On multiple orthogonal polynomials

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Let $\vec{\mu} = (\mu_1, \ldots, \mu_d)$ be a vector of measures supported on the real line, each having infinitely many points in its support and finite moments of all orders.

Let $\vec{n} = (n_1, \ldots, n_d)$ be a multi-index of non-negative integers.

Multiple orthogonal polynomial $P_{\vec{n}}(x)$ (type II) is a polynomial of degree at most $|\vec{n}| = n_1 + \cdots + n_d$ satisfying

$$\int x^k P_{\vec{n}}(x) d\mu_i(x) = 0, \quad k = 0, n_i - 1.$$ 

The multi-index $\vec{n}$ is called normal if $\deg P_{\vec{n}} = |\vec{n}|$. In this case we normalize $P_{\vec{n}}(x)$ to be monic. The vector $\vec{\mu}$ is called perfect if all the multi-indices are normal.
Theorem (Angelesco, 1919)

Let $\Delta_1 < \Delta_2 < \cdots < \Delta_d$, where $\Delta_i$ is the convex hull of the support of $\mu_i$. Then $\vec{\mu}$, called an Angelesco system, is perfect.
Theorem (Angelesco, 1919)

Let $\Delta_1 < \Delta_2 < \cdots < \Delta_d$, where $\Delta_i$ is the convex hull of the support of $\mu_i$. Then $\vec{\mu}$, called an Angelesco system, is perfect.

Theorem (López Lagomasino–Fidalgo Prieto, 2011)

Let $\sigma_i$ be $d$ auxiliary measures and $F_i$ be the convex hulls of their supports. Assume that $F_i \cap F_{i+1} = \emptyset$. Write

$$d \langle \sigma, \nu \rangle (x) := \hat{\nu}(x) d\sigma(x), \quad \hat{\nu}(x) := \int (z - y)^{-1} d\nu(y).$$

Let $\langle \sigma_j, \ldots, \sigma_k \rangle := \langle \sigma_j, \langle \sigma_{j+1}, \ldots, \sigma_k \rangle \rangle$. Put

$$\mu_1 := \sigma_1$$
$$\mu_2 := \langle \sigma_1, \sigma_2 \rangle$$
$$\ldots$$
$$\mu_d := \langle \sigma_1, \ldots, \sigma_d \rangle.$$

Then $\vec{\mu}$, called a Nikishin system, is perfect.
Let \( \{ \vec{e}_i \}_{i=1}^d \) be the standard basis in \( \mathbb{R}^d \). If \( \vec{n} \) and \( \vec{n} + \vec{e}_j \) are normal, then

\[
xP_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_j}(x) + b_{\vec{n},j} P_{\vec{n}}(x) + \sum_{i=1}^{d} a_{\vec{n},i} P_{\vec{n}-\vec{e}_i}(x)
\]

for some coefficients \( b_{\vec{n},i}, a_{\vec{n},i} \). These coefficients satisfy consistency conditions

\[
b_{\vec{n}+\vec{e}_i,j} - b_{\vec{n}+\vec{e}_j,i} = b_{\vec{n},j} - b_{\vec{n},i},
\]

\[
\sum_{k=1}^{d} a_{\vec{n}+\vec{e}_j,k} - \sum_{k=1}^{d} a_{\vec{n}+\vec{e}_i,k} = b_{\vec{n}+\vec{e}_j,i} b_{\vec{n},j} - b_{\vec{n}+\vec{e}_i,j} b_{\vec{n},i},
\]

\[
a_{\vec{n},i}(b_{\vec{n},j} - b_{\vec{n},i}) = a_{\vec{n}+\vec{e}_j,i}(b_{\vec{n}-\vec{e}_i,j} - b_{\vec{n}-\vec{e}_i,i}).
\]

When \( d = 1 \), these relations simply become classical recurrence relations for monic orthogonal polynomials

\[
xP_n(x; \mu) = P_{n+1}(x; \mu) + b_n(\mu) P_n(x; \mu) + a_n^2(\mu) P_{n-1}(x; \mu).
\]
Theorem (Filipuk–Haneczok–Van Assche, 2015)

