Ergodic Theory of $\mathbb{Z}^d$-actions
London Math. Soc. Lecture Note Series 228, 1996, 1–61

Ergodic Ramsey Theory–an Update

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5. Some open problems and conjectures.

Our achievements on the theoretical front will be very poor indeed if...we close our eyes to problems and can only memorize isolated conclusions or principles...

–Mao Tsetung, "Rectify the Party's style of work", [Mao], p. 212.

A mathematical discipline is alive and well if it has many exciting open problems of different levels of difficulty. This section's goal is to show that this is the case with Ergodic Ramsey Theory.

To warm up we shall start with some results and problems related to single recurrence. The following result ([K2]) is usually called Khintchine's recurrence theorem (cf. [Pa], p. 22; [Pe], p. 37).

Theorem 5.1. For any invertible probability measure preserving system $(X, \mathcal{B}, \mu, T)$, $\epsilon > 0$, and any $A \in \mathcal{B}$ the set $\{n \in \mathbf{Z} : \mu(A \cap T^n A) \geq \mu(A)^2 - \epsilon\}$ is syndetic.

One possible way of proving Theorem 5.1 is to use the uniform version of von Neumann's ergodic theorem: if U is a unitary operator acting on a Hilbert space $\mathcal{H}$, then for any $f \in \mathcal{H}$

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} U^n f = P_{inv} f = f^*,$$

where the convergence is in norm and P_{inv} is the orthogonal projection onto the subspace of U-invariant elements.

Noting that $\langle f^*, f\rangle = \langle f^*, f^*\rangle$ and taking $\mathcal{H} = L^2(X, \mathcal{B}, \mu)$, $(Ug)(x) :=$

$g(Tx)$, $g \in L^2(X, \mathcal{B}, \mu)$, and $f = 1_A$ one has

$$\begin{aligned}
&\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \mu(A \cap T^n A) \\
=&\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \langle U^n f, f\rangle = \langle f^*, f\rangle \\
=&\langle f^*, f^*\rangle = \langle f^*, f^*\rangle\langle 1, 1\rangle \geq \big(\langle f^*, 1\rangle\big)^2 = \big(\langle f, 1\rangle\big)^2 = \mu(A)^2
\end{aligned}$$

(cf. [Ho]).

The following alternative way of proving Theorem 5.1 is more elementary and has two additional advantages: it enables one to prove a stronger fact, namely the IP*-ness of the set $\{n \in \mathbf{Z} : \mu(A \cap T^n A) \geq \mu(A)^2 - \epsilon\}$ and is easily adjustable to measure preserving actions of arbitrary (semi)groups.

Note first that if A_k, $k = 1, 2, \cdots$ are sets in a probability measure space such that $\mu(A_k) \geq a > 0$ for all $k \in \mathbf{N}$ then for any $\epsilon > 0$ there exist $i < j$ such that $\mu(A_i \cap A_j) \geq a^2 - \epsilon$. Indeed, if this would not be the case, the following inequality would be contradictive for sufficiently large n:

$$n^2 a^2 \leq \Big(\int \sum_{i=1}^n 1_{A_i}\Big)^2 \leq \int \Big(\sum_{i=1}^n 1_{A_i}\Big)^2 = \sum_{i=1}^n \mu(A_i) + 2 \sum_{1\leq i<j\leq n} \mu(A_i \cap A_j)$$

(cf. [G]).

To show that $\{n \in \mathbf{Z} : \mu(A \cap T^n A) \geq \mu(A)^2 - \epsilon\}$ is an IP*-set, let $(n_i)_{i=1}^\infty$ be an arbitrary sequence of integers and let $A_k = T^{n_1+\cdots+n_k} A$, $k \in \mathbf{N}$. By the above remark, there exist $i < j$ such that

$$a^2 - \epsilon \leq \mu(A_i \cap A_j) = \mu(T^{n_1+\cdots+n_i} A \cap T^{n_1+\cdots+n_j} A) = \mu(A \cap T^{n_{i+1}+\cdots+n_j} A).$$

This shows that

$$\{n \in \mathbf{Z} : \mu(A \cap T^n A) \geq \mu(A)^2 - \epsilon\} \cap FS(n_i)_{i=1}^\infty \neq \emptyset$$

and we are done. We remark also that the IP*-ness of the set $\{n \in \mathbf{Z} : \mu(A \cap T^n A) \geq \mu(A)^2 - \epsilon\}$ is equivalent to the "linear" case of Theorem 3.11.

