SPECIAL α-LIMIT SETS

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Abstract. We investigate the notion of the special α-limit set of a point. For a continuous selfmap of a compact metric space, it is defined as the union of the sets of accumulation points over all backward branches of the map. The main question is whether a special α-limit set has to be closed. We show that it is not the case in general. It is unknown even whether a special α-limit set has to be Borel or at least analytic (it is in general an uncountable union of closed sets). However, we answer this question affirmatively for interval maps for which the set of all periodic points is closed. We also give examples showing how those sets may look like and we provide some conjectures and a problem.

1. Introduction

Let (X, f) be a dynamical system given by a compact metric space X and a continuous map $f : X \to X$.

The ω-limit set of $x \in X$, $\omega(x)$, is the set of points of accumulation of the sequence $(f^n(x))_{n=0}^{\infty}$. If f is a homeomorphism, one can define the α-limit set of x, $\alpha(x)$, as the ω-limit set of x for f^{-1}. However, if f is not a homeomorphism, this simple way does not work.

The standard solution is to define the *α-limit set* $\alpha(x)$ of x as the set of limits of all convergent sequences $(x_{n_i})_{i=0}^{\infty}$ such that $f^{n_i}(x_{n_i}) = x$ and $\lim_{i\to\infty} n_i = \infty$. However, one can also think of different solutions.

A *backward branch* of x is a sequence $(x_n)_{n=0}^{\infty}$ such that $x_0 = x$ and $f(x_n) = x_{n-1}$. The *α-limit set of the backward branch* $(x_n)_{n=0}^{\infty}$ is the set of points of accumulation of this sequence.

In 1992, M. Hero [6] explored still different path. He defined the *special α-limit set* $s\alpha(x)$ of x as the union of the α-limit sets over all backward branches of x. Clearly, always $s\alpha(x) \subset \alpha(x)$. If we want to specify that we are using the map f, we will write $s\alpha(x, f)$ instead of $s\alpha(x)$ (and similarly $\omega(x, f)$ and $\alpha(x, f)$ instead of $\omega(x)$ and $\alpha(x)$, respectively). We denote $\mathrm{SA}(f) = \bigcup_{x\in X} s\alpha(x, f)$.

To formulate some known facts on special α-limit sets, we recall some notions and fix notations. By $\mathrm{Orb}(x)$ or $\mathrm{Orb}(x, f)$ we denote the orbit of the point x, i.e. the set $\{x, f(x), f^2(x), \dots\}$. By $\mathrm{Fix}(f)$, $\mathrm{Per}(f)$, $\mathrm{Rec}(f)$ and $\Omega(f)$ we denote, respectively, the set of fixed points, periodic points, recurrent points and nonwandering points of f.

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Recall that x is a *recurrent point* if $x \in \omega(x)$ and it is a *nonwandering point* if for every neighbourhood U of x there is $n \geq 1$ such that $f^n(U) \cap U \neq \emptyset$.

By $\Lambda^\infty = \Lambda^\infty(f)$ we denote the *attracting center* defined as follows. First, for a subset Y of X, put $\Lambda(Y, f) = \bigcup_{x \in Y} \omega(x, f)$. Then $\Lambda^\infty(f) = \bigcap_{n=1}^\infty \Lambda^n(f)$, where $\Lambda^1(f) = \Lambda(X, f)$ and $\Lambda^n(f) = \Lambda(\Lambda^{n-1}(f))$ for every $n > 1$. Further denote $\Gamma(f) = \bigcup_{x \in X} \gamma(x, f)$ where $\gamma(x, f) = \omega(x, f) \cap \alpha(x, f)$. For *interval* maps, by [13], we have

$$\mathrm{Per}(f) \subset \mathrm{Rec}(f) \subset \Gamma(f) = \Lambda^2(f) = \Lambda^\infty(f) \subset \overline{\mathrm{Per}(f)} \subset \Lambda^1(f) \subset \Omega(f) \tag{1}$$

and, by [5], $x \in \Omega(f)$ if and only if $x \in \alpha(x)$.

For a continuous selfmap f of the *interval* $I = [0, 1]$, Hero's main results are the following ones.

- (H1) The following are equivalent:
  - (1) $x \in s\alpha(y)$ for some y,
  - (2) $x \in s\alpha(x)$,
  - (3) $x \in \Lambda^\infty$.
- (H2) $\mathrm{SA}(f) \setminus \mathrm{Rec}(f) \neq \emptyset$ if and only if f has a periodic point with period not a power of two.

To get (H1) and (H2), Hero proved, among others, the following facts for interval maps.

- (H3) If $x \in \mathrm{Rec}(f)$ then $x \in s\alpha(x)$.
- (H4) If $x \in \mathrm{SA}(f)$ then $x \in s\alpha(x)$ and $x \in \Gamma(f)$.
- (H5) If $x \in \Gamma(f)$ then $x \in s\alpha(x)$.

Combining (H3–H5) with (1), we get for interval maps

$$\mathrm{Rec}(f) \subset \mathrm{SA}(f) = \Gamma(f) = \Lambda^2(f) = \Lambda^\infty(f) \subset \Lambda^1(f).$$

Notice that the inclusion $\mathrm{SA}(f) \subset \Lambda^1(f)$ can be reformulated as follows:

- (H6) Every special α-limit point is an ω-limit point.[1]

Another interesting result is the following.

- (H7) If $y \in s\alpha(x)$ then the orbit closure $\overline{\mathrm{Orb}(y)} = \mathrm{Orb}(y) \cup \omega(y) \subset s\alpha(x)$.

Hero proved (H7) for interval maps but one can see that the proof works in general.

The special α-limit sets have been studied also on graphs and dendrites, see [10, 11, 12]. Let us also mention that the α-limit sets of the backward branches $(x_n)_{n=0}^\infty$ of a map have been studied in [3].

To study special α-limit sets is more complicated than to study ω-limit sets or α-limit sets. While it is clear that the ω-limit sets, α-limit sets and α-limit sets of backward branches are always closed, the situation with the special α-limit sets is unclear. In general, those sets are uncountable unions of closed sets, so a priori their topology may be very complicated. Are those sets closed? If not, are they Borel or at least analytic? And are there any other constraints on them? Those questions can be asked in a general case, but as always, the special case when $X = I = [0, 1]$ is the closed unit interval promises more results than the general one. Thus, in this paper we will concentrate mainly on this special case.