If the recurrence coefficients $\{a_{\vec{n},i}, b_{\vec{n},i}\}$ satisfy

- consistency conditions,
- $a_{n\vec{e},i} > 0$ and $a_{n\vec{e},j} = 0$, $i \neq j$,
- $b_{\vec{n},i} \neq b_{\vec{n},j}$ for $i \neq j$,

then there exists $\vec{\mu}$ for which $\{a_{\vec{n},i}, b_{\vec{n},i}\}$ are the recurrence coefficients.
Theorem (Filipuk–Haneczok–Van Assche, 2015)

If the recurrence coefficients \( \{a_{\vec{n},i} b_{\vec{n},i}\} \) satisfy

- consistency conditions,
- \( a_{n\vec{e},i} > 0 \) and \( a_{n\vec{e},j} = 0, \ i \neq j \),
- \( b_{\vec{n},i} \neq b_{\vec{n},j} \) for \( i \neq j \),

then there exists \( \vec{\mu} \) for which \( \{a_{\vec{n},i} b_{\vec{n},i}\} \) are the recurrence coefficients.

Conversely, \( \{a_{\vec{n},i} b_{\vec{n},i}\} \) can be constructively recovered from \( \{b_n(\mu_i), a_n^2(\mu_i)\} \) and the initial conditions

\[
b_{n\vec{e},i} = b_n(\mu_i), \quad a_{n\vec{e},i} = a_n^2(\mu_i), \quad a_{n\vec{e},j} = 0, \quad j \neq i,
\]

provided \( b_{\vec{n},i} \neq b_{\vec{n},j} \) for \( i \neq j \).

The condition \( b_{\vec{n},i} \neq b_{\vec{n},j} \) holds for multiple Hermite \( (e^{-x^2-c_i x}) \), Laguerre \( (x^{\alpha_j} e^{-x}, x^{\alpha} e^{-c_j x}) \), and Charlier \( (a_i^k/k!) \) polynomials as well as for Angelesco systems where \( b_{\vec{n},i} < b_{\vec{n},j}, \ i < j \) (Aptekarev–Denisov-Ya., 2020).
Theorem (Van Assche, 2016)

Let \( \vec{n} = ([c_1 n], \ldots, [c_d n]) \) for some \( \vec{c} \in (0, 1)^d \) with \( |\vec{c}| = 1 \). Assume that

\[
\lim_{n \to \infty} n^{-2\gamma} a_{\vec{n}, i} = A_{\vec{c}, i} \quad \text{and} \quad \lim_{n \to \infty} n^{-\gamma} b_{\vec{n}, i} = B_{\vec{c}, i}
\]

for some \( \gamma \geq 0 \) with \( B_{\vec{c}, i} \neq B_{\vec{c}, j} \). Then

\[
\lim_{n \to \infty} \frac{P_{\vec{n} + \vec{e}_i}(n^\gamma z)}{n^\gamma P_{\vec{n}}(n^\gamma z)} = \chi_{\vec{c}}(z) - B_{\vec{c}, i},
\]

where \( z = \chi_{\vec{c}} + \sum_i \frac{A_{\vec{c}, i}}{\chi_{\vec{c}} - B_{\vec{c}, i}} \) such that \( \chi_{\vec{c}}(z) - z \to 0 \) as \( z \to \infty \).

When \( a_n^2(\mu) \to A^2, b_n(\mu) \to B \), the theorem recovers

\[
2\chi(z) = (z + B) + \sqrt{(z - B - 2A)(z - B + 2A)}.
\]
**Theorem (Gonchar–Rakhmanov, 1985)**

Let $F = \{F_i\}$ be a collection of intervals, $\theta = \{\theta_i\}$, $\theta_i > 0$, and $A = [a_{ij}]$ be a positive definite symmetric matrix with $a_{ii} > 0$ and $a_{ij} = 0$ if $F_i \cap F_j \neq \emptyset$, $i \neq j$.