Since in mixing measure preserving systems for any $A \in \mathcal{B}$ one has $\lim_{n\to\infty} \mu(A \cap T^n A) = \mu(A)^2$, we see that in a sense, Khintchine's recurrence theorem is the best possible. We have however the following.

Question 1. Is it true that for any invertible mixing measure preserving system $(X, \mathcal{B}, \mu, T)$ there exists $A \in \mathcal{B}$ with $\mu(A) > 0$ such that for all $n \neq 0$, $\mu(A \cap T^n A) < \mu(A)^2$? How about the reverse inequality $\mu(A \cap T^n A) > \mu(A)^2$?

Definition 5.1. A set $R \subset \mathbf{Z}$ is called a set of *nice recurrence* if for any invertible probability measure preserving system $(X, \mathcal{B}, \mu, T)$ and any $A \in \mathcal{B}$ one has $\limsup_{n\to\infty, n\in R} \mu(A \cap T^n A) \geq \mu(A)^2$.

Exercise 19. Check that all the sets of recurrence mentioned in Sections 1 through 4 are sets of nice recurrence.

A natural question arises whether any set of recurrence at all is actually a set of nice recurrence. Forrest showed in [Fo] that this is not always so. See also [M] for a shorter proof.

We saw in Section 1 that sets of recurrence have the Ramsey property: if R is a set of recurrence and $R = \bigcup_{i=1}^r C_i$ then at least one of C_i, $i = 1, \cdots, r$ is itself a set of recurrence.

Question 2. Do sets of nice recurrence possess the Ramsey property?

A natural necessary condition for a set $R \subset \mathbf{Z} \setminus \{0\}$ to be a set of recurrence is that for any $a \in \mathbf{Z}$, $a \neq 0$, $R \cap a\mathbf{Z} \neq \emptyset$. In particular, the set $\{2^n 3^k : n, k \in \mathbf{N}\}$ is not a set of recurrence. But what if one restricts oneself to some special classes of systems?

Question 3. Is it true that for any invertible weakly mixing system $(X, \mathcal{B}, \mu, T)$ and any $A \in \mathcal{B}$ with $\mu(A) > 0$ there exist $n, k \in \mathbf{N}$ such that $\mu(A \cap T^{2^n 3^k} A) > 0$?

Some sets of recurrence have an additional property that the ergodic averages along these sets exhibit regular behavior. For example, we saw in Section 2 that for any $q(t) \in \mathbf{Z}[t]$ and for any unitary operator $U: \mathcal{H} \to \mathcal{H}$ the norm limit $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^N U^{q(n)} f$ exists for every $f \in \mathcal{H}$. The following theorem, due to Bourgain, shows that much more delicate pointwise convergence also holds along the polynomial sets.

Theorem 5.2 ([Bo3]). For any measure preserving system $(X, \mathcal{B}, \mu, T)$, for any polynomial $q(t) \in \mathbf{Z}[t]$ and for any $f \in L^p(X, \mathcal{B}, \mu)$, where $p > 1$, $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^N f(T^{q(n)} x)$ exists almost everywhere.

Question 4. Does Theorem 5.2 hold true for any $f \in L^1(X, \mathcal{B}, \mu)$?

Another interesting question related to ergodic averages along polynomials is concerned with uniquely ergodic systems. A topological dynamical system (X, T), where X is a compact metric space and T is a continuous self mapping of X is called uniquely ergodic if there is a unique T-invariant probability measure on the σ-algebra of Borel sets in X. The following well known result appeared for the first time in [KB]:

Theorem 5.3. A topological system (X, T) is uniquely ergodic if and only if for any $f \in C(X)$ and any $x \in X$ one has

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \int f \, d\mu,$$

where μ is the unique T-invariant Borel measure.

Question 5. Assume that a topological dynamical system (X, T) is uniquely ergodic and let $p(t) \in \mathbf{Z}[t]$ and $f \in C(X)$. Is it true that for all but a first category set of points $x \in X$ $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^{p(n)}x)$ exists?

