *Suppose that Q is not a multiple of a rational form. Then the gaps between consecutive elements of the set $\{Q(x) : x \in \mathbb{Z}^d\}$ go to zero as $Q(x) \to \infty$.*

Conjecture 44 was proved in [16] for $d \geq 9$, and recently the method in [16] was extended to $d \geq 5$ as well. The case $d = 3, 4$ is still open.

When Q is a nondegenerate indefinite definite quadratic form of dimension $d \geq 3$ which is not a multiple of a rational quadratic form, the set $\{Q(x) : x \in \mathbb{Z}^d\}$ is dense in $\mathbb{R}$. This is the Oppenheim conjecture proved by Margulis in [172]. However, the proof in [172] is not effective.

Question 45 (G. Margulis [173]). *Give an effective estimate on $T = T(\varepsilon)$ such that there exists $x \in \mathbb{Z}^d$ with*

$$0 < |Q(x)| < \varepsilon \quad \textit{and} \quad \|x\| < T.$$

This question is especially difficult since the estimate on T should depend on the Diophantine properties of coefficients of the quadratic form Q. An easier question with x satisfying conditions

$$|Q(x)| < \varepsilon \quad \text{and} \quad \|x\| < T$$

is treated for $d \geq 5$ in an upcoming work of G. Margulis and F. Götze.

9.6. Let $Q(x) = ax_1^2 + bx_1x_2 + cx_2^2$ be a nondegenerate indefinite quadratic form with rational coefficients that does not represent zero over $\mathbb{Q}$. For $x \in \mathbb{R}^2$, define

$$m(Q, x) = \inf_{z \in \mathbb{Z}^2} |Q(x+z)| \;\text{ and }\; m(Q) = \sup_{x \in \mathbb{R}^2} m(Q, x).$$

If the supremum $m(Q)$ is isolated, we also define

$$m_2(Q) = \sup\{m(Q, x) : x \in \mathbb{R}^2, m(Q, x) < m(Q)\}.$$

The interest in the quantity $m(Q)$ was motivated by the study of existence of a Euclidean algorithm in quadratic fields $\mathbb{Q}(\sqrt{m})$, $m > 0$. If Q represents the norm of $\mathbb{Q}(\sqrt{m})$ computed with respect to an integral basis, then Euclidean algorithm exists iff $m(Q) < 1$.

The following conjecture was communicated by A. Pollington:

Conjecture 46 (E. Barnes, H. Swinnerton-Dyer [11]). *For any quadratic form Q as above, the supremum $m(Q)$ is rational and isolated. Both $m(Q)$ and $m_2(Q)$ are attained at points with coordinates in the the splitting field of Q.*

Conjecture 46 is based on numerous computations performed in [10, 11]. The supremum $m_2(Q)$ need not be isolated (see [95]).

9.7. The following question was communicated by D. Kleinbock:

Question 47 (Y. Bugeaud). *Let*

$$\mathcal{B}_s = \left\{ (\alpha, \beta) \in \mathbb{R}^2 : \inf_{n \geq 1} n(\min\{\|n\alpha\|, \|n\beta\|\})^s (\max\{\|n\alpha\|, \|n\beta\|\})^{2-s} > 0 \right\}.$$

Compute the Hausdorff dimension of the set $\mathcal{B}_s$, $0 < s < 1$.

Note that $\mathcal{B}_0$ is the set of badly approximable vectors and its Hausdorff dimension is 2. On the other hand, $\mathcal{B}_1$ is the set of exceptions of the Littlewood conjecture, and its Hausdorff dimension is 0.

9.8. Some other interesting open problems on Diophantine approximation are stated in [144], Section 13.

10. Quantum ergodicity and quantum chaos

10.1. The term "quantum chaos" refers to the study of quantizations of Hamiltonian systems whose dynamics is chaotic. More generally, one is interested in connections between properties classical dynamical systems and corresponding quantum systems. We concentrate on the case of the geodesic flow on compact (or, more generally, finite volume) Riemannian manifold X possibly with piecewise smooth boundary (e.g. billiards in $\mathbb{R}^2$). The geodesic flow on the boundary is defined as elastic reflection. Denote by Δ the Laplace-Beltrami operator on X and by dV the normalized Riemannian volume on X. Let $0 = \lambda_0 < \lambda_1 \le \lambda_2 \dots$ be the eigenvalues of $-\Delta$ and ϕ_i, $i \ge 0$, the corresponding eigenfunctions with the Dirichlet boundary condition such that $\|\phi_i\|_2 = 1$:

$$-\Delta\phi_i = \lambda_i\phi_i, \ \ \phi_i|_{\partial X} = 0.$$

One is interested in the semiclassical limit of this system, i.e., in the behavior of the eigenvalues and the eigenfunctions as $i \to \infty$. According to the correspondence principle in quantum mechanics, certain properties of the classical dynamical system are inherited by the semiclassical limit of its quantization.

Consider the probability measures

$$d\mu_i(x) = |\phi_i(x)|^2 dV(x)$$

on X. One of the fundamental questions is to describe all possible weak* limits of the sequence $\{\mu_i\}$ as $i \to \infty$, which are called *quantum limits*. It was discovered by A. Shnirelman [237] and later proved by S. Zelditch [260] and Y. Colin de Verdière [41] (see also [264]) that if the geodesic flow is ergodic on X, then $\mu_{i_k} \to dV$ in the weak* topology as $i_k \to \infty$ along a subsequence $\{i_k\}$ of density one. This property is referred as *quantum ergodicity*. We refer to [220, 21, 179] for a more detailed discussion.

In general, it might be possible that some of the quantum limits are not absolutely continuous and even assign positive measure to an unstable periodic orbit (this is called a *scar*) or to a family of marginally stable periodic orbits (this is called a *bouncing ball mode*). However, it seems that no rigorous proof of this phenomena has been given for the systems discussed here (see [51] for a partial result and [180] for a result on escape of mass to infinity). Existence of scars was proved for quantum cat maps of [70]. For the stadium billiard, there are substantial numerical and heuristic evidences of the existence of scars and bouncing ball modes (see, for example, [106, 107, 121, 158, 8, 242, 263] and references therein). On the other hand, the numerical data in [12] suggest that no scarring occurs for some dispersive billiards.

A. Katok constructed an example of a Finsler metric on 2-dimensional sphere such that the corresponding geodesic flow is ergodic and has only two periodic orbits (see [122]). If it is constructed with a certain care (along the line of [71, Section 3]), then there are only three ergodic invariant measures corresponding to closed orbits and the volume measure.

Question 48 (A. Katok)**.** *Do the closed geodesics in this example correspond to scars?*

Some numerical experiments were performed for $X = \Gamma\backslash\mathbb{H}^2$, where Γ is an arithmetic lattice, and no scars were observed (see [105, 104, 5]). Z. Rudnick and P. Sarnak [215] formulated the following conjecture:

Conjecture 49 (Quantum unique ergodicity)**.** *Suppose that X has negative sectional curvature. Then*

$$\mu_i \to dV \quad as \quad i \to \infty.$$

A recent important breakthrough was made by N. Anantharaman [3] who gave lower bounds on the entropy of the limits of the sequence $\{\mu_i\}$ in some cases. Besides this result, very little is known about general negatively curved manifolds, and we concentrate on the case of arithmetic manifolds $X = \Gamma\backslash\tilde{X}$, where $\tilde{X}$ is a symmetric space of noncompact type and Γ an arithmetic lattice. The arithmeticity assumption implies that there is an infinite set of Hecke operators acting on X, which commute with the left invariant differential operators (in rank one the Laplacian is the only such operator). We assume that ϕ_i, $i \ge 0$, are joint eigenfunctions of the invariant differential operators and Hecke operators. Then the weak* limits of the sequence of measures $\{\mu_i\}$ are called *arithmetic quantum limits*. It is believed (see [34]) that the Laplace-Beltrami operator on $X = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}^2$ has simple cuspidal spectrum; then the assumption on Hecke operators is automatic.