Let $M_\theta(F)$ be the set of vector measures $\vec{\nu} = (\nu_1, \ldots, \nu_d)$ such that $\nu_i$ is supported on $F_i$ and $|\nu_i| = \theta_i$. Define

$$I(\vec{\nu}) := - \sum a_{ij} \int \int \log |x - y| d\nu_i(x) d\nu_j(y).$$

Then there exists a unique $\vec{\omega} \in M_\theta(F)$, the vector equilibrium measure, such that $I(\vec{\omega}) = \min_{M_\theta(F)} I(\vec{\nu})$. 
Let $G$ be a rooted tree with $d + 1$ vertices $V_0, V_1, \ldots, V_d$, where $V_0$ is the root. To each $V_i$, $i > 0$, associate an interval $F_i$ such that $F_i \cap F_j = \emptyset$ if $V_i$ and $V_j$ are either siblings or one is a child of the other.

$V_1, V_2$ are siblings and children of $V_0$;  
$F_1 \cap F_2 = \emptyset$;

$V_3, V_4$ are siblings and children of $V_1$;  
$F_i \cap F_j = \emptyset$, $i, j \in \{1, 3, 4\}$;

$V_5, V_6$ are siblings and children of $V_2$;  
$F_i \cap F_j = \emptyset$, $i, j \in \{2, 5, 6\}$. 

$\begin{align*}
V_3 & \quad V_4 \\
V_1 & \quad V_5 \\
V_0 & \quad V_2 \\
V_6 & \\
\end{align*}$
On each interval $F_i$, choose an auxiliary measure $\sigma_i$. Given $V_m$, let

$$V_0 \to V_{i_1} \to V_{i_2} \to \cdots \to V_{i_k} = V_m$$

be the path connecting $V_0$ and $V_m$. A **GN system** is a vector $\bar{\mu}$ with

$$\mu_m := \langle \sigma_{i_1}, \ldots, \sigma_{i_k} \rangle.$$

\[\begin{array}{c}
V_0 & \rightarrow & V_1 & \rightarrow & V_2 & \rightarrow & V_3 & \rightarrow & V_4 & \rightarrow & V_5 & \rightarrow & V_6
\end{array}\]

$\mu_1 = \sigma_1$, $\mu_2 = \sigma_2$, $\mu_3 = \langle \sigma_1, \sigma_3 \rangle$, $\mu_4 = \langle \sigma_1, \sigma_4 \rangle$, $\mu_5 = \langle \sigma_2, \sigma_5 \rangle$, $\mu_6 = \langle \sigma_2, \sigma_6 \rangle$.

Aptekarev-Lysov generalized this construction to graphs where multiple edges between vertices are allowed.
Given $\vec{c} \in (0, 1)^d$, set $\theta_m = c_m + \sum c_i$, where the sum is over all descendants $V_i$ of $V_m$.

$$
\begin{align*}
\theta_3 &= c_3 \\
\theta_4 &= c_4 \\
\theta_5 &= c_5 \\
\theta_6 &= c_6 \\
\theta_1 &= c_1 + c_3 + c_4 \\
\theta_2 &= c_2 + c_5 + c_6
\end{align*}
$$

Further, let $a_{ii} = 2$, $a_{ij} = -1$ if $V_i, V_j$ is a child/parent pair, $a_{ij} = 1$ if $V_i, V_j$ are siblings, and otherwise $a_{ij} = 0$. 
**Theorem (Gonchar–Rakhmanov–Sorokin, 1997)**

Assume that \( d\sigma_i/dx > 0 \) a.e. on \( F_i \). Suppose further that \( \vec{n} \) is such that \( n_i \leq n_j + 1 \) if \( V_i \) is a child of \( V_j \) and that \( \vec{n}/|\vec{n}| \to \vec{c} \) as \( |\vec{n}| \to \infty \). Let \( \vec{\omega}_{\vec{c}} \) be the vector equilibrium measure. Then the normalized counting measures of zeros of \( P_{\vec{n}}(z) \) converge weak* to \( \sum \omega_{\vec{c},i} \) where the sum is taken over the children of \( V_0 \).
**Theorem (Gonchar–Rakhmanov–Sorokin, 1997)**

Assume that $d\sigma_i/dx > 0$ a.e. on $F_i$. Suppose further that $\vec{n}$ is such that $n_i \leq n_j + 1$ if $V_i$ is a child of $V_j$ and that $\vec{n}/|\vec{n}| \to \vec{c}$ as $|\vec{n}| \to \infty$. Let $\vec{\omega}_c$ be the vector equilibrium measure. Then the normalized counting measures of zeros of $P_{\vec{n}}(z)$ converge weak* to $\sum \omega_{\vec{c},i}$ where the sum is taken over the children of $V_0$.