The next question that we would like to pose is concerned with the possibility of extending results like Theorem 4.2 and 4.3 to polynomial expressions involving infinitely many commuting operators. We shall formulate it for a special "quadratic" case which is a measure theoretic analogue of Theorem 4.12 for $k = 1$. Recall that an indexed family $\{T_w : w \in \mathcal{W}\}$ of measure preserving transformations of a probability measure space $(X, \mathcal{B}, \mu)$ is said to have the R-property if for any $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $w \in \mathcal{W}$ such that $\mu(A \cap T_w^{-1} A) > 0$.

Question 6. Let $(T_{ij})_{(i,j)\in \mathbf{N}\times\mathbf{N}}$ be commuting measure preserving transformations of a probability measure space $(X, \mathcal{B}, \mu)$. For any finite non-empty set $\alpha \subset \mathbf{N} \times \mathbf{N}$ let $T_\alpha = \prod_{(i,j)\in\alpha} T_{ij}$. Is it true that the family of measure preserving transformations

$$\{T_{\gamma\times\gamma} : \emptyset \neq \gamma \subset \mathbf{N}, \gamma \text{ finite}\}$$

has the R-property?

We move on now to questions related to multiple recurrence.

Definition 5.2. Let $k \in \mathbf{N}$. A set $R \subset \mathbf{Z}$ is a set of *k-recurrence* if for every invertible probability measure preserving system $(X, \mathcal{B}, \mu, T)$ and any $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $n \in R$, $n \neq 0$, such that $\mu(A \cap T^n A \cap T^{2n} A \cap \cdots \cap T^{kn} A) > 0$.

One can show that items (i), (ii), (iv) and (v) of Exercise 6 are examples of sets of k-recurrence for any k. On the other hand, an example due to Furstenberg ([F], p. 178) shows that not every infinite set of differences (item (iii) of Exercise 6) is a set of 2-recurrence (although every such is a set of 1-recurrence).

Question 7. Given $k \in \mathbf{N}$, $k \geq 2$, what is an example of a set of k-recurrence which is not a set of $(k+1)$-recurrence?

Question 8. Given a set of 2-recurrence S, is it true that for any pair T_1, T_2 of invertible commuting measure preserving transformations of a probability measure space $(X, \mathcal{B}, \mu)$ and a set $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $n \in S$ such that $\mu(A \cap T_1^n A \cap T_2^n A) > 0$? (The answer is very likely *no*.) Same question for S a set of k-recurrence for any k.

Question 9. Let $k \in \mathbf{N}$, let $T_1, T_2, \cdots, T_k$ be commuting invertible measure preserving transformations of a probability measure space $(X, \mathcal{B}, \mu)$ and let $p_1(t), p_2(t), \cdots, p_k(t) \in \mathbf{Z}[t]$. Is it true that for any $f_1, \cdots, f_k \in L^\infty(X, \mathcal{B}, \mu)$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T_1^{p_1(n)}x) f_2(T_2^{p_2(n)}x) \cdots f_k(T_k^{p_k(n)}x)$$

exists in L^2-norm? Almost everywhere?

Remark. The following results describe the status of current knowledge: The answer to the question about L^2-convergence is *yes* in the following cases:

(i) $k = 2$, $p_1(t) = p_2(t) = t$ ([CL1]).
(ii) $k = 2$, $T_1 = T_2$, $p_1(t) = t$, $p_2(t) = t^2$ ([FW2]).
(iii) $k = 3$, $T_1 = T_2 = T_3$, $p_1(t) = at$, $p_2(t) = bt$, $p_3(t) = ct$, $a, b, c \in \mathbf{Z}$ ([CL2], [FW2]).

The answer to the question about almost everywhere convergence is *yes* for $k = 2$, $T_1 = T_2$, $p_1(t) = at$, $p_2(t) = bt$, $a, b \in \mathbf{Z}$ ([Bo4]).

