Nuttall’s Theorem for Padé Approximants

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In 1844 Liouville\textsuperscript{1} constructed the first example of a transcendental number by using continued fractions.

Studying similarities between simultaneous diophantine approximation of real numbers and rational approximation of holomorphic functions, Hermite\textsuperscript{2} proved in 1873 that $e$ is transcendental.

\textsuperscript{1}Sur des classes très étendues de quantités dont la valeur n’est ni algébrique, ni même réductible à des irrationelles algébriques, 1844

\textsuperscript{2}Sur la fonction exponentielle. C.R. Acad. Sci. Paris, 1873
Hermite’s proof is based on the following criterion.

**Lemma**

\( \alpha \) is transcendental if for any \( m \in \mathbb{N} \) and any \( \varepsilon > 0 \) there exist \( m + 1 \) linearly independent vectors of integers \((q_j, p_{j1}, \ldots, p_{jm})\), \( j = 0, m \), such that \( |q_j \alpha^k - p_{jk}| \leq \varepsilon, \ k = 1, m \).

If \( \alpha \) is algebraic, then for some \( m \in \mathbb{N} \) there exist \( a_k \in \mathbb{Z}, \ k = 0, m \), such that \( \sum_{k=0}^{m} a_k \alpha^k = 0 \). Hence,

\[
\sum_{k=1}^{m} a_k (q_j \alpha^k - p_{jk}) + a_0 q_j + \sum_{k=1}^{m} a_k p_{jk} = 0.
\]

Then for some \( 0 \leq j_0 \leq m \), it holds that

\[
1 \leq \left| \sum_{k=1}^{m} a_k (q_{j_0} \alpha^k - p_{j_0 k}) \right| \leq \varepsilon \sum_{k=1}^{m} |a_k|.
\]
Let $n_0, n_1, \ldots, n_m$ be non-negative integers. Set $N := n_0 + \cdots + n_m$ and consider the following system:

$$Q(z)e^{kz} - P_k(z) = \mathcal{O}(z^{N+1}),$$

where $\deg(Q) \leq N - n_0$ and $\deg(P_k) \leq N - n_k$.

Hermite proceeded to explicitly construct these polynomials, which as it turned out have integer coefficients. By evaluating these polynomials at 1 he succeeded in applying the above criterion.
Let \( F(z) = \sum_{k=0}^{\infty} f_k z^k \) be a function holomorphic at the origin. Consider the following system:

\[
Q(z)F(z) - P(z) = \mathcal{O}(z^{m+n+1}),
\]

where \( \text{deg}(Q) \leq n \) and \( \text{deg}(P) \leq m \). This system always has a solution. Indeed,

\[
Q(z)F(z) = \sum_{k=0}^{\infty} \left( \sum_{j+i=k, i \leq n} f_j q_i \right) z^k.
\]
Set $f_{-k} := 0$ for $k > 0$. Then

$$\begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_m \end{pmatrix} = \begin{pmatrix} f_0 & f_{-1} & \cdots & f_{-n} \\ f_1 & f_0 & \cdots & f_{1-n} \\ \vdots & \vdots & \ddots & \vdots \\ f_m & f_{m-1} & \cdots & f_{m-n} \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_n \end{pmatrix}$$

and

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} f_{m+1} & f_m & \cdots & f_{m+1-n} \\ f_{m+2} & f_{m+1} & \cdots & f_{m+2-n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m+n+1} & f_{m+n} & \cdots & f_{m+1} \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_n \end{pmatrix}$$

The latter is a linear system of $n$ equations with $n + 1$ unknowns. Such a system always has a solution. A solution may not be unique, but the ratio $[m/n]_F := P/Q$ always is.
Indeed, let $Q_1(z), P_1(z)$ and $Q_2(z), P_2(z)$ be solutions. Then

\[
Q_2(z)(Q_1(z)F(z) - P_1(z)) = O(z^{m+n+1})
\]

and

\[
Q_1(z)(Q_2(z)F(z) - P_2(z)) = O(z^{m+n+1}).
\]

Therefore,

\[
Q_2(z)P_1(z) - Q_1(z)P_2(z) = O(z^{m+n+1}).
\]

However,

\[
\deg (Q_2P_1 - Q_1P_2) \leq m + n.
\]
Theorem (de Montessus de Ballore$^3$)

Let $F(z)$ be a meromorphic function in $|z| \leq R$ with $N$ poles contained in $0 < |z| < R$. Then $[m/N]F(z)$ converge to $F(z)$ in $|z| \leq R$ in the spherical metric as $m \to \infty$.