**Theorem (Gonchar–Rakhmanov, 1981)**

For Angelesco systems it holds that the support of $\omega_{\vec{c},i}$ is an interval.

Aptekarev-Lysov claim that this is true for all GN systems.
Theorem (Geronimo–Kuijlaars–Van Assche, 2001)

Let $d\mu_i(x) = \rho_i(x)dx$. Consider the following Riemann-Hilbert problem for $(d+1) \times (d+1)$ matrices:

(a) $Y(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$ and $\lim_{z \to \infty} Y(z)z^{-\sigma(\vec{n})} = I$, where $I$ is the identity matrix and $\sigma(\vec{n}) := \text{diag}(|\vec{n}|, -n_1, \ldots, -n_d)$;

(b) on the real line it holds that $Y_+(x) = Y_-(x)(I + \sum \rho_i(x)E_{1,i+1})$, where $E_{1,i+1}$ has all zero entries except for $(1, i+1)$, which is 1.

This problem has a unique solution whose $(1, 1)$-entry is $P_{\vec{n}}(z)$.

The proof is the modification of the one by Fokas–Its–Kitaev in the case $d = 1$. 
Let $\vec{\mu}$ be an Angelesco system corresponding to intervals $\Delta_1 < \Delta_2 < \cdots < \Delta_d$.

Given $\vec{c} \in (0, 1)^d$, let $\vec{\omega}_{\vec{c}}$ be the vector equilibrium measure constructed before. Denote by $\Delta_{\vec{c},i} \subseteq \Delta_i$ the support of $\omega_{\vec{c},i}$, which is an interval.

We shall assume that $d\mu_i(x) = \rho_i(x)dx$, where $\rho_i(x)$ extends to a holomorphic and non-vanishing function in a neighborhood of $\Delta_i$ (we can also consider Fisher-Hartwig perturbations).

When $\vec{c} = \vec{n}/|\vec{n}|$, we shall simply write $\vec{\omega}_{\vec{n}}$ and $\Delta_{\vec{n},i}$. 
The surface $\mathcal{R}_{\vec{n}}$ constructed w.r.t to cuts $\Delta_{\vec{n},i}$ and has genus 0. Let $\Phi_{\vec{n}}(z)$ be the rational function on $\mathcal{R}_{\vec{n}}$ such that

$$\Phi_{\vec{n}}(z) \sim z^{\vec{n}}.$$ 

$$\Phi_{\vec{n}}(z^{(0)}) \sim z^{\vec{n}}$$

$$\Phi_{\vec{n}}(z^{(1)}) \sim z^{-n_1}$$

$$\Phi_{\vec{n}}(z^{(2)}) \sim z^{-n_2}$$

$$(\Phi_{\vec{n}}) = n_1 \infty^{(1)} + \cdots + n_d \infty^{(d)} - |\vec{n}| \infty^{(0)}, \quad \prod \Phi_{\vec{n}}(z^{(k)}) \equiv 1.$$
**Theorem (Ya., 16)**

If \( \frac{\vec{n}}{|\vec{n}|} \to \vec{c} \in (0, 1)^d \) as \( |\vec{n}| \to \infty \), then

\[
P_{\vec{n}}(z) \sim \left( \Phi_{\vec{n}}S \right)(z^{(0)}),
\]

where \( S(z) \) is a Szegő-type function on \( \mathbb{H}_{\vec{c}} \).

Similar result for Nikishin systems with \( d = 2 \) and \( \vec{n} = n\vec{c} \) for \( \vec{c} \in \mathbb{Q}^2 \cap (0, 1)^2 \)
was proven by Lópex Lagomasino–Van Assche, 2018.