Question 10. Let $k \in \mathbf{N}$. Assume that $(X, \mathcal{B}, \mu, T)$ is a totally ergodic system (i.e. $(X, \mathcal{B}, \mu, T^k)$ is ergodic for any $k \neq 0$). Is it true that for any set of polynomials $p_i(t) \in \mathbf{Z}[t]$, $i = 1, 2, \cdots, k$ having pairwise distinct (non-zero) degrees, and any $f_1, \cdots, f_k \in L^\infty(X, \mathcal{B}, \mu)$ one has:

$$\lim_{N\to\infty} \Big\| \frac{1}{N} \sum_{n=1}^{N} f_1(T^{p_1(n)}x) f_2(T^{p_2(n)}x) \cdots f_k(T^{p_k(n)}x) - \prod_{i=1}^{k} \int f_i \, d\mu \Big\|_{L^2(X,\mathcal{B},\mu)} = 0?$$

Remark. It is shown in [FW2] that the answer is *yes* when $k = 2$, $p_1(t) = t$, $p_2(t) = t^2$. See also Theorem 4.1.

We now formulate a few problems related to partition Ramsey theory. A unifying property that many configurations of interest (such as arithmetic progressions or sets of the form $FS(x_j)_{j=1}^n$) have is that they constitute sets of solutions of (not necessarily linear) diophantine equations or systems thereof. A system of diophantine equations is called *partition regular* if for any finite coloring of $\mathbf{Z} \setminus \{0\}$ (or of $\mathbf{N}$) there is a monochromatic solution. For example, the following systems of equations are partition regular:

$$\begin{aligned} x_1 + x_3 &= 2x_2 \\ x_2 + x_4 &= 2x_3 \\ x_3 + x_5 &= 2x_4 \\ x_4 + x_6 &= 2x_5 \end{aligned} \qquad\qquad \begin{aligned} x + y &= t \\ x + z &= u \\ z + y &= v \\ x + y + z &= w \end{aligned}$$

A general theorem due to Rado gives necessary and sufficient conditions for a system of *linear* equations to be partition regular (cf. [Ra], [GRS] or [F2]). The results involving polynomials brought forth in Sections 1-4 hint that there are some nonlinear equations that are partition regular too. For example, the equation $x - y = p(z)$ is partition regular for any $p(t) \in \mathbf{Z}[t]$, $p(0) = 0$. To see this, fix $p(t)$ and let $\mathbf{N} = \bigcup_{i=1}^{r} C_i$ be an arbitrary partition. Arguing as in [B2] one can show that one of the cells C_i, call it C, has the property that it contains an IP-set and has positive upper density. Let $\{n_\alpha\}_{\alpha \in \mathcal{F}}$ be an IP-set in C. According to Theorem 3.11, $\{p(n_\alpha)\}_{\alpha \in \mathcal{F}}$ is a set of recurrence. This together with Furstenberg's correspondence principle gives that for some $\alpha \in \mathcal{F}$,

$$\overline{d}\Big(C_i \cap \big(C_i - p(n_\alpha)\big)\Big) > 0.$$

If $y \in \big(C_i \cap \big(C_i - p(n_\alpha)\big)\big)$ then $x = y + p(n_\alpha) \in C_i$. This establishes the partition regularity of $x - y = p(z)$. In accordance with the third principle of Ramsey theory one should expect that there are actually many x, y, z having the same color and satisfying $x - y = p(z)$. This is indeed so: using the fact that $\{p(n_\alpha)\}_{\alpha \in \mathcal{F}}$ is a set of nice recurrence one can show, for example, that for any $\epsilon > 0$ and any partition $\mathbf{N} = \bigcup_{i=1}^{r} C_i$ one of C_i, $i = 1, 2, \cdots, r$ satisfies

$$\overline{d}\Big(\big\{z \in C_i : \overline{d}\big(C_i \cap \big(C_i - p(z)\big)\big) \geq \big(\overline{d}(C_i)\big)^2 - \epsilon\big\}\Big) > 0$$

(cf. [B2], see also Theorem 0.4 in [BM1]).

Question 11. Are the following systems of equations partition regular?
(i) $x^2 + y^2 = z^2$.
(ii) $xy = u$, $x + y = w$.
(iii) $x - 2y = p(z)$, $p(t) \in \mathbf{Z}[t]$, $p(0) = 0$.

The discussion in this survey so far has concentrated mainly on topological and measure preserving $\mathbf{Z}^n$-actions. Ergodic Ramsey theory of actions of more general, especially non-abelian groups is much less developed and offers many interesting problems.