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$^3$Sur les fractions continues algébriques, 1902.
$^4$Poles of rows of the Padé table and meromorphic continuation of functions, 1982
$^5$On poles of the $m$-th row of a Padé table, 1984
$^6$On an inverse problem for the $m$-th row of a Padé table, 1985
Theorem (de Montessus de Ballore\textsuperscript{3})

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Theorem (Gonchar\textsuperscript{4} and Suetin\textsuperscript{5,6})

Let $F(z)$ be a holomorphic function at the origin. If the poles of Padé approximants $[m/N]F(z)$ converge to the points $z_1, \ldots, z_N$ as $m \to \infty$, then $F(z)$ can be meromorphically continued to $|z| < R_N := \max |z_k|$ and all the points $z_k$ are singularities of $F(z)$ (polar if $|z_k| < R_N$).

\textsuperscript{3} Sur les fractions continues algébriques, 1902.
\textsuperscript{4} Poles of rows of the Padé table and meromorphic continuation of functions, 1982
\textsuperscript{5} On poles of the $m$-th row of a Padé table, 1984
\textsuperscript{6} On an inverse problem for the $m$-th row of a Padé table, 1985
Theorem (Lubinsky and Saff\textsuperscript{7})

They constructed a one-parameter family of functions $F_q$, holomorphic in $\{|z| < 1\}$ with the \textit{unit circle being the boundary of analyticity}, such that the Padé approximants $[m/N]_{F_q}(z), \ N \geq 1$, had poles clustering on $\{|z| = R_q < 1\}$ as $m \to \infty$.

\textsuperscript{7} Convergence of Padé approximants of partial theta function and Rogers-Szegő polynomials, 1987.

\textsuperscript{8} Convergence of Padé approximants in the general case, 1971.
Theorem (Lubinsky and Saff\textsuperscript{7})

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Theorem (Zinn-Justin\textsuperscript{8})

Let $F(z)$ be a meromorphic function in $|z| \leq R$ with $n$ poles contained in $0 < |z| \leq R$. Then $[m/N]_{F}(z)$ converge to $F(z)$ in measure in $|z| < R$ for any $N \geq n$ as $m \to \infty$.

\textsuperscript{7} Convergence of Padé approximants of partial theta function and Rogers-Szegő polynomials, 1987.

\textsuperscript{8} Convergence of Padé approximants in the general case, 1971.
Let $f$ be a function holomorphic and vanishing at infinity:

$$f(z) = \frac{f_1}{z} + \frac{f_2}{z^2} + \cdots + \frac{f_n}{z^n} + \cdots.$$ 

Further, let $p_n, q_n$ be a pair of polynomials of degree at most $n$ solving the linear system

$$(q_n f - p_n)(z) = O\left(z^{-n-1}\right) \quad \text{as} \quad z \to \infty.$$ 

Such a pair always exists but might not be unique. However, the rational function $[n/n]_f := p_n / q_n$ is unique and is called the diagonal Padé approximant to $f$ of order $n$. 


For any probability Borel measure on $\mathbb{C}$, say $\nu$, set

$$I[\nu] := \int \log \frac{1}{|z-u|} \, d\nu(z) d\nu(u)$$

to be \textit{logarithmic energy}. For any compact set $K$ the \textit{logarithmic capacity} of $K$ is defined by

$$cp(K) := \exp \left\{ - \inf_{\text{supp}(\nu) \subseteq K} I[\nu] \right\}.$$

It is known that either $cp(K) = 0$, i.e., $K$ is polar, or there exists the unique measure $\omega_K$, the \textit{logarithmic equilibrium distribution} on $K$, that realizes the infimum. That is,

$$cp(K) = \exp \left\{ - I[\omega_K] \right\}.$$
In particular, if $D$, the unbounded component of the complement of $K$, is simply connected and $\Phi$ is the conformal map of $D$ onto $\{|z| > 1\}$ such that $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$, then

$$\Phi(z) = \frac{z}{\text{cp}(K)} + \text{terms analytic at infinity}.$$

For $K_r := \{|z| = r\}$, it holds that $\Phi(z) = z/r$ and therefore $\text{cp}(K_r) = r$.