[1] This can be alternatively proved as a trivial consequence of a Sharkovsky's result, see [9], saying that a point $c \in [0, 1]$ lies in $\Lambda^1(f)$ if and only if every open interval containing c contains at least three points of some trajectory. Recall another trivial consequence: the set $\Lambda^1(f)$ is closed.

Our partial answers to the main question, is $s\alpha(x)$ necessarily closed, are as follows.

- In the general case $s\alpha(x)$ does not have to be closed, see Example 2.1.
- For interval maps with closed set of periodic points, $s\alpha(x)$ has to be closed, see Theorem 3.3.

Our conjecture is as follows.

Conjecture 1.1. *For all continuous maps* $[0,1] \to [0,1]$ *all sets* $s\alpha(x)$ *are closed.*

Further, we show that

- not all closed subsets of $[0,1]$ are special α-limit sets for interval maps, see Proposition 3.6, see also Proposition 3.7.

Therefore it is natural to state the following problem.

Problem 1.2. *Characterize all subsets/closed subsets* A *of* $[0,1]$ *for which there exists a continuous map* $[0,1] \to [0,1]$ *and a point* $x \in [0,1]$ *such that* $s\alpha(x) = A$.

For completeness recall that the characterization of ω-limit sets for continuous interval maps is nontrivial but known, see [1]: A subset of $[0,1]$ is an ω-limit set for some continuous map $[0,1] \to [0,1]$ if and only if it is nonempty, closed and either nowhere dense or a union of finitely many nondegenerate closed intervals.

Characterization of α-limit sets for continuous interval maps is trivial (and apparently not mentioned in literature): A subset $A \subset [0,1]$ is an α-limit set for some continuous map $[0,1] \to [0,1]$ if and only if it is closed (possibly empty). One implication is trivial. To prove the converse implication, fix a closed set $A \subset [0,1]$. We need to find a continuous map f and a point x such that $\alpha(x) = A$. Examples with $A = \emptyset$ and $A = [0,1]$ are easy. Otherwise, let J be a nondegenerate closed interval disjoint with the nonempty and proper closed set A. Now fix $x \in J$ and choose a continuous map f such that f is identity on J, $f([0,1]) \subset J$ and $f^{-1}(x) = A$. Then $\alpha(x) = A$.

2. General case

We start by providing an example where a special α-limit set is not closed. It will always be clear from the context whether $(0,1)$ means an open interval or the point of the plane with coordinates 0 and 1.

Example 2.1. First for every $a \in (0,1)$ we define the map $\varphi_a : [0,1] \to [0,1]$ by setting $\varphi_a(x) = \min(x/a, 1)$ (see Figure 1). Clearly, the maps φ_a converge uniformly to the identity as a goes to 1.

Now our space $X \subset \mathbb{R}^2$ is the union of straight line segments I_a joining the point $(0,1)$ with $(a,0)$ over all $a \in \{1 - 1/n : n = 2,3,4,\dots\} \cup \{1\}$. We define the map $f : X \to X$ by defining f on I_1 as the identity and on I_a for $a < 1$ as $\pi_a^{-1} \circ \varphi_a \circ \pi_a$, where $\pi_a : I_a \to [0,1]$ is the projection to the second coordinate. Clearly, f is continuous.

For every $a \in (0,1)$ we have $s\alpha(1, \varphi_a) = \{0,1\}$, while for the identity $s\alpha(1, \mathrm{Id}) = \{1\}$. Thus, we get $s\alpha((0,1), f) = \{(0,1)\} \cup \{(1 - 1/n, 0) : n = 2,3,4,\dots\}$. This set is not closed.

Note that we can modify the above example by replacing the set $\{1 - 1/n : n = 2,3,4,\dots\}$ by the interval $[1/2, 1)$, and then our phase space will be a triangle.

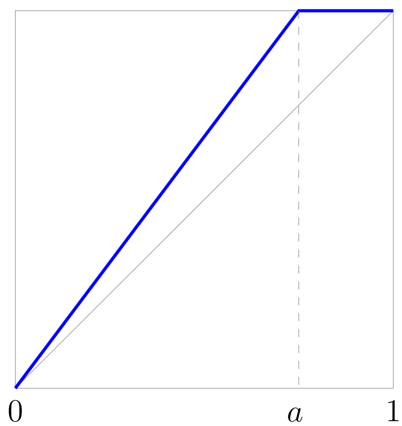

FIGURE 1. The map φ_a.

Now we prove some basic facts that hold in the general case. We work here with a dynamical system (X, f) where X is a compact metric space and $f\colon X \to X$ is continuous.

In general, $s\alpha(x)$ (as well as $\alpha(x)$) may be empty, because for instance it may happen that x has no preimages. Let us investigate closer this problem. For this we need a simple topological lemma.

Lemma 2.2. *If $Y = \bigcap_{n=0}^{\infty} f^n(X)$ then $f(Y) = Y$.*

Proof. Clearly, $f(Y) \subset Y$. Take a point $y \in Y$. Then for every n we have $y \in f^{n+1}(X)$, so there is $x_n \in f^n(X)$ such that $f(x_n) = y$. From the sequence (x_n) we can choose a subsequence convergent to some $x \in X$. For each m almost all points x_n belong to $f^m(X)$, and therefore $x \in f^m(X)$. Thus, $x \in Y$. By continuity, $f(x) = y$. This proves that $Y \subset f(Y)$. □

Note that in the above proof we used the assumption that X is compact. It is easy to construct an example showing that without this assumption the lemma would be false.

The following proposition gives the answer to the question when $s\alpha(x) \neq \emptyset$.

Proposition 2.3. *The following conditions are equivalent:*

(1) $s\alpha(x) \neq \emptyset$,
(2) $\alpha(x) \neq \emptyset$,
(3) $x \in \bigcap_{n=0}^{\infty} f^n(X)$.

In particular, if f is a surjection then $s\alpha(x)$ is nonempty for every $x \in X$.

Proof. Since $s\alpha(x) \subset \alpha(x)$, (1) implies (2). By the definition, if (2) holds then $f^{-n}(x) \neq \emptyset$ for all n, and thus (3) holds. Hence, it remains to prove that (3) implies (1).