In complete analogy with the case of the group $\mathbf{Z}$, given a semigroup G call a set $R \subset G$ a set of recurrence if for any measure preserving action $(T_g)_{g\in G}$ of G on a finite measure space $(X, \mathcal{B}, \mu)$ and for any $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $g \in R$, $g \neq e$, such that $\mu(A \cap T_g^{-1}A) > 0$. Different semigroups have all kinds of peculiar sets of recurrence. For example, one can show that the set $\{1 + \frac{1}{k} : \ k \in \mathbf{N}\}$ is a set of recurrence for the multiplicative group of positive rationals. Sets of the form $\{n^\alpha : n \in \mathbf{N}\}$, where $\alpha > 0$, are sets of recurrence for $(\mathbf{R}, +)$. As a matter of fact, one can show (see [BBB]) that for any measure preserving $\mathbf{R}$-action $(S^t)_{t\in\mathbf{R}}$ on a probability space $(X, \mathcal{B}, \mu)$ one has for every $A \in \mathcal{B}$ that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap S^{n^\alpha} A) \geq \mu(A)^2.$$

On the other hand one has the following negative result.

Theorem 5.4 ([BBB]). Let $(S^t)_{t\in\mathbf{R}}$ be an ergodic measure preserving flow acting on a probability Lebesgue space $(X, \mathcal{B}, \mu)$. For all but countably many $\alpha > 0$ (in particular for all positive $\alpha \in (\mathbf{Q} \setminus \mathbf{Z})$) one can find an L^∞-function f for which the averages $\frac{1}{N}\sum_{n=1}^{N} f(S^{n^\alpha}x)$ fail to converge for a set of x of positive measure.

It is possible that the countable set of "good" α coincides with $\mathbf{N}$. Such a result would follow from a positive answer to the following number-theoretical question which we believe is of independent interest.

Question 12. Let us call an increasing sequence $\{a_n : n \in \mathbf{N}\} \subset \mathbf{R}$ *weakly independent* over $\mathbf{Q}$ if there exists an increasing sequence $(n_i)_{i=1}^{\infty} \subset \mathbf{N}$ having positive upper density such that the sequence $\{a_{n_i} : i \in \mathbf{N}\}$ is linearly independent over $\mathbf{Q}$. Is it true that for every $\alpha > 0$, $\alpha \notin \mathbf{N}$, the sequence $\{n^\alpha : n \in \mathbf{N}\}$ is weakly independent over $\mathbf{Q}$? (It is known that the answer is yes for all but countably many α.)

Definition 5.3. Given a (semi)group G, a set $R \subset G$ is called a set of topological recurrence if for any minimal action $(T_g)_{g\in G}$ of G on a compact metric space X and for any open, non-empty set $U \subset X$ there exists $g \in R$, $g \neq e$, such that $(U \cap T_g^{-1}U) \neq \emptyset$.

Exercise 20. Prove that in an amenable group any set of (measurable) recurrence is a set of topological recurrence.

An interesting result due to Kriz ([Kr], see also [Fo], [M]) says that in $\mathbf{Z}$ there are sets of topological recurrence which are not sets of measurable recurrence. While the same kind of result ought to hold in any abelian group, and while for any amenable group sets of measurable recurrence are, according to Exercise 20, sets of topological recurrence, the situation for more general groups is far from clear. We make the following

Conjecture. A group G is amenable if and only if any set of measurable recurrence $R \subset G$ is a set of topological recurrence.

An intriguing question is, what is the right formulation of the Szemerédi (or van der Waerden) theorem for general group actions. In this connection we want to mention a very nice noncommutative extension of Theorem 1.19 which was recently obtained by Leibman in [L2]: he was able to show that the conclusion of Theorem 1.19 holds if one replaces the assumption about the commutativity of the measure preserving transformations T_i by the demand that they generate a nilpotent group. He also proved earlier in [L1] a topological van der Waerden-type theorem of a similar kind. This should be contrasted with an example due to Furstenberg of a pair of homeomorphisms T_1, T_2 of a compact metric space X generating a metabelian group such that no point of X is simultaneously recurrent for T_1, T_2 (this implies that for metabelian groups one should look for another formulation of a Szemerédi-type theorem).