It is said that a property holds quasi everywhere (q.e.) if it holds everywhere except on a polar set.
### Theorem (Nuttall\(^9\) and Pommerenke\(^{10}\))

Let \( f \) be meromorphic function in the complement of a compact polar set \( F \). Then for any \( E \subset \mathbb{C} \setminus F \) and \( \varepsilon > 0 \), it holds that

\[
\lim_{n \to \infty} \text{cp} \left\{ z \in E : |(f - [n/n]_f)(z)|^{1/2n} > \varepsilon \right\} = 0.
\]

In other words, Padé approximants \([n/n]_f\) converge to \( f \) in capacity and the convergence is faster than geometric.

In the case of Pólya frequency series\(^{11}\) and entire functions of very slow and smooth growth\(^{12}\) the convergence is, in fact, uniform.

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\(^9\) The convergence of Padé approximants of meromorphic functions, 1970

\(^{10}\) Padé approximants and convergence in capacity, 1973

\(^{11}\) Arms and Edrei. The Padé tables and continued fractions generated by totally positive sequences, 1970.

Theorem (Rakhmanov$^{13}$)

Let $D$ be an unbounded domain such that $cp(\partial D) > 0$. Then there exists a function $f$ holomorphic in $D$ such that any $z \in D \setminus \{\infty\}$ has a neighborhood in which $[n/n]_f \Rightarrow \infty$ for $n \in \mathbb{N}_z \subset \mathbb{N}$.

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$^{13}$On the convergence of Padé approximants in classes of holomorphic functions, 1980
Each of the three contours is a valid branch cut for this function.
Theorem (Stahl\textsuperscript{14,15})

Let $F(z)$ be holomorphic at infinity, multi-valued, and with all its singularities contained in a compact polar set $E$. Then

(i) there exists the unique maximal domain $D$, such that $[n/n]_{F(z)}$ converge in capacity to $F(z)$ in $D$ as $n \to \infty$;


\textsuperscript{15} Structure of extremal domains associated with an analytic function, 1985.
Theorem (Stahl\textsuperscript{14,15})

Let $F(z)$ be holomorphic at infinity, multi-valued, and with all its singularities contained in a compact polar set $E$. Then

(i) there exists the unique maximal domain $D$, such that $[n/n] F(z)$ converge in capacity to $F(z)$ in $D$ as $n \to \infty$;

(ii) $\Delta := \overline{\mathbb{C}} \setminus D$ is characterized as the set of the smallest logarithmic capacity among all compact sets that make $F(z)$ single-valued in their complement;

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(iii) it holds that $\Delta = E_0 \cup E_1 \cup \bigcup \Delta_j$, where $E_0 \subseteq E$, $E_1$ is finite, and $\Delta_j$ are open analytic arcs connecting the points in $E_0 \cup E_1$.

\textsuperscript{15} Structure of extremal domains associated with an analytic function, 1985.
In particular, if $F(z)$ is an algebraic function, then

$$\Delta = \{a_1, \ldots, a_p\} \cup \{b_1, \ldots, b_{p-2}\} \cup \bigcup \Delta_j,$$

where $a_j$ are some of the branch points, $b_j$ are not necessarily distinct, and the arcs $\Delta_j$ are the negative critical trajectories of the rational quadratic differential

$$\frac{(z - b_1) \cdots (z - b_{p-2})}{(z - a_1) \cdots (z - a_p)} (dz)^2.$$
The following examples are due to Herbert Stahl\textsuperscript{16}. Take

\[
f(z) = \sqrt{\left( \prod_{j=1}^{4} \left( 1 - \frac{z_j}{z} \right) \right) - c}, \quad f(z) \sim \frac{1}{z} \quad \text{as} \quad z \to \infty,
\]

\[z_j = e^{i\phi_j}, \quad \phi_j \in \left\{ \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6} \right\}.
\]

Then for \( c = \sqrt{.70} \) and \( c = \sqrt{.74} \).