**Theorem (Aptekarev–Denisov–Ya., in prep.)**

When \( d = 2 \), the condition \( \frac{\vec{n}}{|\vec{n}|} \to \vec{c} \in (0, 1)^d \) can be replaced by \( n_i \to \infty \).

There are many other results along the diagonal sequences \( (n, n, \ldots, n) \).
Asymptotics of the Recurrence Coefficients in Angelesco Systems

**Theorem (Aptekarev–Denisov–Ya., 2020)**

Let \( \chi_{\vec{c}} : \mathbb{H}_{\vec{c}} \to \overline{\mathbb{C}} \) be a conformal map such that

\[
\chi_{\vec{c}}(z^{(0)}) = z + \mathcal{O}(z^{-1}) \quad \text{as} \quad z \to \infty.
\]

Define constants \( A_{\vec{c},i}, B_{\vec{c},i} \) by

\[
\chi_{c}(z^{(i)}) = B_{\vec{c},i} + A_{\vec{c},i}z^{-1} + \mathcal{O}(z^{-2}) \quad \text{as} \quad z \to \infty.
\]

Then, as \( |\vec{n}| \to \infty, \vec{n}/|\vec{n}| \to \vec{c} \in (0, 1)^d \), it holds that

\[
\lim a_{\vec{n},i} = A_{\vec{c},i} \quad \text{and} \quad \lim b_{\vec{n},i} = B_{\vec{c},i}.
\]

**Theorem (Aptekarev–Denisov–Ya., in prep.)**

When \( d = 2 \), the limits of \( A_{(c,1-c),i}, B_{(c,1-c),i} \) as \( c \to 0 \) or \( c \to 1 \) exist and

\[
\lim a_{\vec{n},i} = A_{\vec{c},i} \quad \text{and} \quad \lim b_{\vec{n},i} = B_{\vec{c},i}.
\]

holds as \( |\vec{n}| \to \infty, \vec{n}/|\vec{n}| \to \vec{c} \in [0, 1]^2 \).
Let $\mathcal{T}$ be the rooted tree of all possible increasing paths on $\mathbb{N}^d$ starting at $\vec{1}$.

We denote the set of all vertices of $\mathcal{T}$ by $\mathcal{V}$. We let

$$\ell : \mathcal{V} \to \{1, \ldots, d\}, \quad Y \mapsto \ell_Y$$

such that $\Pi(Y) = \Pi(Y_{(p)}) + \vec{e}_{\ell_Y}$,

where $\Pi$ is the natural projection of $\mathcal{V}$ onto $\mathbb{N}^d$. 
Let $\vec{\kappa} \in \mathbb{R}^d$, $|\vec{\kappa}| = 1$. Define two interaction functions $A, B : \mathcal{V} \to \mathbb{R}$ by

\[
A_O := 1, \quad B_O := \sum \kappa_i b_{\vec{1} - \vec{e}_i,i}, \quad Y = O,
\]

\[
A_Y := a_{\Pi(\gamma(p)),\ell_Y}, \quad B_Y := b_{\Pi(\gamma(p)),\ell_Y}, \quad Y \neq O.
\]

Assume now that

\[
0 < a_{\vec{n},j} \text{ for all } \vec{n} \in \mathbb{Z}^d_+ \text{ such that } n_j > 0,
\]

\[
\sup a_{\vec{n},j} < \infty, \quad \sup |b_{\vec{n},j}| < \infty.
\]

This condition is satisfied by Angelesco systems (Aptekarev–Denisov–Ya., 20).

Then, for any function $f \in \ell^2(\mathcal{V})$, the action of the operator $\mathcal{J}_{\vec{\kappa}}$ can be written in the following form

\[
(\mathcal{J}_{\vec{\kappa}} f)_O := (Bf)_O + \sum_i (A^{1/2} f)_{O(\chi h),i}, \quad Y = O,
\]

\[
(\mathcal{J}_{\vec{\kappa}} f)_Y := A^{1/2}_Y f_{\gamma(p)} + (Bf)_Y + \sum_i (A^{1/2} f)_{Y(\chi h),i}, \quad Y \neq O.
\]