A possible way of extending multiple recurrence theorems to a situation involving non-commutative groups is to consider a finite family of pairwise commuting actions of a given group. Results obtained within such framework ought to be called *semicommutative*. We have the following

Conjecture. Assume that G is an amenable group with a Følner sequence $(F_n)_{n=1}^{\infty}$. Let $(T_g^{(1)})_{g\in G}, \cdots, (T_g^{(k)})_{g\in G}$ be k pairwise commuting measure preserving actions of G on a measure space $(X, \mathcal{B}, \mu)$ ("pairwise commuting" means here that for any $1 \le i \ne j \le k$ and any $g, h \in G$ one has $T_g^{(i)} T_h^{(j)} = T_h^{(j)} T_g^{(i)}$). Then for any $A \in \mathcal{B}$ with $\mu(A) > 0$ one has:

$$\liminf_{n\to\infty} \frac{1}{|F_n|} \sum_{g\in F_n} \mu(A \cap T_g^{(1)} A \cap T_g^{(1)} T_g^{(2)} A \cap \cdots \cap T_g^{(1)} T_g^{(2)} \cdots T_g^{(k)} A) > 0.$$

Remarks. (i) We have formulated the conjecture for amenable groups for two major reasons. First of all, the conjecture is known to hold true for $k = 2$ ([BMZ], see also [BeR]). Second, in case the group G is countable, a natural analogue of Furstenberg's correspondence principle, which was formulated in Section 1, holds and allows one to obtain combinatorial corollaries which, should the conjecture turn out to be true for any k, contain Szemerédi's theorem as quite a special case.

(ii) The "triangular" expressions

$$A \cap T_g^{(1)} A \cap T_g^{(1)} T_g^{(2)} A \cap \cdots \cap T_g^{(1)} T_g^{(2)} \cdots T_g^{(k)} A$$

appearing in the formulation of the conjecture seem to be the "right" configurations to consider. See the discussion and counterexamples in [BH2] where a topological analogue of the conjecture is treated (but not fully resolved). We suspect that the answer to the following question is, in general, negative.

Question 13. Given an amenable group G and a Følner sequence $(F_n)_{n=1}^{\infty}$ for G, let $(T_g)_{g\in G}$ and $(S_g)_{g\in G}$ be two commuting measure preserving actions on a probability space $(X, \mathcal{B}, \mu)$. Is it true that for any $A \in \mathcal{B}$ the following limit exists:

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g\in F_n} \mu(T_g A \cap S_g A)?$$

We want to conclude by formulating a conjecture about a density version of the polynomial Hales-Jewett theorem which would extend both the partition results from [BL2] and the density version of the ("linear") Hales-Jewett theorem proved in [FK4]. For $q, d, N \in \mathbf{N}$ let $\mathcal{M}_{q,d,N}$ be the set of q-tuples of subsets of $\{1, 2, \cdots, N\}^d$:

$$\mathcal{M}_{q,d,N} = \big\{(\alpha_1, \cdots, \alpha_q) : \alpha_i \subset \{1, 2, \cdots, N\}^d,\ i = 1, 2, \cdots, q\big\}.$$

Conjecture. For any $q, d \in \mathbf{N}$ and $\epsilon > 0$ there exists $C = C(q, d, \epsilon)$ such that if $N > C$ and a set $S \subset \mathcal{M}_{q,d,N}$ satisfies $\frac{|S|}{|\mathcal{M}_{q,d,N}|} > \epsilon$ then S contains a "simplex" of the form:

$$\big\{(\alpha_1, \alpha_2, \cdots, \alpha_q), (\alpha_1 \cup \gamma^d, \alpha_2, \cdots, \alpha_q), (\alpha_1, \alpha_2 \cup \gamma^d, \cdots, \alpha_q),$$
$$\cdots, (\alpha_1, \alpha_2, \cdots, \alpha_q \cup \gamma^d)\big\},$$

where $\gamma \subset \mathbf{N}$ is a non-empty set and $\alpha_i \cap \gamma^d = \emptyset$ for all $i = 1, 2, \cdots, q$.

Remark. For $d = 1$ the conjecture follows from [FK4]. This paper contains a wealth of related material and is strongly recommended for rewarding reading.

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