\textsuperscript{16} Sets of minimal capacity and extremal domains, manuscript, 2006
It follows from the Cauchy theorem that

\[ 0 = \int_{\Gamma} z^k (q_n f - p_n)(z) \, dz = \int_{\Gamma} z^k (q_n f)(z) \, dz, \quad k \in \{0, \ldots, n-1\}, \]

if \( f \) is holomorphic in the exterior domain of a Jordan curve \( \Gamma \).
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if \( f \) is holomorphic in the exterior domain of a Jordan curve \( \Gamma \). Hence, if

\[ f(z) = \int \frac{d\mu(x)}{x - z} \]

is a Markov function (\( \mu \) is a positive measure compactly supported on \( \mathbb{R} \)), then

\[ \int x^k q_n(x) d\mu(x) = 0, \quad k \in \{0, \ldots, n-1\}. \]
Theorem (Bernstein\textsuperscript{17} and Szeg\text{"o}\textsuperscript{18})

If $p(x)$ is a positive polynomial on $[-1, 1]$ and $d\mu(x) = \frac{dx}{\pi p(x) \sqrt{1-x^2}}$, then

$$
\left( f_p - \left[ \frac{n}{n} \right]_{f_p} \right)(z) = \frac{2}{\sqrt{z^2 - 1}} \frac{\Psi_n^{(1)}(z)}{\left( \Psi_n^{(0)} + p \Psi_n^{(1)} \right)(z)},
$$

where $S_p$ is the unique holomorphic and non-vanishing function in $\overline{\mathbb{C}} \setminus [-1, 1]$ such that $|S_p^\pm|^2 = p$ on $[-1, 1]$ and

$$
\begin{align*}
\Psi_n^{(0)}(z) &:= (z + \sqrt{z^2 - 1})^n S_p(z), \\
\Psi_n^{(1)}(z) &:= (z - \sqrt{z^2 - 1})^n / S_p(z),
\end{align*}
$$

\textit{17 Selected papers, volume 1, 1952}  
\textit{18 Orthogonal Polynomials, volume 23 of Colloquium Publications, 1999}
Notice that

(i) $f_p$ is an algebraic function with two branch points $\pm 1$ and the segment $[-1, 1]$ is the minimal capacity contour for $f_p$;
(ii) the function $\psi_n^{(0)}$ has a pole of order $n$ at infinity, the function $\psi_n^{(1)}$ has a zero of order $n$ there, and $(\psi_n^{(0)})^\pm = p(\psi_n^{(1)})^\mp$ on $[-1, 1]$;
(iii) the Padé approximants $[n/n]_{f_p}$ converge to $f_p$ locally uniformly in $\mathbb{C} \setminus [-1, 1]$.

Strategy

Take an arbitrary algebraic function together with its minimal capacity contour. Find analogs of $\psi_n^{(k)}$. 
Definitions & Classical Results
Weak Convergence & Extremal Domains
Uniform Convergence & Algebraic S-contours

Chebyshëv-type Weight

\[ w_{\Delta}(z) := \prod_{e \in E} \Delta(z - e), \]
where \( E_{\Delta} \subseteq \{a_1, \ldots, a_p\} \cup \{b_1, \ldots, b_{p-2}\} \) is the subset of points with odd bifurcation index, and the function is normalized so \( z - g^{-1} w_{\Delta}(z) \to 1 \) as \( z \to \infty \).
Chebyšëv-type Weight

\[ w^2_\Delta(z) := \prod_{e \in E_\Delta} (z - e), \]

where \( E_\Delta \subseteq \{a_1, \ldots, a_p\} \cup \{b_1, \ldots, b_{p-2}\} \) is the subset of points with odd bifurcation index, and the function is normalized so

\[ z^{-g-1}w_\Delta(z) \to 1 \quad \text{as} \quad z \to \infty. \]
Theorem (Nuttall-Singh\textsuperscript{19} and unknowingly Y)

Let $\Delta$ be the minimal capacity contour for some algebraic function $F$. Further, let $p$ be a non-vanishing polynomial on $\Delta$ and

$$f_p(z) := \frac{1}{\pi i} \int_{\Delta} \frac{1}{x - z} \frac{\text{d}x}{p(x)w_\Delta^+(x)}.$$ 

Then

$$f_p - \left[ \frac{n}{n} \right] f_p = \frac{2}{w_\Delta} \frac{\psi_1^{(1)}}{\psi_0^{(0)} + p \psi_1^{(1)}}.$$ 

\textsuperscript{19}Orthogonal polynomials and Padé approximants associated with a system of arcs, 1977
Denote by $\mathcal{R}$ be the Riemann surface of $w_\Delta$. The genus of $\mathcal{R}$ is $g$.

