Large Deviations, Linear Statistics, and Scaling Limits for Mahler Ensemble of Complex Random Polynomials

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joint work with

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The Mahler measure of a polynomial $p(z) = a \prod (z - \alpha_n)$ is given by

$$M(p) := \exp \left\{ \int_{T} \log |p(\tau)| \frac{|d\tau|}{2\pi} \right\},$$

where $T := \{|z| = 1\}$. It follows from Jensen’s formula that

$$M(p) = |a| \prod \max\{1, |\alpha_n|\} = |a| \prod \exp \{ \log^+ |\alpha_n| \}.$$
Associate to each element $\mathbf{v} \in \mathbb{C}^{N+1}$ a polynomial $p_{\mathbf{v}}$. The following quantity is of number theoretic interest:

$$\# \{ \mathbf{v} \in \mathbb{Z}[i]^{N+1} : M(p_{\mathbf{v}}) \leq T \}.$$

Clearly, this quantity is equal to

$$\# \mathbb{Z}[i]^{N+1} \cap \{ \mathbf{v} \in \mathbb{C}^{N+1} : M(p_{\mathbf{v}}) \leq T \}.$$

Chern & Vaaler\(^1\) have shown that this quantity is bounded by

$$T^{2N+2} \text{vol} \{ \mathbf{v} \in \mathbb{C}^{N+1} : M(p_{\mathbf{v}}) \leq 1 \}.$$

\(^1\) The distribution of values of Mahler's measure, J. Reine Angew. Math., 540:1—47, 2001
They further computed that

\[
\text{vol}\{ \mathbf{v} \in \mathbb{C}^{N+1} : M(\mathbf{v}) \leq 1 \} = \frac{\pi}{N+1} H_N(N+1),
\]

where

\[
H_N(s) := \int_{\mathbb{C}^N} M^{-2s}(P_\mathbf{u}) dA^{\otimes N} = \frac{\pi^N}{N!} \prod_{n=1}^{N} \frac{s}{s - n}
\]

and \( P_\mathbf{u} \) is the monic polynomial of degree \( N + 1 \) with the non-leading coefficients described by the vector \( \mathbf{u} \in \mathbb{C}^N \).
Random Polynomials

Mahler Ensemble of Complex Random Polynomials

Question
Where do the zeros of a typical polynomial from this volume lie? (Is $z^N - 1$ or $(z - 1)^N$ more typical?)

Definition
By a random polynomial from a complex Mahler ensemble we will mean a polynomial chosen according to the density $M^{-2s}(P_u)/H_N(s)$.

Remark
True interest of a number theorists lies in polynomials with integer coefficients which leads to real Mahler ensemble. Please, stay for the talk by Chris Sinclair where this more complicated case is addressed.
As was observed by Chern & Vaaler, a change of variables from the coefficients of polynomials to their roots, gives

\[ H_N(s) := \frac{1}{N!} \int_{\mathbb{C}^N} D_{N,s}(\alpha_1, \ldots, \alpha_N) dA \otimes^N (\alpha_1, \ldots, \alpha_N), \]

where

\[
D_{N,s} := \prod_n \exp \left\{ -2s \log^+ |\alpha_n| \right\} \prod_{n < m} |\alpha_n - \alpha_m|^2 \\
= \prod_n \exp \left\{ -2s \int_{\mathbb{T}} \log |\tau - \alpha_n| \frac{|d\tau|}{2\pi} \right\} \prod_{n < m} |\alpha_n - \alpha_m|^2.
\]
For any probability Borel measure on $\mathbb{C}$, say $\nu$, set

$$I[\nu] := \int \log \frac{1}{|z-u|} \, d\nu \otimes 2(z, u)$$

to be its logarithmic energy. For any compact set $K$ the logarithmic capacity of $K$ is defined by

$$\text{cp}(K) := \exp \left\{ - \inf_{\text{supp}(\nu) \subseteq K} I[\nu] \right\}.$$ 

