Asymptotics Uniqueness of Best Rational Approximants in $L^2(\mathbb{T})$ to Cauchy Transforms

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Motivation

Meromorphic Approximation

Rational Approximation

Uniqueness

Symmetric Contours

“Crack” Problem
Motivation
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“Crack” Problem

\[ \int_{\gamma} \Phi \, ds = 0 \]
\[ \int_{\Gamma} \Phi \, ds = 0 \]
Let $u$ be the equilibrium distribution of heat or current. Then

$$
\begin{cases}
\Delta u = 0 & \text{in } D \setminus \gamma \\
\frac{\partial u}{\partial n_\Gamma} = \Phi & \text{on } \Gamma := \partial D, \\
\frac{\partial u^\pm}{\partial n^\pm_\gamma} = 0 & \text{on } \gamma \setminus \{\gamma_0, \gamma_1\}
\end{cases}
$$

where $\Delta u$ is the Laplacian of $u$. 
Methods of crack identification:

- iterative methods: solve direct problem, use some minimizing criteria, crack needs to be localized in advance;
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- semi-explicit methods: localization through approximation of $u$ in the whole domain $D$;
Methods of crack identification:

- iterative methods: solve direct problem, use some minimizing criteria, crack needs to be localized in advance;

- semi-explicit methods: localization through approximation of $u$ in the whole domain $D$;

- method of meromorphic approximants introduced by L. Baratchart and E. B. Saff.
## Idea of the Method

It can be shown that $u$ has well-defined conjugate in $D \setminus \gamma$ and

$$\mathcal{F}(\xi) = u(\xi) - i \int_{\xi_0}^{\xi} \Phi ds, \quad \xi \in \Gamma.$$
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Further,

$$\mathcal{F}(z) = h(z) + \frac{1}{2\pi i} \int_{\gamma} \frac{(\mathcal{F}^- - \mathcal{F}^+)(t)}{z - t} dt, \quad z \in D \setminus \gamma,$$

where $h$ is analytic in $D$ and continuous in $\overline{D}$. 
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where $h$ is analytic in $D$ and continuous in $\overline{D}$.

One approximates $\mathcal{F}$ on $\Gamma$ by meromorphic in $D$ functions and observes the asymptotic behavior of their poles as the number of poles growth large.
Idea of the Method
Let $\mu$ be a complex measure whose support, $S_\mu$, is a subset of the unit disk, $\mathbb{D}$.

Define the **Cauchy transform** of $\mu$ by

$$\mathcal{F}(z) = \mathcal{F}(\mu; z) := \int \frac{d\mu(t)}{z - t}$$

and denote

$$D_\mathcal{F} := \mathbb{C} \setminus S_\mu.$$
Let $h$ be a complex-valued function on the unit circle, $\mathbb{T}$. Then

$$h \in L^p \quad \text{iff} \quad \|h\|_p^p := \sum |h_j|^p < \infty, \quad h_j := \frac{1}{2\pi} \int_{\mathbb{T}} \xi^{-j} h(\xi) |d\xi|,$$

$$h \in L^\infty \quad \text{iff} \quad \|h\|_\infty := \operatorname{ess. sup}_{\mathbb{T}} |h| < \infty.$$
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$$
\begin{align*}
\text{If } h & \in L^p \text{ iff } \|h\|_p^p := \sum |h_j|^p < \infty, \ h_j := \frac{1}{2\pi} \int_{\mathbb{T}} \xi^{-j} h(\xi) |d\xi|, \\
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\end{align*}
$$

Let $p \in [2, \infty]$. The Hardy spaces are defined by

$$
\begin{align*}
H^p & := \{ h \in L^p : h_j = 0, \ j < 0 \}, \\
\overline{H}^p & := \{ h \in L^p : h_j = 0, \ j > -1 \}.
\end{align*}
$$
Fix $p \in [2, \infty]$ and $n \in \mathbb{N}$. The space of meromorphic functions of the degree $n$ is defined as

$$H_n^p := H^p + \mathcal{R}_n,$$

where $\mathcal{R}_n$ is the set of rational functions of type $(n - 1, n)$ with all their poles in $\mathbb{D}$. 
Meromorphic approximation problem:

$$\| \mathcal{F} - g_n \|_p = \inf_{g \in H^n_p} \| \mathcal{F} - g \|_p.$$
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\[ \| \mathcal{F} - g_n \|_p = \inf_{g \in H^p_n} \| \mathcal{F} - g \|_p. \]

This problem always admits a solution:

- **Adamjan, Arov, and Krein**\(^a\), \( p = \infty \);
- **Baratchart and Seyfert**\(^b\) & **Prokhorov**\(^c\), \( p \in [1, \infty) \).