By Lemma 2.2, if $x \in \bigcap_{n=0}^{\infty} f^n(X)$ then x has an infinite backward branch. From this branch we can choose a convergent subsequence, so $s\alpha(x) \neq \emptyset$. □

Recall that $f : X \to X$ is *minimal* if every orbit is dense and is *topologically exact* if for every nonempty open set $U \subset X$ there is a positive integer n such that

$f^n(U) = X$. Further, a set $E \subset X$ is f-invariant if $f(E) \subset E$. By $B(x,\delta)$ denote the ball with radius δ, centered at x.

Proposition 2.4. *Let $E \subset X$ be an f-invariant set such that $\bigcup_{n=1}^{\infty} f^n(U) \supset X \setminus E$ for every nonempty open set $U \subset X$. Then*

$$s\alpha(x) = X \quad \text{for every } x \in X \setminus E.$$

Thus, if $E = \emptyset$ then $s\alpha(x) = X$ for every $x \in X$. In particular, this is true if f is topologically exact or minimal.

Proof. Fix $x \in X \setminus E$ and $y \in X$. We prove that $y \in s\alpha(x)$.

Due to the assumption, in the ball $B(y,1)$ there is a point y_1 whose f^{k_1}-image (for some $k_1 \geq 1$) is x. Since E is f-invariant and $x \in X \setminus E$, also $y_1 \in X \setminus E$. By the same argument as above, there is a point $y_2 \in B(y,1/2) \setminus E$ with $f^{k_2}(y_2) = y_1$ (for some $k_2 \geq 1$). Continuing this way, we find a sequence of points $(y_n)_{n=1}^{\infty}$ in $X \setminus E$ and a sequence of positive integers $(k_n)_{n=1}^{\infty}$ such that $y_n \to y$ and $f^{k_n}(y_n) = y_{n-1}$ for $n \geq 2$ and $f^{k_1}(y_1) = x$. Hence $y \in s\alpha(x)$. $\square$

Note that if the set E in this proposition is not equal to X then it has *empty interior*. Otherwise it contains a nonempty open set U for which then trivially $\bigcup_{n=1}^{\infty} f^n(U) \subset E$. Since $E \neq X$, this contradicts the assumption that $\bigcup_{n=1}^{\infty} f^n(U) \supset X \setminus E$. The proposition thus shows that, under those assumptions, for almost every (in topological sense) point x, the set $s\alpha(x)$ equals the whole space X.

Let us now study what happens if we replace a point with its image or a preimage.

Lemma 2.5. *If $y \in s\alpha(x)$ then*

(1) $\overline{\mathrm{Orb}(y)} \subset s\alpha(x)$*; in particular $f(y) \in s\alpha(x)$,*
(2) $y \in s\alpha(f(x))$ *and* $f(y) \in s\alpha(f(x))$.

Proof. (1) This follows from (H7) and our comment after it.

(2) The first property follows from the definition and the second one is a consequence of it and of (1). $\square$

Lemma 2.6. *If $y \in s\alpha(x)$ then $f^{-1}(y) \cap s\alpha(x) \neq \emptyset$.*

Proof. If $y \in s\alpha(x)$ then there is a sequence of points $(y_n)_{n=1}^{\infty}$ and a sequence of positive integers $(k_n)_{n=1}^{\infty}$ such that $y_n \to y$ and $f^{k_n}(y_n) = y_{n-1}$ for $n \geq 2$ and $f^{k_1}(y_1) = x$. Set $z_n = f^{k_n - 1}(y_n)$. By passing to a subsequence if necessary, we may assume that $z_n \to z$. Then $f(z) = \lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} f^{k_n}(y_n) = \lim_{n\to\infty} y_{n-1} = y$. Hence $z \in f^{-1}(y)$. Since $f^{k_n}(z_{n+1}) = z_n$ for $n \geq 1$ and $f(z_1) = x$, we also have $z \in s\alpha(x)$. $\square$

From Lemmas 2.5 and 2.6 we get immediately the following corollary.

Corollary 2.7. *We have $f(s\alpha(x)) = s\alpha(x)$. In particular, if $s\alpha(x)$ is finite then it is a union of periodic orbits.*

The following lemma will be useful when constructing examples. By $\mathrm{Int}\, C$ we denote the interior of C.

Lemma 2.8. *Suppose that there is an f-invariant (not necessarily closed or open) set C and a point $y \in X$ with the following two properties:*

(a) *y has positive distance from C,*

(b) *y is pre-trapped by C, meaning that there exists a positive integer t such that $f^t(y) \in \operatorname{Int} C$.*

Then $y \notin \mathrm{SA}(f) := \bigcup_{x\in X} s\alpha(x)$.

Proof. Suppose, on the contrary, that $y \in s\alpha(x)$ for some x. Then there is a sequence of points $(y_n)_{n=1}^\infty$ and a sequence of positive integers $(k_n)_{n=1}^\infty$ such that $y_n \to y$, $f^{k_n}(y_n) = y_{n-1}$ for $n \ge 2$ and $f^{k_1}(y_1) = x$. Since f is continuous, by (a) and (b) there is a neighbourhood U of y such that $U \cap C = \emptyset$ and $f^t(U) \subset C$. Then, since $f(C) \subset C$, we get $f^\ell(U) \subset C$ for all $\ell \ge t$. Since $y_n \to y$, there is n_0 such that for all $n \ge n_0$ we have $y_n \in U$. Thus,

$$n \ge n_0 \text{ and } \ell \ge t \Rightarrow f^\ell(y_n) \in C. \tag{2}$$

Consider $t+1$ points $y_{n_0}, y_{n_0+1}, \dots, y_{n_0+t} \in U$. For $L := k_{n_0+1} + k_{n_0+2} + \cdots + k_{n_0+t}$ we have

$$f^L(y_{n_0+t}) = y_{n_0} \in U. \tag{3}$$

However, $n_0 + t \ge n_0$ and for the integer L, which is the sum of t positive integers, we have $L \ge t$. Therefore we can use (2) to get $f^L(y_{n_0+t}) \in C$. This contradicts (3). ☐

Now we investigate how the special α-limit sets behave when we consider the iterations of the map.

Proposition 2.9. *Let n be a positive integer. The special α-limit sets have the following properties:*

(1) $s\alpha(x, f^n) \subset s\alpha(x, f)$,
(2) $s\alpha(x, f) \subset s\alpha(f^n(x), f)$,
(3) $s\alpha(x, f) = \bigcup_{i=0}^{n-1} \bigcup_{f^i(y)=x} s\alpha(y, f^n)$,
(4) $s\alpha(x, f) \subset \bigcup_{j=0}^{n-1} s\alpha(f^j(x), f^n)$.
(5) $\mathrm{SA}(f^n) = \mathrm{SA}(f)$.