It is known that either $\text{cp}(K) = 0$ ($K$ is polar) or else there exists the unique measure $\omega_K$, the logarithmic equilibrium distribution on $K$, that realizes the infimum. The measure $\frac{|d\tau|}{2\pi} |_{\mathbb{T}}$ is the equilibrium distribution on both $\mathbb{T}$ and $\mathbb{D}$. 
Green’s function $g_K$, with a pole at $\infty$ for the unbounded component of $K^c$, the complement of a compact set $K$, is the unique harmonic function which is zero q.e. on $\partial K^c$ and behaves like $\log |z|$ at $\infty$. In particular,

$$g_D(z) = g_T(z) = \log^+ |z|.$$ 

Put $g_K \equiv 0$ in $\mathbb{C} \setminus \overline{K^c}$. If it is continuous in $\mathbb{C}$, $K$ is called regular w.r.t. Dirichlet problem.

Let $K$ be such that $cp(K) = 1$. The Mahler measure of a polynomial $p$ with respect to $K$ is defined by

$$M_K(p) := \exp \left\{ \int \log |p| d\omega_K \right\} = |a| \exp \left\{ \sum g_K(\alpha_n) \right\}.$$
Let $K$ be a compact set. The joint density of \textit{random configurations} (zeros of random polynomials or equivalently eigenvalues of normal random matrices) is defined by

$$\Omega_{N,s}(z) := \frac{1}{Z_{N,s}} \exp \left\{ -2s \sum_{n=1}^{N} g_K(z_n) \right\} \prod_{m<n} |z_n - z_m|^2,$$

where $s - N + 1 > 1 + c_0$ for some $c_0 > 0$ and

$$Z_{N,s} = \int_{C^N} \exp \left\{ -2s \sum_{n=1}^{N} g_K(z_n) \right\} \prod_{m<n} |z_n - z_m|^2 \, dA^\otimes N.$$
Let \( \eta = \{\eta_1, \ldots, \eta_N\} \) be a random configuration chosen according to the law \( \Omega_{N,s} \). To any such configuration we associate the \textit{empirical measure} defined as

\[
\omega_\eta := \frac{1}{N} \sum_{k=1}^{N} \delta_{\eta_k},
\]

where \( \delta_z \) is the classical Dirac delta with the unit mass at \( z \).

\textbf{Question}

Where is it most likely to find \( \omega_\eta \) when \( N \) is large? That is, where it is most likely for random polynomials to have their zeros?
Let $\nu$ and $\mu$ be two probability Borel measures on $\mathbb{C}$. The distance between them is defined by

$$\text{dist}(\nu, \mu) = \sup_{f} \left| \int f \, d\nu - \int f \, d\mu \right|,$$

where the supremum is taken over all functions $f$ that are bounded by 1 in modulus and satisfy the Lipschitz condition with constant 1 on $\text{supp}(\nu) \cup \text{sup}(\mu)$.

For measures supported on a compact set it holds that $\text{dist}(\nu, \nu_n) \to 0$ as $n \to \infty$ if and only if $\nu_n \to^* \nu$, where $\to^*$ stands for the convergence in the weak* topology of measures.
The following theorem takes place.\(^2\)

**Theorem (M.Y.)**

Let \( K \) be a compact set with connected complement which is regular with respect to the Dirichlet problem and such that \( K = \overline{K^c} \). Then

\[
\lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{N^2} \log \text{Prob} \{ \text{dist} (\nu, \omega_\eta) < \epsilon \} = - (I_{\ell}[\nu] - I[\omega_K]),
\]

for any probability Borel measure \( \nu, \text{supp}(\nu) \subset \mathbb{C} \), where

\[
l_{\ell}[\nu] := I[\nu] + \frac{2}{\ell} \int g_K \, d\nu, \quad \ell := \lim_{N \to \infty} s^{-1} N,
\]

and it holds that \( l[\omega_K] = I[\omega_K] < l[\nu], \nu \neq \omega_K \).