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\(^b\) An \( L^p \) analog of AAK theory for \( p \geq 2 \). *J. Funct. Anal.*, 191(1): 52-122, 2002

Let $g_n = h_n + r_n$, $h_n \in H^2$ and $r_n \in R_n$, be a best approximant for $\mathcal{F}$ in MAP with $p = 2$. Then

$$\|\mathcal{F} - g_n\|_2^2 = \|h_n\|_2^2 + \|\mathcal{F} - r_n\|_2^2.$$
Let $g_n = h_n + r_n$, $h_n \in H^2$ and $r_n \in R_n$, be a best approximant for $F$ in MAP with $p = 2$. Then

$$\|F - g_n\|_2^2 = \|h_n\|_2^2 + \|F - r_n\|_2^2.$$ 

Therefore, we arrive at

**Rational Approximation Problem**

$$\|F - r_n\|_2 = \inf_{r \in R_n} \|F - r\|_2.$$
We say that \( r \in \mathcal{R}_n \) is a \textbf{critical point} in RAP for \( \mathcal{F} \) if

\[
D\Theta(r) = 0,
\]

where \( \Theta(r) := \Theta_{\mathcal{F},n}(r) = \|\mathcal{F} - r\|_2^2 \).
We say that $r \in \mathcal{R}_n$ is a critical point in RAP for $\mathcal{F}$ if

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where $\Theta(r) := \Theta_{\mathcal{F},n}(r) = \|\mathcal{F} - r\|_2^2$.

We say that $r_n$ is irreducible critical point if $r_n$ has exactly $n$ poles. (It is known that all best and locally best rational approximants are always irreducible critical points.)
Let $r_n = p_{n-1}/q_n$ be a critical point in RAP to $\mathcal{F}$. Then

Rational function $r_n$ interpolates $\mathcal{F}$ at the reflections of the zeros of $q_n$ with order 2 in the Hermite sense.
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Rational function \( r_n \) interpolates \( \mathcal{F} \) at the reflections of the zeros of \( q_n \) with order 2 in the Hermite sense.

In other words, \( r_n \) is a multipoint Padé approximant with the implicitly defined interpolation set. Furthermore,

\[
\int t^j q_n(t) \frac{d\mu(t)}{\tilde{q}_n^2(t)} = 0, \quad j = 0, \ldots, n - 1,
\]

where \( \tilde{q}_n(z) = z^n q_n(1/\bar{z}) \) is the reciprocal polynomial.
Let $F$ be an interval contained in $(-1,1)$ with the endpoints $a$ and $b$. Set

- $w(z) = w(F, z) := \sqrt{(z - a)(z - b)}$ to be a holomorphic outside of $F$ function such that $w(z)/z \to 1$ as $z \to \infty$. Then $w^+ = -w^-$ on $F$;
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- $\phi$ to be the conformal map $\overline{\mathbb{C}} \setminus (F \cup F^{-1})$ onto an annulus $\{\rho \leq |z| \leq 1/\rho\}$ such that $\phi(\mathbb{T}) = \mathbb{T}$ and $\phi(\pm 1) = \pm 1$;
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- $\mu$ to be of the form $d\mu(t) = \frac{h(t)dt}{w^+(t)}$, where $h$ is a non-vanishing Dini-continuous function on $F$.

Then the following theorem takes place.
Theorem 1 (Baratchart and Y.)