Proof. Property (1) follows immediately from the definition, while (2) follows from Lemma 2.5. To prove (3), observe that a backward branch of x for f decomposes into n backward branches of preimages of x for f^n. Property (4) follows from (3) and (2). Finally, one inclusion in (5) follows from (1) and the other one from (3) or (4). ☐

3. Interval maps

Here we prove the main results about the special α-limit sets for interval maps. Thus, $I = [0, 1]$ will be the closed unit interval, and $f : I \to I$ a continuous map.

Lemma 3.1. *Let f be an interval map. If $y \in \alpha(x)$ and $f(y) = y$, then $y \in s\alpha(x)$.*

Proof. If $y = x$, then $f(x) = x$, so $y = x \in s\alpha(x)$. Thus, we may assume that $y < x$.

By the assumption, there exists a sequence $x_n \to y$ such that $f^{k_n}(x_n) = x$ for some $k_n \to \infty$. By choosing a subsequence, we may assume that the sequence (x_n) is monotone and $x_n < x$ for all n.

Assume first that the sequence (x_n) is decreasing. By induction we will find a sequence (y_n) such that $f^m(y_1) = x$ for some m, and $f^{k_n}(y_n) = y_{n-1}$, $y < y_n < x_n$, for all $n \ge 2$. We start by taking $y_1 = x_1$ and $m = k_1$. Now, if y_{n-1} is already chosen, we note that $y_{n-1} \in (y, x)$, $f^{k_n}(y) = y$, and $f^{k_n}(x_n) = x$, so there exists $y_n \in (y, x_n)$

such that $f^{k_n}(y_n) = y_{n-1}$. This completes the induction step. Clearly, $y_n \to y$, so by the definition, $y \in s\alpha(x)$.

Now we assume that the sequence (x_n) is increasing. Then $x_n < y < x$ and $f^{k_n}(x_n) = x$, so there exists $0 \leq m_n < k_n$ such that $f^{m_n}(x_n) < y$ and $f^{m_n+1}(x_n) > y$. We will distinguish two cases.

The first case is that $\sup_n f^{m_n}(x_n) < y$. Denote this supremum by a. Since $x_n \to y$, we may assume that $x_n > a$ for all n. Then we repeat the proof from the case of (x_n) decreasing. Namely, by induction we find a sequence (y_n) such that $f^m(y_1) = x$, and $f^{m_n}(y_n) = y_{n-1}$, $x_n < y_n < y$, for all $n \geq 2$. This time in the induction step we have $y_{n-1} \in (a, y)$, $f^{m_n}(y) = y$, and $f^{m_n}(x_n) \leq a$. Again we get $y \in s\alpha(x)$.

The second case is that $\sup_n f^{m_n}(x_n) = y$. Then, by passing to a subsequence, we may assume that $f^{m_n}(x_n) \to y$. By continuity and since $f(y) = y$, we get $f^{m_n+1}(x_n) \to y$. By taking a subsequence again, we may assume that the sequence $(f^{m_n+1}(x_n))$ is decreasing. Then we can use this sequence instead of the sequence (x_n), and we know already that in this case $y \in s\alpha(x)$. $\square$

Theorem 3.2. *Let f be an interval map. If a point $y \in \alpha(x)$ is periodic, then $y \in s\alpha(x)$.*

Proof. Let $y \in \alpha(x, f)$ be a periodic point of f of period p. Then there exists a sequence of points (y_k) convergent to y, such that $f^{m_k}(y_k) = x$ for some sequence (m_k) going to infinity. There is $j \in \{0, 1, \dots, p-1\}$ such that infinitely many numbers m_k are congruent to j modulo p. By passing to a subsequence if necessary, we may assume that for every k there is a positive r_k with $m_k = pr_k + j$. Put $z_k = f^j(y_k)$. By continuity, $z_k \to f^j(y)$. Moreover, $r_k \to \infty$ and for $g = f^p$ we have $g^{r_k}(z_k) = x$. Hence, $f^j(y) \in \alpha(x, g)$. Since $f^j(y)$ is a fixed point of g, Lemma 3.1 gives us $f^j(y) \in s\alpha(x, g) = s\alpha(x, f^p)$. Then, by Proposition 2.9(1), $f^j(y) \in s\alpha(x, f)$ and, by Lemma 2.5(1), $y = f^{p-j}(f^j(y)) \in s\alpha(x, f)$. $\square$

Once we know that $\alpha(x) \cap \mathrm{Per}(f) \subset s\alpha(x)$, we can specify a class of interval maps for which all special α-limit sets are closed.

Theorem 3.3. *For an interval map f, if the set of all periodic points is closed, then $s\alpha(x) = \alpha(x) \cap \mathrm{Per}(f)$ for every x. In particular, all special α-limit sets are closed.*

Proof. Let f be an interval map for which the set $\mathrm{Per}(f)$ is closed. Then, by a theorem of Sharkovsky (see [8]), the ω-limit set of every point is finite and so it is a periodic orbit. Thus, by (H6), all special α-limit points are contained in $\mathrm{Per}(f)$. Hence, for any point x we have $s\alpha(x) \subset \alpha(x) \cap \mathrm{Per}(f)$. Since Theorem 3.2 gives the converse inclusion, $s\alpha(x) = \alpha(x) \cap \mathrm{Per}(f)$. Since both $\alpha(x)$ and $\mathrm{Per}(f)$ are closed, the set $s\alpha(x)$ is also closed. $\square$

An interval map is called *of type* n if it has a periodic orbit of period n but no periodic orbits of periods preceding n in the Sharkovsky ordering. Additionally, it is *of type* 2^∞ if it has periodic orbits of period 2^n for all n and of no other periods. If f is of type 2^n for some finite n, then $\mathrm{Per}(f)$ consists of all fixed points of f^{2^n}, so it is closed. Therefore, we get a corollary to Theorem 3.3.

Corollary 3.4. *For an interval map of type 2^n for some finite n, all special α-limit sets are closed.*

This is still true for interval maps of type 2^∞ with closed set of periodic points. However, there are well known examples of interval maps of type 2^∞ whose set of periodic points is not closed (see, e.g., [8]). Recall also that for the maps having also periodic points whose period is not a power of two, the set of periodic points is never closed.