\(^2\) Large Deviations and Linear Statistics for Potential Theoretic Ensembles Associated with Regular Closed Sets, Probab. Theory Relat. Fields., 2014
Let $\eta$ be a random configuration chosen according to $\Omega_{N,s}$ and $\omega_\eta$ be the corresponding empirical measure. $\omega_\eta$ can be considered as a simple point process on $\mathbb{C}$.

The *correlation functions* of $\omega_\eta$ w.r.t. $\,dA$ are functions (if they exist) $R_n : \mathbb{C}^n \rightarrow [0, \infty)$ such that for any family of mutually disjoint subsets $O_1, \ldots, O_n$ it holds that

$$E \left[ \prod_{k=1}^{n} \omega_\eta(O_k) \right] = \int_{O_1 \times \cdots \times O_n} R_n(z_1, \ldots, z_n) \,dA^\otimes n(z_1, \ldots, z_n)$$

and $R_n(z_1, \ldots, z_n)$ vanishes whenever $z_i = z_k$ for $i \neq k$.

Thus, $\int_O R_1 \,dA$ is the expected number of zeros that lie in the set $O$. 
**Exercise**

\[ R_n(z_1, \ldots, z_n) = \frac{N!}{(N-n)!} \int_{\mathbb{C}^{N-n}} \Omega_N,s dA^{\otimes(N-n)}(z_{n+1}, \ldots, z_N). \]

**Theorem (M.Y.)**

Under the conditions of the previous theorem, it holds that

\[ \lim_{N \to \infty} \frac{(N-n)!}{N!} \int_{\mathbb{C}^n} f R_n dA^{\otimes n} = \int f d\omega_K^{\otimes n} \]

for each \( f \in C_b(\mathbb{C}^n), \ n \in \mathbb{N} \), where \( C_b(\mathbb{C}^n) \) is the Banach space of bounded continuous functions on \( \mathbb{C}^n \).

**Remark**

In particular, \( E(\omega_\eta(O)) \approx N \omega_K(O) \).
Define a sequence of orthonormal polynomials \( \{p_n\} \) such that

\[
\int_{\mathbb{C}} p_n(z)\overline{p_m(z)}e^{-2sg_K(z)}dA = \delta_{nm}.
\]

The following fact is by now standard, see Deift\(^3\) or Mehta\(^4\),

\[
R_n(z_1, \ldots, z_n) = \det \left[ K_N(z_i, z_k) \right]_{i,k=1}^n,
\]

where

\[
K_N(z, w) := e^{-s(g_K(z)+g_K(w))} \sum_{n=0}^{N-1} p_n(z)\overline{p_n(w)}.
\]


Then the following theorem takes place\textsuperscript{5}.

**Theorem (M.Y and C. Sinclair)**

Let $K$ be a Jordan domain whose boundary $\partial K$ is a Jordan curve of class $C^{1,\alpha}$, $\alpha > 1/2$. Then

$$p_n = (1 + o(1)) \sqrt{\frac{n + 1}{\pi}} \left(1 - \frac{n + 1}{s}\right) \Phi^n \Phi'$$

uniformly on $\overline{K^c}$, where $\Phi$ is the conformal map from $K^c \rightarrow \{|z| > 1\}$.

**Remark**

Observe that $|\Phi(z)| = \exp\{g_K(z)\}$ for $z \in K^c$.

\textsuperscript{5}Universality for ensembles of matrices with potential theoretic weights on domains with smooth boundary, J. Approx. Theory, 164(5):682—708, 2012
Denote by $K(z, w)$ the **reproducing kernel** for the Bergman space on $K^\circ$. That is,

$$f(z) = \int_{K^\circ} f(w)K(z, w)dA(w)$$

for every holomorphic $f$ such that $\int_{K^\circ} |f|^2dA < \infty$.

**Theorem (M.Y. and C. Sinclair)**

Under the conditions of the previous theorem, $K_N(z, w)$ converges to $K(z, w)$ locally uniformly in $K^\circ \times K^\circ$.