Conjecture 50 (Arithmetic quantum unique ergodicity)**.** *The Riemannian volume is the only arithmetic quantum limit.*

Positive results towards this conjecture were obtained for the case $X = \Gamma\backslash\mathbb{H}^2$, where Γ is a congruence subgroup in either $\mathrm{SL}_2(\mathbb{Z})$ or in the group of quaternions of norm one. In this case, T. Watson [253] proved Conjecture 50 assuming the generalized Riemann hypothesis. His proof also implies the optimal rate of convergence. Unconditionally, Conjecture 50 for this case was proved by E. Lindenstrauss [159]. The only issue that was not handled in [159] is the escape of the limit measure to the cusp in noncompact case. To handle this difficulty, E. Lindenstrauss suggested the following intermediate problem:

Problem 51 (E. Lindenstrauss)**.** *Let $X = \Gamma\backslash\mathbb{H}^2$ where Γ is a noncocompact arithmetic lattice. Show that for all $f, g \in C_c(X)$,*

$$\frac{\int_X f\, d\mu_i}{\int_X g\, d\mu_i} \to \frac{\int_X f\, dV}{\int_X g\, dV} \quad as \quad i \to \infty.$$

The formula of T. Watson [253] gives an explicit connection between the arithmetic quantum unique ergodicity and subconvexity estimates for L-function. Such subconvexity estimates have been carried out in several case (see, for example, [205, 221, 163]), and they imply Conjecture 50 (and answer Question 51) for some natural subsequences of the sequence $\{\mu_i\}$.

We also mention that analogs of Conjecture 50 for continuous spectrum was proved in [115, 167, 149], and some partial results toward Conjecture 50 for higher-rank symmetric spaces were proved by L. Silberman and A. Venkatesh [233].

10.2. M. Berry [20] conjectured that eigenfunctions of a typical chaotic systems behave like a superposition of plane waves with random amplitude, phase and direction. This model predicts that the eigenfunctions ϕ_i behave like independent Gaussian random variables as $i \to \infty$. In particular, the following conjecture should hold for generic negatively curved compact Riemannian manifolds:

Conjecture 52 (J. Marklof)**.**

$$\mathrm{Vol}(\{x \in X : a \le \phi_i(x) \le b\}) \to \frac{1}{\sqrt{2\pi}} \int_a^b e^{-t^2/2} dt$$

as $i \to \infty$.

Conjecture 52 is supported by numerical experiments (see [105, 106]).

The random wave model also predicts a central limit theorem for the convergence in Conjecture 49 (see [72, 55]). To formulate this, we need some notations. For a smooth function a on the unit cotangent bundle S^*X of X, we denote by $\mathrm{Op}(a)$ a pseudodifferential operator of order zero with principal symbol a. For example, when a is a function on X, then $\mathrm{Op}(a)$ is a multiplication by a. Let λ be the Liouville measure on S^*X and g^t is the geodesic flow. It is expected that for generic negatively curved compact Riemannian manifolds,we have the following. (Suppose w.l.o.g. the surface has area 4π so that Weyl's law reads $N(\lambda) = \#\{i : \lambda_i \le \lambda\} \sim \lambda$.)

Conjecture 53 (J. Marklof (after Feingold-Peres [72]))**.** *Suppose X is "generic". For $a \in C_c^\infty(S^*X, \mathbb{R})$ with*

$$\int_{S^*X} a \, d\lambda = 0,$$

put

$$\xi_i(a) = \lambda_i^{1/4} \left| \langle \mathrm{Op}(a)\phi_i, \phi_i \rangle \right| .$$

Then the sequence $\xi_i(a)$ has a Gaussian limit distribution whose variance is given by the classical autocorrelation function

$$V(a) \stackrel{def}{=} \int_{-\infty}^{\infty} \int_{S^*X} a(xg^t) a(x) \, d\lambda(x) dt.$$

That is, as $\lambda \to \infty$,

(1)
$$\frac{1}{\lambda}\sum_{\lambda_i \le \lambda} \xi_i(a)^2 \to V(a),$$

(2) *for any interval $I \in \mathbb{R}$*
$$\frac{1}{\lambda}\#\left\{\lambda_i \le \lambda : V(a)^{-1/2}\xi_i(a) \in I\right\} \to \frac{1}{\sqrt{2\pi}}\int_I e^{-t^2/2}dt.$$

It was proved by W. Luo and P. Sarnak [168] that for the modular surface $X = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}^2$,
$$\sum_{\lambda_i \le \lambda} |\langle \mathrm{Op}(a)\phi_i, \phi_i\rangle|^2 \sim \sqrt{\lambda}B(a) \;\text{ as }\; \lambda \to \infty,$$
where B is a quadratic form on $C_c^\infty(X)$ which is closely related to but distinct from the form V defined above (see [222]). In this respect the modular and other arithmetic surfaces are ruled out as "generic" examples for the above conjecture.