Let us now continue with other properties of the special α-limit sets for interval maps (so we assume below that $f : [0,1] \to [0,1]$ is a continuous map). By an *interval* we always mean a *nondegenerate* interval (still, we sometimes emphasize that it is nondegenerate).

Lemma 3.5. *If f is constant on some open interval J, then J contains at most one point of $s\alpha(x)$; this point is periodic and its orbit contains x. In particular, if $K \subset s\alpha(x)$ is an interval, then $f^n(K)$ is an interval (and not a singleton) for every positive integer n.*

Proof. Assume that f is constant on an open interval J, and $y \in J$ belongs to $s\alpha(x)$. Then there are points $z, w \in J$ such that $f^k(w) = z$ and $f^n(z) = x$ for some $k, n \geq 1$. Thus, $f^k(y) = f^k(z) = f^k(w) = z$ and $f^n(y) = f^n(z) = f^n(w) = x$. Hence $f^{k+n}(y) = x$. We get $x = f^k(f^n(y)) = f^k(x)$, so x is periodic. Moreover, $f^k(z) = z$, so z is also periodic. However, $f^n(z) = x$, so x and z must belong to the same periodic orbit. Since $y \in s\alpha(x)$, the point z can be chosen arbitrarily close to y, and this proves that y belongs to the orbit of x. Thus y is periodic. Clearly, the orbit of x, since it is periodic, can contain at most one point of J.

Now let $K \subset s\alpha(x)$ be an interval. We already know that then f is not constant on K, i.e. $f(K)$ is an interval. By Corollary 2.7 we have $f(K) \subset s\alpha(x)$, hence also $f^2(K)$ is an interval. By induction, $f^n(K)$ is an interval for every n. □

A subinterval K of $[0,1]$ is called *periodic* or *weakly periodic* if there is a positive integer k such that $K, f(K), \dots, f^{k-1}(K)$ are pairwise disjoint and $f^k(K) = K$ or $f^k(K) \subset K$, respectively. In such a case, the set $\bigcup_{i=0}^{k-1} f^i(K)$ is called a *cycle* or a *weak cycle*, respectively, of intervals of period k.

Proposition 3.6. *Assume that* $\operatorname{Int} s\alpha(x) \neq \emptyset$. *Then there are only finitely many nondegenerate components of $s\alpha(x)$ and they form a cycle containing x.*

Proof. Let J be a nondegenerate component of $s\alpha(x)$. Then there are $y, z \in \operatorname{Int}(J)$ such that $f^n(y) = z$ for some $n \geq 1$. Since, by Corollary 2.7, $f(s\alpha(x)) = s\alpha(x)$, we have $f^n(J) \subset J$ and by Lemma 3.5, $f^i(J)$, $i = 1, \dots, n-1$ are nondegenerate intervals. Thus, we get a weak cycle of intervals in $s\alpha(x)$ of period at most n. If we enlarge those intervals to the components of $s\alpha(x)$ containing them, we get a weak cycle of nondegenerate components of $s\alpha(x)$. Since some points of this weak cycle are eventually mapped to x, the point x itself has to belong to one of those components. If there are two different weak cycles of nondegenerate components of $s\alpha(x)$, one of them does not contain x, a contradiction.

Thus there are an interval K and a positive integer k such that the union of all nondegenerate components of $s\alpha(x)$ equals $\bigcup_{i=0}^{k-1} f^i(K)$, where $K, f(K), \dots, f^{k-1}(K)$ are pairwise disjoint intervals and the interval $f^k(K) \subset K$. We claim that this weak cycle is a cycle, i.e. $f^k(K) = K$. Suppose, on the contrary, that $f^k(K)$ is a proper subinterval of K. Then, regardless of whether K is closed or not, $\operatorname{Int}(K \setminus f^k(K)) \neq \emptyset$.

Let $L \subset K \setminus f^k(K)$ be an open interval. Then $L \subset s\alpha(x)$ and so, by Lemma 3.5, $f^k(L) \subset f^k(K)$ is a nondegenerate interval. Therefore one can choose a point $y \in L$ with $f^k(y) \in \operatorname{Int} f^k(K)$. Since L is open, y has positive distance from $f^k(K)$. Moreover, $f^k(K) \subset K$ and so the set $f^k(K)$ is f^k-invariant. Hence, by Lemma 2.8, $y \notin \mathrm{SA}(f^k)$. By Proposition 2.9(5), $y \notin \mathrm{SA}(f)$. This contradicts the fact that $y \in L \subset s\alpha(x)$. □

Now we will consider simultaneously two situations; if at least one of them occurs, we will say that x is *of cyclic type*. The first one is when the orbit of x is periodic, the second one when there is a nondegenerate component of $s\alpha(x)$. In the second situation, by Proposition 3.6, those components form a cycle and x belongs to one of them. We will call the union of those components (or the periodic orbit of x) the *cycle of x*.

In both cases we can use the standard techniques of combinatorial dynamics (see, e.g., [2]). The closures of the components of the complement of the cycle of x in the convex hull of this cycle are vertices of a *Markov graph*. The graph is directed; there is an arrow from J to K if K is contained in the interval with endpoints $f(a)$ and $f(b)$, where $J = [a, b]$. Then we can use the symbolic dynamics. In particular, for every path in the graph there is a point of the interval with that itinerary.[2] The orbit of this point x goes along the path, that is, $f^n(x)$ belongs to the nth vertex of the path.

Moreover, if the path is finite, of length k, then we can prescribe any point of the kth vertex as $f^k(x)$. Different infinite paths result in different points, except for at most countably many paths. Indeed, if a point y has two different infinite itineraries, then $f^n(y) = x$ for some n, and x is a periodic point. If the period of x is k, then the itinerary of y, starting at $(n + k)$th term or earlier, becomes periodic and goes along the *fundamental loop* of the Markov graph (see [4]). For a given n there are only finitely many such itineraries, so totally this can happen only for countably many itineraries.[3]

We can also speak of the *entropy* of the cycle (or of the graph). It is the minimal possible topological entropy of a continuous interval map with this cycle. It is equal to the logarithm of the spectral radius of the transition matrix of its Markov graph (see Corollary 4.4.8 of [2]). In particular, if the period of the cycle is not a power of 2, then its entropy is positive (see Corollary 4.4.18 of [2]).

