Mike Boyle

Entropy on shrinking scales, and the entropy theory of symbolic extensions

Abstract: This talk is based on the papers [BD, D, DN] listed below. I will expand on the items below.

1. Extension entropy. Consider a homeomorphism T of a compact metric space X. By a symbolic extension of (X,T) we mean a subshift (Y,S) with a factor map φ onto the system (X,T). Given such a map, we define the symbolic extension entropy function h^{φ}_{ext} on the space $\mathcal{M}_T$ of T-invariant Borel probabilities, by setting $h^{\varphi}_{\text{ext}}(\mu) = \max\{h(S,\nu) : \varphi\nu = \mu\}$.

2. Entropy structure. An entropy structure for (X,T) will be an allowed sequence of u.s.c. (upper semicontinuous) functions h_n on $\mathcal{M}_T$, converging to the entropy function h, with all differences $h_n - h_{n-1}$ also u.s.c. The general determination of "allowed" is a complicated business, but here is one example which gives the right intuition. Suppose the system (X,T) admits a sequence of partitions P_n with *small boundaries* (the boundary of the closure of each partition element has μ measure zero for every μ in $\mathcal{M}_T$), and with the maximum diamater of elements of P_n going to zero as $n \to \infty$. Then the sequence h_n defined by $h_n(\mu) = h(\mu, P_n)$ is an entropy structure for (X,T). There are more complicated constructions which provide entropy structures for any system.

3. Sex Entropy Theorem. (We use "sex entropy" to abbreviate "symbolic extension entropy.) Suppose (h_n) is an entropy structure. A bounded function E on $\mathcal{M}_T$ such that every $E - h_n$ is u.s.c. is called a *superenvelope* of the entropy structure. The main result of [BD] is the Sex Entropy Theorem: a bounded function on $\mathcal{M}_T$ is a symbolic extension entropy function if and only if it is affine and a superenvelope of the entropy structure.

4. The function $\mathbf{h_{sex}}$. The infimum of the symbolic extension entropy functions h_φ above is called the *symbolic extension entropy function* and denoted h_{sex}. This u.s.c. function is a very fine probe into the complexity of the system (X,T), and evidently is intimately connected with the way entropy emerges on ever smaller scales. The Sex Entropy Theorem leads to a variety of results, estimates and examples. For one thing: the maximum of h_{sex} need not be assumed on an ergodic measure. For another: the infimum of the topological entropies of symbolic extensions of (X,T) need not be achieved. For another: the infimum of the entropies of symbolic extensions of (X,T) (the topological sex entropy of T) equals the maximum of h_{sex}.

4. The subtlety of sex entropy. Starting with an entropy structure (h_n), there is an inductive construction of functions u_α (with $u_\alpha \le u_\beta$ when $\alpha < \beta$) such that $h_{\text{sex}} - h = u_\alpha$ if $u_\alpha = u_{\alpha+1}$. The construction always terminates at some countable ordinal α — but the induction in general is transfinite. This indicates the subtlety of even the topological sex entropy of T, in contrast to the usual topological entropy.

5. Shrinking scales and entropy structures. As sex entropy is extremely subtle and is still only a reflection of how entropy emerges on shrinking scales, we appreciate the problem of finding appropriate structures to describe "emergence of entropy on shrinking scales". The approach of Downarowicz [D] has been to use entropy structures. He defines an appropriate

equivalence relation on sequences of functions, and declares sequences which are equivalent to the entropy structures of [BD] to be also entropy structures, and constructs them in various natural ways. The equivalence class for a system is a master entropy invariant, determining in particular the entropy function on measures and the set of symbolic entropy functions. It is a previously unseen ghost in the machine, ruling all of entropy; not always accessible, but accessible enough for theorems, examples and sometimes estimates.

6. Smooth examples. Building from [BD], for $1 \leq \alpha < \infty$, Downarowicz and Newhouse constructed certain generic families of C^α diffeomorphisms with topological symbolic extension entropy strictly greater than the topological entropy.

References

[BD] M. Boyle and T. Downarowicz, *The entropy theory of symbolic extensions*, Math. Inventiones to appear.

[D] T. Downarowicz, *Entropy Structure*, preprint (2003).

[DN] T. Downarowicz and S. Newhouse, *Symbolic extensions in smooth dynamical systems*, preprint (2002).