DOES MINIMAL ACTION BY THE GROUP OF MEASURE PRESERVING TRANSFORMATIONS IMPLY THE EXISTENCE OF A UNIQUELY ERGODIC TRANSFORMATION FOR A MEASURE ON A CANTOR SET?

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Assume that a Borel probability measure μ on a Cantor set X satisfies $\mu(A) > 0$ if A is opene (=open and nonempty) and $\mu(A) = 0$ if A is countable, ie. μ is *full* and *nonatomic*. For such a measure the *clopen values set*

$$S(\mu) \quad =_{def} \quad \{\mu(A) : A \ \text{ a clopen subset of } X\}$$

is a countable dense subset of the interval $[0,1]$ which includes the endpoints $0,1$. Such a subset is called *grouplike* if it is the intersection of $[0,1]$ with a dense additive subgroup of the reals. For such a measure μ we define for any clopene (= clopen and nonempty) subset A the relative measure μ_A by $\mu_A(B) = \mu(B)/\mu(A)$ for measurable $B \subset A$. Thus, μ_A is a full, nonatomic probability measure on the Cantor set A.

Definition 1. *A full, nonatomic probability measure μ on a Cantor set X is called* good *if for all clopen subsets A, B of X with $\mu(A) < \mu(B)$ there exists a clopen subset A_1 of B such that $\mu(A) = \mu(A_1)$.*

These measures satisfy the following properties, see Akin (2003) *Good measures on Cantor space* (to appear Trans. AMS).

(1) Among good measures the clopen values set provides a complete topological invariant.
(2) If μ is a good measure on X then $S(\mu)$ is grouplike and every grouplike subset is the clopen values set for some good measure.
(3) If μ is a good measure on X and $A \subset X$ is clopene then μ_A is a good measure on A.
(4) If μ is a good measure on X then exists an element T of the group H_μ of μ preserving homeomorphisms of X on X such that μ is the unique invariant measure for T, ie the topological dynamical system (X,T) is *uniquely ergodic*. A fortiori the group H_μ acts minimally on X. In fact, for every $x, y \in X$ there exists $S \in H_\mu$ such that $S(x) = y$.
(5) If μ is a full, nonatomic probability measure on a Cantor set X and the action of the automorphism group H_μ on X is minimal (or topologically transitive) then the action of group H_{μ_A} on A is minimal (resp. topologically transitive) on A for every clopene subset $A \subset X$.
(6) Assume μ is a full, nonatomic probability measure on a Cantor set X. If the automorphism group H_μ acts minimally and for every clopene subset $A \subset X$ the clopen values set $S(\mu_A)$ is grouplike then the measure μ is good.
(7) There exists μ a full, nonatomic probability measure on a Cantor set X which is not good, but, nonetheless, the action of H_μ on X is topologically

transitive and for every clopene subset $A \subset X$ the clopen values set $S(\mu_A)$ is grouplike.

In Glasner and Weiss (1995) *Weak orbital equivalence of minimal Cantor systems* Internat. J. Math. **6**: 559-579, the authors' Lemma 2.5 implies that if T is a uniquely ergodic minimal transformation on a Cantor set then the unique invariant measure is good. Thus, the good measures are exactly those which arise as the invariant measures of uniquely ergodic minimal systems on Cantor sets. [The assumption of minimality merely ensures that the minimal set which is the support of the invariant measure is all of X, or, equivalently, that the measure is full].

Question 1. *Assume μ is a full, nonatomic probability measure on a Cantor set X. If the automorphism group H_μ acts minimally on X must μ be a good measure?*

Observe that in order to obtain an affirmative answer it suffices to prove

$$H_\mu \text{ acts minimally} \quad \Longrightarrow \quad S(\mu) \text{ is grouplike}$$

For then (5) above would imply that $S(\mu_A)$ is grouplike for every clopene subset A of X and μ would be good by (6) above.

An affirmative answer would imply that if T is any homeomorphism on a Cantor set X such that the system (X, T) is minimal then all invariant measures are good. The obvious way to obtain a negative answer would be to look for such a minimal system which is not uniquely ergodic and show that for some invariant measure the clopen values set is not grouplike. This requires some nice way to compute the clopen values set, which, in my case, I have not got.

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