Proposition 3.7. *Assume that x is of cyclic type and the entropy of its cycle is positive. Then the cardinality of $s\alpha(x)$ minus the cycle of x is continuum.*

Proof. Clearly, this cardinality cannot be larger than continuum. We will show that it is at least continuum.

Since the entropy of the Markov graph is positive, it is easy to see that there is a transitive subgraph G of positive entropy (*transitive* means that there is a path from every vertex to every one). Indeed, if the entropy is positive then there are two distinct loops through some vertex; taking only vertices appearing in those loops

[2]The converse is not true. Due to our definition of the Markov graph, some points may be mapped by f in a way which does not correspond to arrows in the graph.

[3]This is the same situation as with the decimal expansions of the numbers from $[0, 1]$; if we remove countably many expansions, then two different expansions give different numbers.

produces a transitive subgraph with positive entropy. In this situation, there is a vertex J of G from which there are arrows to at least two vertices. This means that $f(J)$ contains at least two vertices, so it contains some element of our cycle. Thus, there is a point $x_0 \in J$ such that $f^j(x_0) = x$ for some $j \geq 0$.

We will write paths as strings of vertices. Let $C = V_0V_1V_2\dots$ be an infinite path in the graph G. Let us construct by induction a sequence of finite paths C_n as follows. Set $C_0 = J$ (a path of length 0; the length of a path is the number of arrows in it). If the path C_{n-1} is defined, then the path C_n is the concatenation of three paths: the beginning of C of length n, i.e. $V_0V_1\dots V_n$, then a connector $W_1^n \dots W_{s(n)}^n$, and then the path C_{n-1}. The connector is chosen in such a way that C_n is a path in G; this is possible since G is transitive.

Recall that $f^j(x_0) = x$ where $x_0 \in J$. During the induction step we can also choose a point $x_n \in V_0$ such that $f^{n+s(n)+1}(x_n) = x_{n-1}$ (remember that x_{n-1} lies in the first vertex of C_n after the connector, i.e. in J if $n = 1$ and in V_0 if $n \geq 2$). In such a way, the sequence (x_n) is a subsequence of the backward branch of x. From this sequence we can choose a subsequence convergent to some point y. Then $y \in s\alpha(x)$. Since $x_m \in V_0$ for all $m \geq 1$, we have $y \in V_0$. Further, if $n \geq 1$ then for all $m \geq n$ the point $f^n(x_m)$ belongs to V_n. Thus, $f^n(y) \in V_n$. Therefore, the path corresponding to y is C.

Since the entropy of G is positive, the number of infinite paths in G has cardinality continuum. Thus, we get a subset of $s\alpha(x)$ of cardinality continuum, and only finitely many of those points can belong to our cycle of points or intervals (if such a point belongs to an interval of the cycle, it has to be its endpoint). □

From Propositions 3.6 and 3.7 we get immediately two corollaries.

Corollary 3.8. *Assume that x is periodic and $s\alpha(x)$ has empty interior. If the orbit of x has positive entropy (in particular, if its period is not a power of 2) then the cardinality of the set of components of $s\alpha(x)$ is continuum.*

Corollary 3.9. *Assume that $s\alpha(x)$ has nonempty interior and the cycle of the non-degenerate components of $s\alpha(x)$ has positive entropy (in particular, if its period is not a power of 2). Then the cardinality of the set of components of $s\alpha(x)$ that are singletons is continuum.*

Now we are going to study special α-limit sets of transitive interval maps. Recall that if $f\colon [0,1] \to [0,1]$ then the endpoint 0 (resp. 1) is *accessible* if there exists $x \in (0,1)$ and $n \geq 1$ such that $f^n(x) = 0$ (resp. $f^n(x) = 1$). If an endpoint is not accessible, it is called *nonaccessible*. Recall also, see e.g. [7, Section 2.1.4], that a transitive interval map is of one of the following two kinds:

- A topologically mixing map. Let $E \subset \{0,1\}$ be the set of those endpoints which are nonaccessible. It can be $E = \emptyset$ or $E = \{0\}$ or $E = \{1\}$ or $E = \{0,1\}$. Always $f(E) = E$ (if $E = \{0,1\}$, then the endpoints are either fixed points or they form a periodic orbit of period 2) and it follows from the definition of an accessible point that also $f^{-1}(E) = E$. Moreover, for every nonempty open set $U \subset [0,1]$, $\bigcup_{n=1}^{\infty} f^n(U) \supset [0,1] \setminus E$.
- A transitive map which is not topologically mixing. Then there is a point $c \in (0,1)$ such that $f([0,c]) = [c,1]$, $f([c,1]) = [0,c]$ and both $f^2|_{[0,c]}$ and $f^2|_{[c,1]}$

are topologically mixing (clearly, then c is the unique fixed point of f). Let E_1 or E_2 be the set of nonaccessible endpoints for $f^2|_{[0,c]}$ or $f^2|_{[c,1]}$, respectively, and let $E = E_1 \cup E_2$. Then $E \subset \{0, c, 1\}$ and both $f(E) = E$ and $f^{-1}(E) = E$. Again, for every nonempty open set $U \subset [0,1]$, $\bigcup_{n=1}^{\infty} f^n(U) \supset [0,1] \setminus E$.

Proposition 3.10. *Let f be a transitive interval map. Except of at most three points, for all other x we have $s\alpha(x) = [0,1]$. For each of those (at most three) exceptional points, $s\alpha(x)$ is nonempty and consists of at most two points.*

Proof. The set E from the above classification of transitive maps has at most three points, is invariant and $\bigcup_{n=1}^{\infty} f^n(U) \supset [0,1] \setminus E$ for every nonempty open set $U \subset [0,1]$. Now use Proposition 2.4 to get that $s\alpha(x) = [0,1]$ for every $x \in [0,1] \setminus E$. Since $f^{-1}(E) = E$ and E is closed, for every $x \in E$ we have $s\alpha(x) \subset E$ (here $s\alpha(x) \neq \emptyset$ since f is surjective). If $E = \{0, c, 1\}$, then we cannot have $s\alpha(x) = E$, since then $f^{-1}(\{c\}) = \{c\}$ and $f^{-1}(\{0,1\}) = \{0,1\}$. □

Corollary 3.11. *Let f be transitive. If $(0,1) \subset s\alpha(x)$ then $s\alpha(x) = [0,1]$.*

Observe, that if $s\alpha(x)$ has to be closed, then the above corollary holds even without the assumption about transitivity.

