FOR MINIMAL SYSTEMS DOES LI-YORKE SENSITIVITY OCCUR EXACTLY FOR EXTENSIONS OF NONTRIVIAL WEAK MIXING SYSTEMS?

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There are, alas, a plethora of possible meanings for the word "chaos" applied to a topological dynamical system (X, T) (with X a compact metric space and T a continuous map).

The Li-Yorke concept in T.Y.Li, J.A.Yorke (1975) *Period three implies chaos*, Amer. Math.Month. **8**2:985-992 is related to the classical concept of proximality. A pair $(x, y) \in X \times X$ is *proximal* if

$$\liminf_{k\to\infty} d(T^k(x), T^k(y)) = 0. \tag{1}$$

If the limit exists and so is zero, then the pair is called *asymptotic*. Call the pair a *Li-Yorke pair* if it is proximal but not asymptotic. That is, it is proximal and

$$\limsup_{k\to\infty} d(T^k(x), T^k(y)) > 0. \tag{2}$$

A subset $A \subset X$ is called *scrambled* if every pair of distinct points in A is a Li-Yorke pair. The system (X, T) is is chaotic in the sense of Li and Yorke, or simply *Li-Yorke chaotic*, when X contains an uncountable scrambled subset.

The notion of *sensitive dependence upon initial conditions* is equivalent to the condition that there exists $\epsilon > 0$ such that for every opene (= open and nonempty) set $U \subset X$ there exist $x, y \in U$ with

$$\limsup_{k\to\infty} d(T^k(x), T^k(y)) > \epsilon. \tag{3}$$

In E.Akin, S.Kolyada (2003) *Li-Yorke sensitivity*, Nonlinearity **1**6:1421-1433 we introduce a concept designed to bridge these two ideas.(In addition, Theorem 3.4 of that paper shows that the definition we have given of sensitive dependence is equivalent to the standard one which uses *sup* in place of *lim sup* in (3)). We call the system *Li-Yorke sensitive* if there exists an $\epsilon > 0$ such that for every $x \in X$ and every neighborhood U of x there exists $y \in U$ so that equations (1) and (3) hold. That is, the pair is proximal but the two orbits are frequently more than ϵ apart. We show that a nontrivial, weak mixing system satisfies Li-Yorke sensitivity. If (Y, S) is any dynamical system and (X, T) is Li-Yorke sensitive then it is easy to see that the product system $(X \times Y, T \times S)$ is again Li-Yorke sensitive. Notice that the first coordinate projection is an open mapping which is an action map onto (X, T). In general, an action map $\pi : (Y, S) \to (X, T)$ between minimal systems is surjective and almost open, i.e. if $V \subset Y$ is opene then $\pi(V)$ has nonempty interior in X. For two minimal systems so related we call (Y, S) an *extension* of (X, T) and (X, T) a *factor* of (Y, S).

Question 1. *Suppose that $\pi : (Y, S) \to (X, T)$ is an action map between nontrivial minimal systems. Does Li-Yorke sensitivity of (X, T) imply the same property for (Y, S)? That is to ask: is every minimal extension of a nontrivial Li-Yorke sensitive system also Li-Yorke sensitive. In particular, is every minimal extension of a nontrivial, weak mixing, minimal system Li-Yorke sensitive? On the other hand, we can also ask the converse question: Does every Li-Yorke sensitive minimal system admit a nontrivial weak mixing factor?*

Remark 1. *Recently Michaela Ciklova answered negatively to the last question in Question 1 (see the paper "Li-Yorke sensitive minimal maps", Nonlinearity, **19**(2006), 517-529).*

Clearly, Li-Yorke sensitivity implies sensitive dependence upon initial conditions. The remaining implications are less clear. Any minimal system which is distal but not equicontinuous has sensitive dependence but is neither Li-Yorke sensitive nor Li-Yorke chaotic since it has no off-diagonal proximal pairs. If X contains a distal point x then (x, y) is proximal iff $x = y$ and so the system is not Li-Yorke sensitive. There exist minimal systems which are almost one-to-one extensions of distal systems but with some uncountable fibers which are scrambled. Such a system is Li-Yorke chaotic but not Li-Yorke sensitive. Currently open is the following:

Question 2. *For minimal systems does Li-Yorke sensitivity imply Li-Yorke chaos? That is, if (X, T) is a Li-Yorke sensitive minimal system then does X contain an uncountable scrambled set?*

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