This problem is formulated in [D-L] = Downarowicz & Lacroix, *The Law of Series*, preprint.

Let $(X, \Sigma, \mu, T, \mathcal{P})$ be an invertible process on finitely many states, i.e., $\mathcal{P}$ is a finite partition of a probability measure preserving invertible transformation (X, Σ, μ, T), and we distinguish points up to their $\mathcal{P}$-names. Choose $B \in \Sigma$ such that $\mu(B) > 0$. Consider the *induced transformation* defined almost everywhere on B, $T_B(x) = T^{n_x}(x)$, where $n_x = \min\{n > 0 : T^n(x) \in B\}$. ($n_x$ is called *the first return time.*) The conditional measure μ_B, given by $\mu_B(A) = \frac{\mu(A\cap B)}{\mu(B)}$, is invariant under T_B, so we have a new probability measure preserving invertible transformation (X, Σ, μ_B, T_B) called *the induced system.* (Because $\mu_B(B) = 1$, in this system we can replace X by B.)

We will consider two processes defined on this system. One is $(X, \Sigma, \mu_B, T_B, \mathcal{P})$, the process generated by the finite partition $\mathcal{P}$ (note that this is NOT the *full* induced process – it only tell us about ONE symbol in the $\mathcal{P}$ name of x at each moment when x visits B), the other is $(X, \Sigma, \mu_B, T_B, \mathcal{R})$, generated by the countable *first return time partition* $\mathcal{R} = \{R_n : n \in \mathbb{N}\}$, where $R_n = \{x : n_x = n\}$.

We will use the following language: a set $B \in \bigvee_{i=1}^{n} T^i\mathcal{P}$ will be called an *n-cylinder.* We will say that a property $\mathfrak{P}$ holds for n-cylinders *with μ-tolerance ϵ* if the measure μ of the union of all n-cylinders which do not satisfy $\mathfrak{P}$ is smaller than ϵ. A (finite or countable) partition $\mathcal{P}$ is *ϵ-independent* of a sub-sigma-field $\mathcal{A} \subset \Sigma$ if for any finite (equivalently countable) $\mathcal{A}$-measurable partition α, holds

$$\sum_{A\in\alpha, B\in\mathcal{P}} |\mu(A \cap B) - \mu(A)\mu(B)| < \epsilon.$$

A process $(X, \Sigma, \mu, T, \mathcal{P})$ is called an *ϵ-independent process* it the "present" $\mathcal{P}$ is ϵ-independent for the "past" $\mathcal{P}^- = \bigvee_{i>0} T^i\mathcal{P}$. For a measure preserving system (X, Σ, μ, T), the process generated by $\mathcal{P}$ is ϵ-independent of the process generated by another partition $\mathcal{R}$ if for every n holds $\mathcal{P}^n = \bigvee_{i=0}^{n-1} T^{-i}\mathcal{P}$ is $n\epsilon$-independent of the *full history* of the process generated by $\mathcal{R}$, $\bigvee_{i=-\infty}^{\infty} T^i\mathcal{R}$.

In [D-L] we have proved a partial "ϵ-independence" result:

Lemma 3. *Let $(X, \Sigma, \mu, T, \mathcal{P})$ be a process with positive entropy. Given $\epsilon > 0$ and $K \in \mathbb{N}$ there exists n_0 such that for every $n \geq n_0$ the following is true for n-cylinders with μ-tolerance ϵ: with respect to μ_B, the partition $\mathcal{P}$ is ϵ-independent from jointly the past $\mathcal{P}^-$ (of the "master" process $(X, \Sigma, \mu, T, \mathcal{P})$) and K first return times $n_x,\ n_{T_B x},\ \ldots,\ n_{T_B^K x}$.*

Although it is not included in the formulation, the lemma holds trivially for processes of entropy zero: in such case for majority of sufficiently long blocks B the "presence" is, up to a very small ϵ, determined by B, i.e., with respect to μ_B the partition $\mathcal{P}$ is nearly a one-element partition, hence it is ϵ-independent from anything.

Because the full past of the process contains the past of the induced process, Lemma 3 implies that the process $(X, \Sigma, \mu_B, T_B, \mathcal{P})$ is not only an ϵ-independent process but also that it is ϵ-independent from the past, present and finite future (up to step K) of the process of return times. We were unable to remove the parameter K (pass with it to infinity). Hence the natural question is this:

Question. *Does the following hold, for sufficiently large n, and for n-cylinders B with μ-tolerance ϵ: On (X, Σ, μ_B, T_B) the processes generated by $\mathcal{P}$ and $\mathcal{R}$ are ϵ-independent?*