4. Examples

In this section we provide examples of special α-limit sets of various kinds for interval maps. We also formulate some conjectures for interval maps.

Example 4.1. While an ω-limit set for a continuous interval map cannot be the disjoint union of a closed interval and a singleton, Figure 2 shows that this is possible for a special α-limit set.

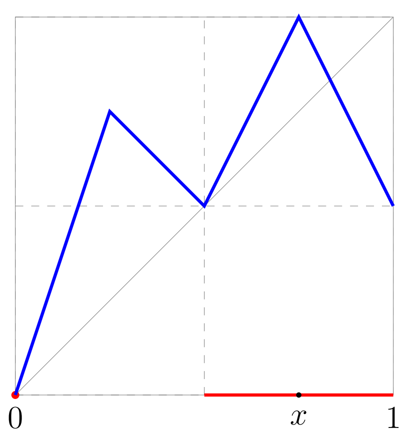

Figure 2. Here $s\alpha(x) = \{0\} \cup [1/2, 1]$

Example 4.2. A preimage of a point in $s\alpha(x)$ need not belong to $s\alpha(x)$. In Figure 2 there are points outside $s\alpha(x)$ which are mapped to $s\alpha(x)$. However, see Lemma 2.6.

Example 4.3. For maps φ_a from Example 2.1, we have $s\alpha(1) = \{0, 1\}$, while $\alpha(1) = [0,1]$ is much larger.

Example 4.4. A constant piece in the previous example is not that important, see Figure 3.

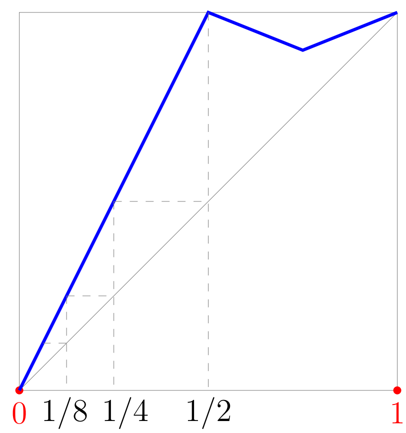

FIGURE 3. Here $s\alpha(1) = \{0, 1\}$ and $\alpha(1) = \{1/2^{n-1} : n \in \mathbb{N}\} \cup \{0\}$

Example 4.5. Let $f(0) = 0$, $f(1/4) = f(3/4) = 1/2$, $f(1) = 1$ and let f be linear in between, see Figure 4. Then it is a continuous map $[0, 1] \to [0, 1]$ and $s\alpha(1/2) = \{0, 1/2, 1\}$, all the three points being fixed for f.

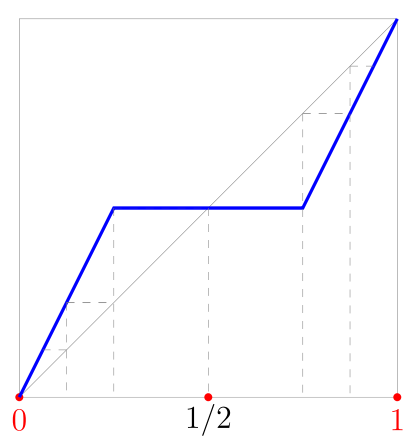

FIGURE 4. Here $s\alpha(1/2) = \{0, 1/2, 1\}$

Example 4.6. For a continuous map $f\colon [0, 1] \to [0, 1]$, $s\alpha(x)$ can be countable infinite. To get $s\alpha(1) = \{1, 1/2, 1/4, \dots\} \cup \{0\}$, consider the map from Figure 5.

Example 4.7. We construct a continuous map $f : [0, 1] \to [0, 1]$ with $s\alpha(1)$ being the middle third Cantor set, see Figure 6. In the construction, the peak over the middle third goes to the very top. In each step, only the peak over the rightmost contiguous interval of that rank goes to the very top.

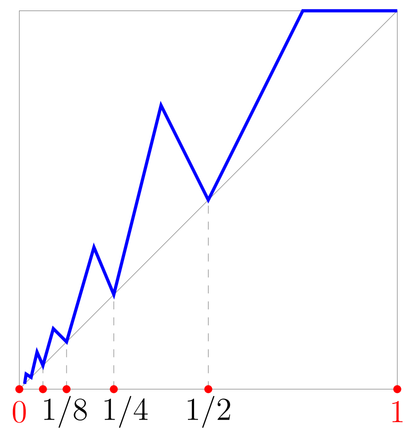

FIGURE 5. Here $s\alpha(1)$ is countable infinite

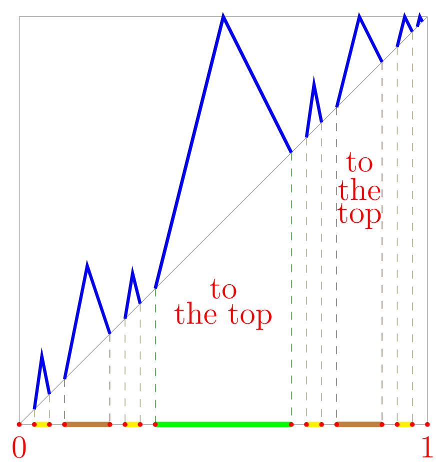

FIGURE 6. Here $s\alpha(1)$ is the middle third Cantor set

Example 4.8. Also it is possible to get $s\alpha(1)$ equal to the union of $\{1\}$ and a Cantor set, see Figure 7.

Example 4.9. If two special α-limit sets intersect each other, they need not be equal, see Figure 8. Compare this with properties of minimal sets.

Conjecture 4.10. *If* $\mathrm{Int}(s\alpha(x) \cap s\alpha(y)) \neq \emptyset$ *then* $s\alpha(x) = s\alpha(y)$.

Example 4.11. If there exists x such that $s\alpha(x) = [0,1]$, f need not be transitive, see Figure 9.

Conjecture 4.12. *If there are* $x_1 \neq x_2$ *with* $s\alpha(x_1) = s\alpha(x_2) = [0,1]$, *then* f *is transitive.*

Example 4.13. In examples of transitive maps in Figure 10 we have $s\alpha(x) = [0,1]$ for all x, with possible exceptions of at most three points, as claimed in Proposition 3.10.