Let $\{r_n\}$ be a sequence of irreducible critical points in RAT for $F$ with $\mu$ as described. Then

$$(F - r_n)(z) = (D + o(1)) \frac{w^*(z)}{w(z)} \left( \frac{\rho}{\phi(z)} \right)^{2n} D_n(z)$$

locally uniformly in $D_F$, where

- $w^*(z) = zw(1/\bar{z})$;
- $D$ is some constant;
- $\{D_n\}$ is a sequence of outer functions in $\overline{\mathbb{C}} \setminus (F \cup F^{-1})$;
- $|D_n|$ are uniformly bounded away from zero and infinity.
The proof of the above stated result utilizes:

- a priori knowledge of the behavior of the arguments of $q_n$ on $F$ (B, Küstner, Totik$^a$);
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- formulae of strong asymptotics for polynomials satisfying non-Hermitian orthogonality relations with varying measures on arcs (last section and almost Aptekarev\textsuperscript{b});
The proof of the above stated result utilizes:

- a priori knowledge of the behavior of the arguments of $q_n$ on $F$ (B, Küstner, Totik\(^a\));

- formulae of strong asymptotics for polynomials satisfying non-Hermitian orthogonality relations with varying measures on arcs (last section and almost Aptekarev\(^b\));

- special connection (reciprocity) between the polynomial part of the weight, $\tilde{q}_n^2$, and the orthogonal polynomials $q_n$ (B, Stahl, Wielonsky\(^c\)).

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Numerical search of best rational approximants is a nonconvex optimization problem and therefore it often gets trapped in local minima. However, if there is only one local minimum, the descent algorithms converge.
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Definitions

- A critical point $r$ is called **nondegenerate** if $D^2\Theta(r)$ is a nonsingular quadratic form.

- The **Morse index** of a nondegenerate critical point $r$, $M(r)$, is the number of negative eigenvalues of $D^2\Theta_F(r)$. 

Theorem (Baratchart and Olivi)

If all the critical points are nondegenerate and neither of them interpolates $F$ on $T$, then there are only finitely many such points and $\sum (-1)^{M(r)} = 1$.
Definitions

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- The **Morse index** of a nondegenerate critical point \( r \), \( M(r) \), is the number of negative eigenvalues of \( D^2 \Theta_F(r) \).

Theorem (Baratchart and Olivi)\(^a\)

If all the critical points are nondegenerate and neither of them interpolates \( F \) on \( \mathbb{T} \), then there are only finitely many such points and

\[
\sum (-1)^{M(r_c)} = 1.
\]

Theorem (Adopted from Baratchart, Stahl, Wielonsky)\(^a\)

Let \( r_n \) be an irreducible critical point of order \( n \) that does not interpolate \( \mathcal{F} \) on \( \mathbb{T} \). If there exists a meromorphic function \( \Pi \) with at most of \( n - 1 \) poles in \( \mathbb{D} \), continuous on \( \mathbb{T} \), such that

\[
2|\mathcal{F} - r_n| \leq |\Pi - r_n| \quad \text{on} \quad \mathbb{T},
\]

and the winding number

\[
w_{\mathbb{T}}(\mathcal{F} - \Pi) \leq 1 - 2n,
\]

then \( r_n \) is a local minimum, i.e. \( D^2\Theta(r) \) is positive definite.

Set

- \( \varphi_i(z) = z - w(z); \)
- \( \varphi(z) = z + w(z); \)
- \( E_n \) to be a set of \( 2n \) points in \( D := \mathbb{C} \setminus F; \)
- \( \Psi_n(z) := \prod_{e \in E_n} \frac{\varphi(z) - \varphi(e)}{1 - \varphi(z)\varphi(e)}; \)
Definition

A system of sets \( \{E_n\} \) is called **admissible** if, to each \( n \in \mathbb{N} \), there is a one-to-one correspondence \( \Delta_n : E_n \rightarrow E_n \) such that

\[
\sup_{n \in \mathbb{N}} \sum_{e \in E_n} \frac{|\tilde{\varphi}_i(e) - \Delta_n(\varphi_i(e))|}{(1 - |\varphi_i(e)||)(1 - |\Delta_n(\varphi_i(e))||)} < \infty
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\[
\sup_{n \in \mathbb{N}} \sum_{e \in E_n} \frac{|\bar{\varphi}(e) - \Delta_n(\varphi(e))|}{(1 - |\varphi(e)||)(1 - |\Delta_n(\varphi(e))||)} < \infty
\]

and

\[
\lim_{n \to \infty} \sum_{e \in E_n} (1 - |\varphi(e)|) = \infty.
\]
Note

- Admissibility implies that $\psi_n = o(1)$ in $\mathbb{C} \setminus F$ and $|\psi_n^{\pm}|$ are uniformly bounded above on $F$. 
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- Admissibility implies that $\psi_n = o(1)$ in $\mathbb{C} \setminus F$ and $|\psi_n^\pm|$ are uniformly bounded above on $F$.

