GEOMETRIC MODELS OF PISOT SUBSTITUTIONS AND NON-COMMUTATIVE ARITHMETIC

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Unimodular Substitutions on 2 letters. *Conjecture*: the dynamical system associated with a primitive substitution on 2 letters, with matrix in $SL(2,\mathbb{Z})$, is measurably isomorphic to a circle rotation. There is a very convenient criterium, due to B.Host: Definition: the substitution σ has strong coincidence if there exists n and k such that $\sigma^n(0)$ and $\sigma^n(1)$ have same letter of index k, and the 2 prefixes of order k have the same number of 0 and 1 (in other word, these two prefixes have the same abelianization). Host proved that every unimodular substitution on 2 letters with strong coincidence generates a dynamical system that is isomorphic to a circle rotation. No example of unimodular substitution without strong coincidence is known, and the above conjecture says that such an example does not exist. Remark that there are primitive non unimodular substitutions without coincidences, the simplest being Morse substitution ($0 \mapsto 01$, $1 \mapsto 10$). It is tempting to generalize for more letters; the simplest natural setting is that of unimodular Pisot substitution, generalized strong coincidence conditions are easy to define, and again no example of a unimodular Pisot substitution without strong coincidences is known. In that case, strong coincidence condition implies that the dynamical system is isomorphic to an exchange of pieces by translation, and an additional condition implies that it is isomorphic to a toral translation. One can again *conjecture* that every unimodular Pisot substitution is isomorphic to a toral rotation, although a counterexample seems more likely in 3 or more letters. See [S] and the forthcoming papers of Anne Siegel for more information on the subject. *The above conjecture has been recently proved by Barge and Diamond, see [BD]. They proved that all Pisot substitutions, even not unimodular, and on d letters, have at least one strong coincidence. This proves that all Pisot substitutions on 2 letters have discrete spectrum, see [HS]. Extension of this result to 3 or more letters seems difficult.*

Sequences of complexity $2n+1$. Sequences of complexity $2n+1$ with only one left special word and one right special word of any length are very well understood. They are linked to some kind of generalized continued fraction, and one can prove that, in the case when this continued fraction is periodic, they are symbolic dynamics of a toral translation with respect to a partition with fractal boundary. On the other hand, Ferenczi, Cassaigne and Zamboni proved that some sequences with unbounded partial quotients cannot be associated with such a rotation, since they are totally unbalanced [FCZ]. One should like to understand which sequences are associated to toral rotations, and what rotations can be obtained in this way; this might lead to interesting generalized continued fractions algorithms.

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Positive integral matrices and non-commutative arithmetic. The set $SL(2,N)$ of positive integral matrices of determinant 1 is obviously a monoid for the multiplication, and it is easy to prove that it is a free monoid on 2 generators. This is deeply linked to the continued fraction expansion: the unique decomposition of a matrix in $SL(2,N)$ gives the period of the continued fraction expansion of the slope of the positive eigenvector of this matrix. It is again tempting to extend this in higher dimension; in $SL(3,N)$, one can define units (matrices of permutations), and indecomposable matrices (that is, matrices M that cannot be written $M = AB$, where A and B are not units). In $SL(2,N)$, the fact that it is a free monoid implies that there are only 2 indecomposable matrices, giving a rather trivial arithmetic. By contrast, it is known (Rivat) that there are infinitely many indecomposable matrices in $SL(3,N)$. *Question*: Can one characterize indecomposable matrices in $SL(3,N)$ or

$SL(k,N)$? Can one give theorem on the possible decompositions in

indecomposable elements? It should be also interesting to know the repartition of these indecomposable matrices, and the number of decompositions of a given matrices as a product of indecomposable matrices.

Monoid of substitutions and doubly non-commutative arithmetic. Instead of matrices, one can study substitutions (morphisms of the free monoid on d generators), which project to matrices by abelianization. This set is naturally a monoid, that contains the submonoid of unimodular substitutions (with abelianization of determinant $+1$ or -1) and the submonoid of invertible substitution (that extend to an automorphism of the free group). The structure of the set of invertible substitutions on 2 letters (the Sturm monoid) is very well understood (see [BS], [Wen], [Lo]); however, even in that case, the set of unimodular substitution is already not completely understood. few things are known on 3 or more letters; Wen and Zhang [WZ] have proved that there are infinitely many indecomposable elements in the monoid of invertible substitutions on 3 letters; they have also proved that its image under abelianization is strictly included in the set of matrices of determinant +1 or -1: one can exhibit an 3 by 3 invertible matrix with positive integer coefficient that is not the matrix of any invertible substitution on 3 letters. It would be very interesting to understand better these monoids for many questions in low complexity dynamics. There are useful references about this subject (see [SAI], [CS1],[CS2],[S]).

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