Further, let $\Delta$ be the chain on $\mathcal{R}$ that lies above $\Delta$. 
(i) Given \( \{P_1, \ldots, P_k\} \) and \( \{Z_1, \ldots, Z_{k-g}\} \) for some \( k > g \), there exist \( \{Z_{k-g+1}, \ldots, Z_g\} \) such that the divisor \( \mathcal{D} = \sum_{j=1}^{k} Z_j - \sum_{j=1}^{k} P_j \) is principal. The collection \( \{Z_{k-g+1}, \ldots, Z_g\} \) is either unique or special.
(i) Given \( \{P_1, \ldots, P_k\} \) and \( \{Z_1, \ldots, Z_{k-g}\} \) for some \( k > g \), there exist \( \{Z_{k-g+1}, \ldots, Z_g\} \) such that the divisor \( D = \sum_{j=1}^{k} Z_j - \sum_{j=1}^{k} P_j \) is principal. The collection \( \{Z_{k-g+1}, \ldots, Z_g\} \) is either unique or special.

(ii) A collection of points \( \{P_1, \ldots, P_l\} \), \( l \leq g \), from \( \mathcal{R} \) is called special if there exists a rational function on \( \mathcal{R} \) with poles only among the points \( P_j \) counting multiplicities. On \( \mathcal{R} \) as described, it happens iff it contains at least one pair of involution-symmetric points.
(i) Given \( \{P_1, \ldots, P_k\} \) and \( \{Z_1, \ldots, Z_{k-g}\} \) for some \( k > g \), there exist \( \{Z_{k-g+1}, \ldots, Z_g\} \) such that the divisor \( \mathcal{D} = \sum_{j=1}^{k} Z_j - \sum_{j=1}^{k} P_j \) is principal. The collection \( \{Z_{k-g+1}, \ldots, Z_g\} \) is either unique or special.

(ii) A collection of points \( \{P_1, \ldots, P_l\}, l \leq g \), from \( \mathbb{R} \) is called special if there exists a rational function on \( \mathbb{R} \) with poles only among the points \( P_j \) counting multiplicities. On \( \mathbb{R} \) as described, it happens iff it contains at least one pair of involution-symmetric points.

(iii) The problem of finding \( \{Z_{k-g+1}, \ldots, Z_g\} \), given \( \{P_1, \ldots, P_k\} \) and \( \{Z_1, \ldots, Z_{k-g}\} \), is a particular case of the more general Jacobi Inversion Problem. Solution of JIP is either special or unique.
Proposition

Denote by $\mathcal{D}_n$ the unique solutions of a special JIP that depends on the periods of Green and holomorphic differentials on $\mathcal{R}$, the weight $p$, and the index $n$ whenever the solution is unique. Denote further by $\mathbb{N}_{JIP}$ the subsequence of indices for which JIP is uniquely solvable and does not contain $\infty^{(0)}$. It holds that $\mathbb{N}_{JIP}$ has gaps of size at most $g$. 
Proposition

Denote by $D_n$ the unique solutions of a special JIP that depends on the periods of Green and holomorphic differentials on $\mathcal{H}$, the weight $p$, and the index $n$ whenever the solution is unique. Denote further by $N_{JIP}$ the subsequence of indices for which JIP is uniquely solvable and does not contain $\infty^{(0)}$. It holds that $N_{JIP}$ has gaps of size at most $g$.

Let $n \in N_{JIP}$. Then there exists unique (up to normalization) function $\psi_n$, sectionally meromorphic in $\mathcal{H} \setminus \Delta$, whose zeros and poles are described by the divisor $(n - g)\infty^{(1)} + D_n - n\infty^{(0)}$, and which has continuous traces on $\Delta \setminus E_{\Delta}$ that satisfy $\psi^+_n = p\psi^-_n$. For $n \notin N_{JIP}$, set $\psi_n := \psi_{\tilde{n}}$, where $\tilde{n}$ is the largest integer in $N_{JIP}$ smaller than $n$. 
Recall that

\[ f_p - \left[ \frac{n}{n} \right] f_p = \frac{2}{\omega \Delta} \frac{\Psi_n^{(1)}}{\Psi_n^{(0)} + p \Psi_n^{(1)}}, \]

where \( \Psi_n^{(k)} := \Psi_n|_{D^{(k)}} \).
Recall that
\[
f_p - \left[ \frac{n}{n} \right]_{f_p} = \frac{2}{w_\Delta} \frac{\Psi_n^{(1)}}{\Psi_n^{(0)} + p \Psi_n^{(1)}},
\]
where \( \Psi_n^{(k)} := \Psi_n|_{D^{(k)}} \). Write \( D_n = \sum_{j=1}^{g} Z_{nj} \). Therefore,