- Let $r_n$ be an irreducible critical point in RAP to $F$ of order $n$ and let $\{\xi_{j,n}\}$ be its poles. Then $E_n^* := \{1/\bar{\xi}_{j,n}\}$ form an admissible sequence of sets. We shall denote associated “rational” functions by $\psi_n^*$. 
Theorem 2 (Baratchart and Y.)

Let \( \{E_n\} \) be an admissible sequence of sets and \( \mathcal{F} \) be as in Theorem 1. Further, let \( \Pi_n \) be the diagonal multipoint Padé approximant of order \( n \) with the interpolation set \( E_n \). Then

\[
(\mathcal{F} - \Pi_n)(z) = (G + o(1)) \frac{\psi_n(z)}{w(z)} S_n(z)
\]

locally uniformly in \( D_{\mathcal{F}} \), where

- \( G \) is some constant;
- \( \{S_n\} \) is a sequence of outer functions in \( \mathbb{C} \setminus F \);
- \( |S_n| \) are uniformly bounded away from zero and infinity.
We take $\Pi = \Pi_{n-1}$ for some admissible interpolation scheme $\{E_n\}$. By the previous theorem $w(F - \Pi_{n-1}) = 1 - 2n$ whenever $E_n \subset \mathbb{C} \setminus \overline{D}$. Thus, points $\{E_n\}$ need to be chosen in $\mathbb{C} \setminus \overline{D}$ so

$$\left| 1 - \frac{F - \Pi_{n-1}}{F - r_n} \right| > 2 \quad \text{on} \quad \mathbb{T},$$
We take \( \Pi = \Pi_{n-1} \) for some admissible interpolation scheme \( \{E_n\} \). By the previous theorem \( w(\mathcal{F} - \Pi_{n-1}) = 1 - 2n \) whenever \( E_n \subset \mathbb{C} \setminus \overline{\mathbb{D}} \). Thus, points \( \{E_n\} \) need to be chosen in \( \mathbb{C} \setminus \overline{\mathbb{D}} \) so

\[
\left| 1 - \frac{\mathcal{F} - \Pi_{n-1}}{\mathcal{F} - r_n} \right| > 2 \quad \text{on} \quad \mathbb{T},
\]

i.e.

\[
\left| \Psi_{n-1}(z)/\Psi_n^*(z) \right| > 2.
\]
One can construct \( \{ E_n \} \) based on \( \{ E^*_n \} \) so that functions \( \log |\psi_{n-1}/\psi_n^*| \) approximate the Green potential of any signed measure of total mass 2 supported on \( F^{-1} \);
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- there exists a measure on \( F^{-1} \) whose Green potential satisfies \(|1 - G| > 2\) everywhere on \( \mathbb{T} \).
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Theorem 3 (Baratchart and Y.)

Let \( \mathcal{F} \) be as in Theorem 1. Then for all \( n \) large enough there exists a unique critical point of order \( n \).
Let $F$ be now any oriented smooth arc connecting $\pm 1$. Set

- $w(z) := w(F, z)$ defined as before;
- $\varphi(z) = z + w(z)$;
- $E_n$ to be a set of $2n$ points in $D := \mathbb{C} \setminus F$;
- $v_n$ to be a polynomial with zeros at finite points of $E_n$;
- $\Psi_n(z) := \prod_{e \in E_n} \frac{\varphi(z) - \varphi(e)}{1 - \varphi(z)\varphi(e)}$;
- $h$ to be a Dini-continuous non-vanishing function on $F$. 