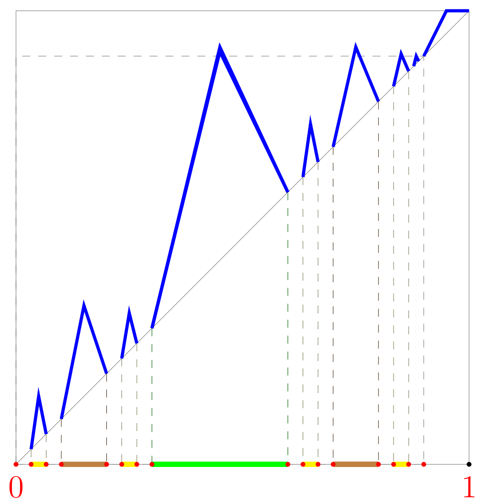

FIGURE 7. Here $s\alpha(1)$ is the union of the Cantor set and the singleton $\{1\}$.

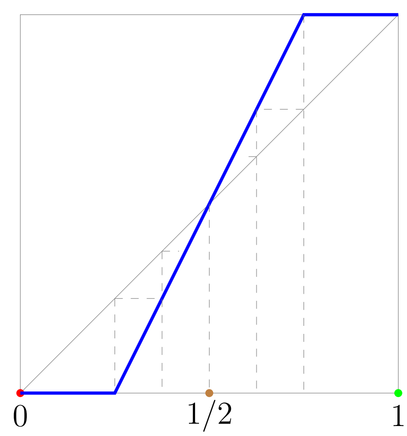

FIGURE 8. Here $s\alpha(1) = \{1/2, 1\}$ and $s\alpha(0) = \{0, 1/2\}$

Conjecture 4.14. *If $s\alpha(x) = [0,1]$ then either f is transitive or there is $c \in (0,1)$ such that $f|_{[0,c]}$ and $f|_{[c,1]}$ are transitive.*

Example 4.15. Consider the map f from Figure 11. Then $y > 1/2$ is pre-trapped by the f-invariant interval $[0, 1/2]$. Hence by Lemma 2.8, $y \notin \bigcup_{x\in[0,1]} s\alpha(x)$.

Conjecture 4.16. *Any isolated point of $s\alpha(x)$ is periodic.*

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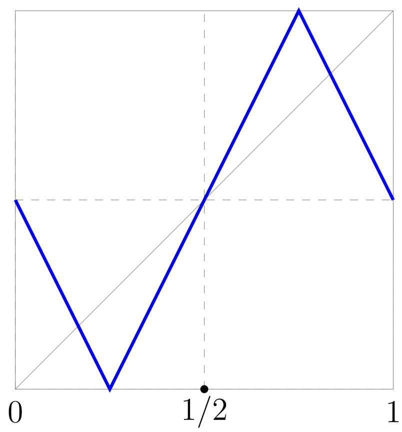

FIGURE 9. Here $s\alpha(1/2) = [0, 1]$ but f is not transitive.

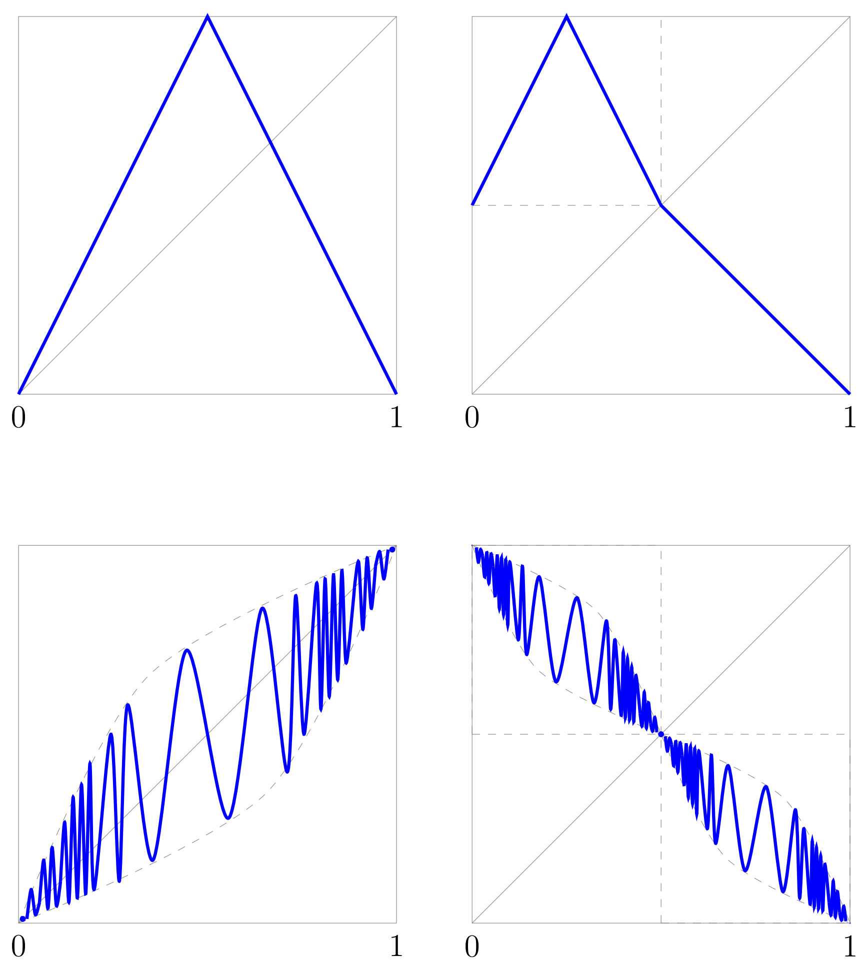

FIGURE 10. At most three points x whose $s\alpha(x)$ is not equal to $[0, 1]$.

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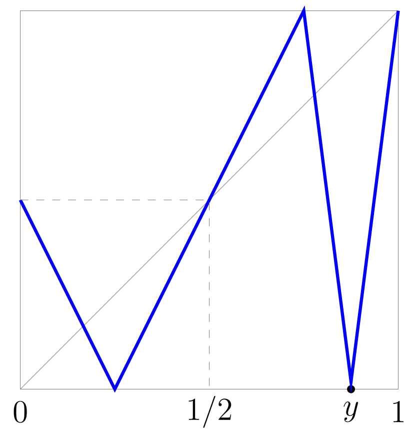

FIGURE 11. Here $y \notin \bigcup_{x\in[0,1]} s\alpha(x)$

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