Density of periodic orbit measures for piecewise monotonic interval maps

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Let $T : [0,1] \to [0,1]$ be a piecewise monotonic map, this means there exists a partition $\mathcal{Z}$ of $[0,1]$ into finitely many pairwise disjoint open intervals with $\bigcup_{Z\in\mathcal{Z}} \overline{Z} = [0,1]$, such that for every $Z \in \mathcal{Z}$ the map $T|_Z$ is continuous and strictly monotonic. (Note that T need not be continuous at the endpoints of the intervals in $\mathcal{Z}$.) Denote the set of all T-invariant Borel probability measures on $[0,1]$ by $\mathcal{M}(T)$, and let $\mathcal{P}(T)$ be the set of all T-invariant Borel probability measures concentrated on a periodic orbit of T. The set $\mathcal{M}(T)$ is endowed with the weak star-topology.

Problem. Suppose that T is topologically transitive and $h_{\mathrm{top}}(T) > 0$. Is $\mathcal{P}(T)$ dense in $\mathcal{M}(T)$?

It is known that the above problem has a positive answer in the following cases:

- The map T is continuous (Alexander Blokh, see e.g. [1]). (*Remark*: In this case T need not be piecewise monotonic.)
- The map T is a monotonic mod one transformation (also called Lorenz map), this means there exists a strictly increasing and continuous function $f : [0,1] \to \mathbb{R}$, such that $Tx = f(x) \pmod 1$ (Franz Hofbauer in [2]).
- The map T has two intervals of monotonicity, this means $\operatorname{card} \mathcal{Z} = 2$, where $\mathcal{Z}$ is as above ([3]).

References

[1] A. Blokh, *The 'spectral' decomposition for one–dimensional maps*, Dynamics Reported, vol. 4 (C. K. R. T. Jones, V. Kirchgraber, H. O. Walther, eds.), Springer, Berlin, 1995, pp. 1–59.

[2] F. Hofbauer, *Generic properties of invariant measures for simple piecewise monotonic transformations*, Israel J. Math. **59** (1987), 64–80.

[3] F. Hofbauer, P. Raith, *Density of periodic orbit measures for transformations on the interval with two monotonic pieces*, Fund. Math. **157** (1998), 221–234.