For \( h \) as above we define geometric mean:

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G_h := \exp \left\{ \int \log h(t) \frac{idt}{\pi w^+(t)} \right\}
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$$G_h := \exp \left\{ \int \log h(t) \frac{idt}{\pi w^+(t)} \right\}$$

and **Szegő function**:

$$S_h(z) := \exp \left\{ \frac{w(z)}{2} \int \frac{\log(h(t)/G_h)}{t - z} \frac{idt}{\pi w^+(t)} \right\}.$$
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\]

Then \( S_h \) is an outer function in \( \overline{\mathbb{C}} \setminus F \), \( S_h(\infty) = 1 \), and \( S^\pm_h \) are continuous functions on \( F \) such that

\[
h = G_h S^+_h S^-_h.
\]
Orthogonal polynomials:

\[ \int_{F} t^{j} q_{n}(t) w_{n}(t) \frac{dt}{w^{+}(t)} = 0, \quad j = 0, \ldots, n - 1. \]
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Functions of second kind:

$$R_{n}(z) := \frac{1}{\pi i} \int_{F} \frac{q_{n}(t) w_{n}(t)}{t - z} \frac{dt}{w^{+}(t)}, \quad z \in \mathbb{C} \setminus F.$$
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Weights:

\[ w_n(t) = \frac{h(t)}{v_n(t)}, \]

where \( E_n \) (that is \( v_n \)) are such that \( \Psi_n = o(1) \) locally uniformly in \( D \) and \( |\psi_n^\pm| = O(1) \) uniformly on \( F \).
Theorem (Baratchart and Y.)

Let \( \{q_n\}_{n \in \mathbb{N}} \) be a sequence of polynomials as above.

Then each polynomials \( q_n \) has exact degree \( n \) for all \( n \) large enough and therefore can be normalized to be monic.

Under such a normalization we have

\[
\begin{cases}
q_n &= (1 + o(1))/S_n \\
R_n w &= (1 + o(1))\gamma_n S_n
\end{cases}
\]

locally uniformly in \( D \)

and

\[
\frac{q_n^2(t)w_n(t)}{\gamma_n w^+(t)} dt \overset{*}{\to} \frac{dt}{w^+(t)},
\]

where \( S_n := S_{w_n}(2/\varphi)^n \), \( \gamma_n := 2^{1-2n}G_{w_n} \), and \( \overset{*}{\to} \) stands for the weak* converges of measures.
Theorem (BY)

Further,

\[ \begin{align*}
q_n &= \frac{(1 + d_n^-)}{S_n^+} + \frac{(1 + d_n^+)}{S_n^-} \\
(R_n w)^\pm &= (1 + d_n^\pm) \gamma_n S_n^\pm
\end{align*} \]

on \( F \),

where \( d_n^\pm \) are continuous on \( F \) and satisfy

\[
\int_F \frac{|d_n^-(t)|^p + |d_n^+(t)|^p}{\sqrt{|1 - t^2|}} |dt| \to 0 \text{ as } n \to \infty
\]

for any \( p \in [1, \infty) \).
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Remarks

- smoothness of $F$ can be reduced. Most likely we can handle quasismooth arcs without twisting points;

- function $h$, in fact, can vanish at a finite number of points in a "controlled manner";

- we can consider a compact family $\{h_n\}$ instead of $h$. 
For any $\alpha \in \mathbb{R}$ denote

$$F_\alpha := \left\{ \frac{i\alpha + x}{1 + i\alpha x} : x \in [-1, 1] \right\}.$$

and for any point $e \in \mathbb{C}$ define

$$e^* = \frac{2i\alpha + (1 - \alpha^2)\bar{e}}{(1 - \alpha^2) + 2i\alpha \bar{e}}.$$

Then

$$e^* = e \quad \text{for any} \quad e \in F_{\alpha}^{-1}$$

and

$$|(\Psi_e \Psi_{e^*})^\pm| = 1 \quad \text{on} \quad F_{\alpha},$$

where

$$\Psi_e(z) := \frac{\varphi(z) - \varphi(e)}{1 - \varphi(z)\varphi(e)}.$$
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Numerics

\[ w_n(t) = \exp \left\{ \frac{2it - 1}{2i - t} \pi \right\} / (t - 2i)^{2n} \]

Zeros of \( q_{10} \) (black) and \( q_{15} \) (red).
\[ w_n(t) = t^{-n}(t + 4i/3)^{-n} \]

Zeros of \( q_{10} \) (black), \( q_{15} \) (red), and \( q_{20} \) (blue).