10.3. A. Katok suggested polygonal billiards as a promising model for quantum chaos.

Question 54 (A. Katok)**.** *Do periodic orbits in a triangular billiard correspond to scars? More precisely, are there quantum limits supported on periodic orbits?*

Based on the investigation [23], it seems likely that the answer to this question is 'yes' for rational billiards.

Problem 55 (J. Marklof)**.** *Classify all quantum limits of the eigenfunctions of a polygonal billiard.*

10.4. According to the Berry-Tabor conjecture (see [176]), the eigenvalues of the Laplacian for generic integrable dynamical system have the same statistical properties as a Poisson process. For 2-dimensional torus, the set of eigenvalues is $\{Q(x) : x \in \mathbb{Z}^2\}$ where Q is a positive definite quadratic form.

Question 56 (J. Marklof)**.** *What is the distribution of the set*
$$\{(Q(x_1) - Q(x_2), Q(x_2) - Q(x_3)) : x_i \in \mathbb{Z}^2, x_i \ne x_j \textit{ for } i \ne j\} \subset \mathbb{R}^2?$$
More precisely, determine the asymptotics of
$$N_T((a,b),(c,d)) \stackrel{def}{=} \#\left\{ (x_1,x_2,x_3) \in (\mathbb{Z}^2)^3 : \begin{array}{l} a < Q(x_1) - Q(x_2) < b, \\ c < Q(x_2) - Q(x_3) < d, \\ x_i \ne x_j \textit{ for } i \ne j,\ \|x_i\| < T \end{array} \right\}.$$

The distribution of the set $\{Q(x) - Q(y) : x, y \in \mathbb{Z}^2\}$ was studied in [223, 64, 65], and in [177, 178] in the case of rational forms over shifted lattice points. It depends on Diophantine properties of coefficients of the quadratic form. The Berry-Tabor conjecture predicts that the set in Question 56 should be equidistributed in $\mathbb{R}^2$ for a generic quadratic form Q, i.e, after a suitable normalization, $N_T((a,b),(c,d))$ converges to $(b-a)(d-c)$. However, it is not even known whether the set in Conjecture 56 is dense in $\mathbb{R}^2$.

11. André-Oort conjecture

We refer to [189] for an introduction to Shimura varieties, to to [247] for an accessible account of André-Oort conjecture, to [198] for a recent survey.

A *Shimura datum* is a pair (G, X) where G is a reductive algebraic group defined over $\mathbb{Q}$ and X is a $G(\mathbb{R})$-conjugacy class of homomorphisms $h : \mathbb{C}^\times \to G(\mathbb{R})$ such that

(1) The adjoint action of $h(\mathbb{C}^\times)$ on $\mathrm{Lie}(G^{\mathrm{ad}}(\mathbb{R}))$[4] decomposes as a direct sum of eigenspaces with characters $z/\bar{z}$, 1, $\bar{z}/z$.
(2) $\mathrm{ad}\, h(i)$ acts as a Cartan involution on $G^{\mathrm{ad}}(\mathbb{R})$.
(3) $G^{\mathrm{ad}}(\mathbb{R})$ has no factors on which the adjoint action of $h(\mathbb{C}^\times)$ is trivial.

Morphisms $(\tilde{G}, \tilde{X}) \to (G, X)$ of Shimura datums are induced by morphisms $\tilde{G} \to G$ of algebraic groups in obvious way. Note that X has a natural structure of complex manifold such that its connected components are Hermitian symmetric domains, $G(\mathbb{R})$ acts on X by holomorphic automorphisms, and morphisms are equivariant holomorphic maps.

Let $\mathbb{A}_f$ denote the ring of finite adeles, and K is an open compact subgroup in $G(\mathbb{A}_f)$. Define

$$\mathrm{Sh}_K(G, X) = G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}_f))/K$$

One can show that $\mathrm{Sh}_K(G, X)$ is a finite disjoint union of Hermitian locally symmetric domains. In particular, by the Baily-Borel theorem, $\mathrm{Sh}_K(G, X)$ has a natural structure of an algebraic variety. In fact, it is canonically defined over a number field.