**Remark**

For all $N$ large, random polynomials are expected to have a “fixed” number of zeros in each set of positive Lebesgue measure.
**Theorem (M.Y. and C. Sinclair)**

Under the conditions of the previous theorem, it holds that

\[
|\Phi(z)\Phi(w)|^s \frac{K_N(z, w)}{s - N} \to \frac{1}{\pi} \frac{\Phi'(z)\Phi'(w)}{\Phi(z)\Phi(w) - 1} \left[ 1 + \frac{c^{-1}}{\Phi(z)\Phi(w) - 1} \right]
\]

locally uniformly in $K^c \times K^c$, where $c := \lim_{N \to \infty} (s - N)$. In particular, $K_N(z, w) \to 0$ when $s - N \to \infty$.

**Remark**

When $c < \infty$ and $N$ is large, random polynomials are expected to have a “fixed” number of zeros in each set of positive Lebesgue measure.
From the linear statistics we know that

\[ E[\omega_\eta(D_\varepsilon(\tau))] \sim N\omega_K(D_\varepsilon(\tau)) \sim \varepsilon N\omega'_K(\tau). \]

Thus, to see a non-trivial behavior around \( \tau \) we need to scale \( \varepsilon \sim N^{-1} \).

We also know that

\[ E\left[ \omega_\eta(D_{1/N}(\tau)) \right] = \int_{D_{1/N}(\tau)} R_1(z) dA(z) = \int_{D_{1/N}(\tau)} K_N(z, z) dA(z) \]

\[ = \int \frac{1}{N^2} K_N \left( \tau + \frac{z}{N}, \tau + \frac{z}{N} \right) dA(z). \]

Thus, we expect integrand to converge and therefore set

\[ K_\tau(z, w) := \lim_{N \to \infty} \frac{1}{N^2} K_N \left( \tau + \frac{z}{N}, \tau + \frac{w}{N} \right). \]
Theorem (M.Y. and C. Sinclair)

Under the conditions of the previous theorem, it holds that

$$K_{\tau}(z, w) = \frac{\omega(\tau, z)\omega(\tau, w)}{\pi} \int_0^1 x(1 - \ell x)e^{a(\tau, z) + a(\tau, w)x}dx,$$

where $a(\tau, z) := z\Phi'(\tau)\overline{\Phi(\tau)}$ (the argument of $a(\tau, z)$ is equal to the angle between $z$ and the outward normal to $\partial K$), $\ell = \lim_{N \to \infty} s^{-1}N$, and

$$\omega(\tau, z) := \lim_{N \to \infty} \exp \left\{-s g_K \left( \tau + \frac{z}{N} \right) \right\}.$$

Remark

$$\int_0^1 x(1 - \ell x)e^{\eta x}dx = (1 - \ell) \frac{e^{\eta(\eta - 1)} + 1}{\eta^2} + \ell \frac{e^{\eta(\eta - 2)} + \eta + 2}{\eta^3}.$$
\[ K^{\sin}(a, b) = \frac{\sin(a - b)}{a - b} = \frac{e^\eta - e^{-\eta}}{2\eta}, \quad \eta = i(a - b) \]

Plots of \( R_{2}^{\sin}(a, b) = 1 - K^{\sin}(a, b)^2 \) as a function of \( 2(a - b) \). The second plot is an enlargement of the shaded region.
\[
\frac{\pi K_\tau(z, w)}{\omega(\tau, z) \omega(\tau, w)} = (1 - \ell) \frac{e^\eta(\eta - 1) + 1}{\eta^2} + \ell \frac{e^\eta(\eta - 2) + \eta + 2}{\eta^3}
\]

where \( \eta = a(\tau, z) + \overline{a(\tau, w)} \) which can be parametrized as \( i(a - b) \) in the tangential direction (in which case \( \omega(\tau, z) = 1 \)).
Plot of the interpolation between $R_2^0$ and $R_2^1$ along a tangent line.