(i) if \( Z_{nj} \in D^{(1)} \), then \( \left[ \frac{n}{n} \right]_{f_p} \) overinterpolates \( f_p \) at the projection of \( Z_{nj} \);

(ii) if \( Z_{nj} \in D^{(0)} \), then \( \left[ \frac{n}{n} \right]_{f_p} \) has a pole next to the projection of \( Z_{nj} \).

Generically, the collection \( \left\{ \left\{ Z_{nj} \right\}_{j=1}^{g} \right\}_n \) is dense in \( \mathcal{M} \).
Almost a Theorem

Let $\Delta$ be the minimal capacity contour for some algebraic function $F$. Further, let $\rho$ be a non-vanishing Hölder continuous function on $\Delta$ and

$$f_{\rho}(z) := \frac{1}{\pi i} \int_{\Delta} \frac{1}{x - z} \rho(x) w_{\Delta}^+(x).$$

Then for $n \in \mathbb{N}_{JIP}$ it holds that

$$f_{\rho} - \left[ \frac{n}{n} \right] f_{\rho} = \frac{2}{\psi_n^{(0)} 1 + E_n^{(0)}} + p_n \psi_n^{(1)} \left[ 1 + E_n^{(1)} \right],$$

where $p_n$, $\deg(p_n) \leq n$, is the polynomial of best uniform approximation to $\rho$ on $\Delta$, $E_n$ is sectionally meromorphic on $\mathcal{H} \setminus \Delta$ with at most $g$ poles only among the elements of $\mathcal{D}_n$, and $\|L_n E_n^\pm\|_{2, \Delta} \ll \|\rho - p_n\|_{\Delta}$. 
The previous theorem has been verified when

(i) $\Delta = [-1, 1]$ by Nuttall\textsuperscript{20};
(ii) $\Delta$ consists of disjoint arcs by Suetin\textsuperscript{21};
(iii) $\Delta$ consists of 3 arc with a common endpoint by Baratchart-Y\textsuperscript{22};
(iv) $\Delta$ is any algebraic S-contour tentatively by Y.

\textsuperscript{20} Padé polynomial asymptotic from a singular integral equation, 1990
\textsuperscript{21} Uniform convergence of Padé diagonal approximants for hyperelliptic functions, 2000
\textsuperscript{22} Asymptotics of Padé approximants to a certain class of elliptic-type functions, 2013
Theorem (Aptekarev-Y$^{24}$)

Let

(i) $\Delta$ be a minimal capacity contour such that no more than three arcs $\Delta_j$ have a common endpoint;

(ii) the weight $\rho$ be such that $\rho|_{\Delta_j}$ is a Jacobi weight modified by a non-vanishing holomorphic function;

(iii) $\mathbb{N}_{JIP}^* \subset \mathbb{N}_{JIP}$ be such that the elements of $\mathcal{D}_{n-1}$ and $\mathcal{D}_n$ are uniformly bounded away from $\infty^{(1)}$ and $\infty^{(0)}$, respectively.

Then for $n \in \mathbb{N}_{JIP}^*$ it holds that

$$f_\rho - \left[\frac{n}{n}\right] f_\rho = \left[1 + O(1/n)\right] \frac{2}{\mathcal{W}_\Delta} \frac{\Psi_n^{(1)}}{\psi_n^{(0)}}$$

in $\mathcal{D} \setminus \bigcup U_\epsilon(Z_{nj})$, where $U_\epsilon(Z)$ is the $\epsilon$-neighborhood of the projection of $Z$ in and $O(1/n)$ is uniform for each fixed $\epsilon > 0$.

$^{24}$Padé approximants for functions with branch points – strong asymptotics of Nuttall-Stahl polynomials.