Example: Let $G = \mathrm{GL}_2$, $h(a + bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and $K = \mathrm{GL}_2(\hat{\mathbb{Z}})$. Then $\mathrm{Sh}_K(G, X) \simeq \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}^2$ parameterizes isomorphism classes of elliptic curves over $\mathbb{C}$.

The *Shimura variety* associated to (G, X) is the projective limit of $\mathrm{Sh}_K(G, X)$ where K runs over open compact subgroups of $G(\mathbb{A}_f)$. A point $h \in X$ is called *special* if there exists a torus T of G defined over $\mathbb{Q}$ such that $h(\mathbb{C}^\times) \subset T(\mathbb{R})$. One can check that in the above example, the special points are imaginary

[4] G^{ad} is the adjoint group which is the factor of G by its center.

quadratic irrationals that correspond to elliptic curves with complex multiplication.

For $g \in G(\mathbb{A}_f)$, we have natural projection maps

$$\pi_1 : \mathrm{Sh}_{K \cap gKg^{-1}}(G, X) \to \mathrm{Sh}_K(G, X)$$
$$\pi_2 : \mathrm{Sh}_{K \cap gKg^{-1}}(G, X) \to \mathrm{Sh}_{gKg^{-1}}(G, X).$$

with finite fibers. This defines *Hecke correspondence*

$$T_g(x) = \pi_2(\pi_1^{-1}(x))g : \mathrm{Sh}_K(G, X) \to \mathrm{Sh}_K(G, X).$$

Let $(\tilde{G}, \tilde{X}) \to (G, X)$ be morphism of Shimura datums that induces the map $\mathrm{Sh}_{\tilde{K}}(\tilde{G}, \tilde{X}) \to \mathrm{Sh}_K(G, X)$. The *special subvarieties* (also called subvarieties of Hodge type) are the irreducible components of the image

$$\mathrm{Sh}_{\tilde{K}}(\tilde{G}, \tilde{X}) \to \mathrm{Sh}_K(G, X) \xrightarrow{T_g} \mathrm{Sh}_K(G, X).$$

Using Hecke correspondences, one shows that the set of special points in a special subvariety is dense with respect to Zariski (or even analytic) topology. The following conjecture is the converse of this fact.

Conjecture 57 (Y. André-F. Oort [4, 202])**.** *Zariski closure of a set of special point is a finite union of special subvarieties.*

Recently, B. Klinger and A. Yafaev announced a prove of Conjecture 57 assuming the generalized Riemann Hypothesis.

There have been two main approaches to Conjecture 57. One is number-theoretic and is due to S. J. Edixhoven and A. Yafaev [56, 57, 259]. It uses Hecke correspondences and requires lower estimates on the size of Galois orbits of special points. The other approach is ergodic-theoretic and is due to L. Clozel and E. Ullmo [38, 39]. It uses the Ratner theory of unipotent flows and in particular [194].

Conjecture 57 was partially motivated by an analogy with the theory of abelian varieties, according to which special points correspond to torsion points and special subvarieties correspond to translates of abelian subvarieties by torsion points. Analogous conjectures for abelian varieties is due to S. Lang, Yu. Manin, and D. Mumford. These conjectures were settled (see [245] for a survey). One of the proofs (see [246, 265]) is based on equidistribution of Galois orbits of “generic” sequences of points, which was established in [241] (see also [22]). This approach may also lead to a proof of Conjecture 57 (see, for example, [266]).

Conjecture 58. *Let $\{x_n\}$ be a sequence of special points on a Shimura variety. Suppose that x_n lies outside of any special subvariety for sufficiently large n. Then the Galois orbits of x_n become equidistributed as $n \to \infty$ with respect to the normalized Haar measure.*

For some partial results on this conjecture using convexity and subconvexity bounds for L-functions see [53, 102, 39, 40, 266]. Note that for the above example, Conjecture 58 was established in [53].

Question 59 (L. Silberman)**.** *Give an ergodic-theoretic proof of the equidistribution of special points on* $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}^2$*.*

Recently, M. Einsidler, E. Lindenstrauss, P. Michel, and A. Venkatesh [62] gave an ergodic-theoretic argument, which proves the equidistribution of closed geodesic established in [53], under the condition that mass does not escape to infinity.

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