$\mathcal{J}_{\vec{\kappa}}$ is a bounded and self-adjoint operator on $\ell^2(\mathcal{V})$. 
Proposition (Aptekarev-Denisov-Ya., 2020)

Let $\nu_{\vec{\kappa}}$ be the spectral measure of $J_{\vec{\kappa}}$ associated to an Angelesco system $(\mu_1, \mu_2)$. Then

$$\Theta_{\nu_{\vec{\kappa}}}(z) = \Xi(\mu_1, \mu_2) \frac{\Theta_{\mu_1}(z) - \Theta_{\mu_2}(z)}{\kappa_2 \Theta_{\mu_1}(z) + \kappa_1 \Theta_{\mu_2}(z)},$$

where $\Theta_{\mu}(z) := \int (x - z)^{-1} d\mu(x)$ and

$$\Xi(\mu_1, \mu_2) := \left( \int t (d\mu_2(t) - d\mu_1(t)) \right)^{-1}.$$

If the measures $\mu_i$ are absolutely continuous w.r.t. the Lebesgue measure, then

$$\nu'_{\vec{\kappa}}(x) = \frac{\Theta_{\mu_2}(x)\mu_1'(x) - \Theta_{\mu_1}(x)\mu_2'(x)}{\left| \kappa_1 \Theta_{\mu_1}(x) + \kappa_2 \Theta_{\mu_2}(x) \right|^2}.$$

Proposition (Aptekarev-Denisov-Ya., 2020)

If $\nu_{\vec{\kappa}}$ and $\Xi(\mu_1, \mu_2)$ are known, then $\mu_1$, $\mu_2$, and $J_{\vec{\kappa}}$ can be found uniquely.
Theorem (Aptekarev–Denisov–Ya., in prep.)

Let $\Delta_1 < \Delta_2$ be two intervals. Write $\Delta_{c,i}$ for the support of the $i$-th component of the vector equilibrium measure $\vec{\omega}_{c,1-c}$.

Let $\chi_c(z)$ be the above constructed conformal map on $\mathbb{C}_c$ that defines constants $A_{c,1}, A_{c,2}, B_{c,1}, B_{c,2}$ together with their limits as $c \to 0$ and $c \to 1$.

Let $\mathcal{J}_{\vec{\kappa}}$ be a Jacobi operator corresponding to some constants $\{a_{\vec{n},i}, b_{\vec{n},i}\}$. If for any $c \in [0, 1]$ it holds that

$$\lim a_{\vec{n},i} = A_{c,i} \quad \text{and} \quad \lim b_{\vec{n},i} = B_{c,i}$$

where the limit is taken along any sequence $\vec{n}/|\vec{n}| \to (c, 1 - c)$ as $|\vec{n}| \to \infty$, then $\sigma_{\text{ess}}(\mathcal{J}_{\vec{\kappa}}) = \Delta_1 \cup \Delta_2$. 
We shall say that \((\mu_1, \mu_2)\) forms a symmetric Stahl system if

\[
\text{supp}(\mu_1) = [-1, a], \quad \text{supp}(\mu_2) = [-a, 1], \quad a \in (0, 1).
\]

Let \(h\) be an algebraic function given by

\[
A(z)h^3 - 3B_2(z)h - 2B_1(z) = 0,
\]

where \(A(z) := (z^2 - 1)(z^2 - a^2), \quad B_2(z) := z^2 - p^2, \quad \text{and} \quad B_1(z) := z,\) for some parameter \(p > 0.\)
Let $\mathcal{R}$ be the Riemann surface of $h$. We are looking for the surface such that

$$\text{Re} \left( \int \mathfrak{z} h(t) \, dt \right)$$

is a single-valued and harmonic function on $\mathcal{R}$.

**Theorem (Aptekarev–Van Assche–Ya., 2017)**

(I) If $a \in (0, 1/\sqrt{2})$, then there exists $p \in (a, \sqrt{(1 + a^2)/3})$ such that the condition is fulfilled. In this case $\mathcal{R}$ has 8 ramification points whose projections are $\{\pm 1, \pm a\}$ and $\{\pm b, \pm ic\}$ for some uniquely determined $b \in (a, p)$ and $c > 0$.

(II) If $a = 1/\sqrt{2}$, then the condition is fulfilled for $p = 1/\sqrt{2}$. In this case $\mathcal{R}$ has 4 ramification points whose projections are $\{\pm 1, \pm 1/\sqrt{2}\}$.

(III) If $a \in (1/\sqrt{2}, 1)$, then the condition is fulfilled for $p = \sqrt{(1 + a^2)/3}$. In this case $\mathcal{R}$ has 6 ramification points whose projections are $\{\pm 1, \pm a\}$ and $\{\pm b\}$, $b \in (p, a)$. 
(a) Case I

(b) Case II

(c) Case III
Let $\Phi(z) := \exp \left\{ \int^z h(t) \, dt \right\}$. It is a multiplicatively multi-valued function on $\mathbb{R}$ with the divisor $\infty^{(1)} + \infty^{(2)} - 2\infty^{(0)}$.

Let $\rho_1$ and $\rho_2$ be functions holomorphic and non-vanishing in a neighborhood of $[-1, 1]$. In Case I, assume also that the ratio $\rho_1/\rho_2$ extends holomorphically to a non-vanishing function in a neighborhood of $\mathbb{R}^{(1)} \cap \mathbb{R}^{(2)}$. Then $\Psi_n \leftrightarrow \Phi^n$, where

\[
\begin{align*}
(\Psi_n^{(1)})^\pm &= \pm (\Psi_n^{(0)})^\mp \rho_1, \\
(\Psi_n^{(2)})^\pm &= \mp (\Psi_n^{(0)})^\mp \rho_2, \\
(\Psi_n^{(2)})^\pm &= \pm (\Psi_n^{(0)})^\mp \rho_2, \\
(\Psi_n^{(2)})^\pm &= \pm (\Psi_n^{(1)})^\mp (\rho_2/\rho_1).
\end{align*}
\]

$\Psi_n(z)$ has a wandering zero (2 in Case I) and there exists a subsequence $\mathbb{N}_*$ such that

- $|\Psi_n| \leq C(\mathbb{N}_*) |\Phi^n|$ uniformly away from the branch points of $\mathbb{R}$;
- $|\Psi_n| \geq C(\mathbb{N}_*)^{-1} |\Phi^n|$ uniformly in a neighborhood of $\infty^{(0)}$. 
Theorem (Aptekarev-Van Assche-Ya., 2017)

Let $d\mu_i(x) = \rho_i(x)dx$ be a symmetric Stahl system, where $\rho_i(x)$ are as before and we assume in addition that the ratio $(\rho_2/\rho_1)(x)$ extends from $(-a,a)$ to a holomorphic and non-vanishing function

- in a domain that contains in its interior the closure of all the bounded components of the regions $\Omega_{ijk}$ in Case I;
- in a domain whose complement is compact and belongs to the right-hand component of $\Omega_{021}$ in Cases II and IIIa;
- in the extended complex plane, i.e., the ratio is a non-zero constant, in Case IIIb,

where

$$\Omega_{ijk} := \left\{ z : |\Phi^{(i)}(z)| > |\Phi^{(j)}(z)| > |\Phi^{(k)}(z)| \right\}.$$

Then for multi-indices $\vec{n} = (n,n)$ it holds that

$$P_{\vec{n}}(z) \sim \psi_n^{-1}(\infty^{(0)})\psi_n(z^{(0)}), \quad n \in \mathbb{N}_*.$$
Symmetric Stahl Systems

Case I:

\[ \Omega_{012} \]

\[ \Omega_{021} \]

\[ \Omega_{201} \]

\[ \Omega_{012} \]

\[ \Omega_{021} \]

\[ \Omega_{201} \]

Case II:

\[ \Omega_{012} \]

\[ \Omega_{021} \]

\[ \Omega_{201} \]

\[ \Omega_{012} \]
Case IIIa:

\[ \Omega_{021} \quad \Omega_{102} \quad \Omega_{012} \quad \Omega_{021} \]

Case IIIb:

\[ \Omega_{102} \quad \Omega_{012} \quad \Omega_{102} \]