Root statistics of random polynomials with bounded Mahler measure

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The **Mahler measure** of a polynomial \( P(z) = a \prod_{n=1}^{N} (z - \alpha_n) \) is defined as

\[
M(P) := |a| \prod_{n=1}^{N} \max \{1, |\alpha_n|\}
\]

\[
= \exp \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} \log |P(e^{i\theta})| \, d\theta \right\}.
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**Theorem (Kronecker, 1857)**

\( M(P) = 1 \) for a polynomial \( P \) with integer coefficients iff \( P \) is a product of monomials and cyclotomic polynomials (divisors of \( z^n - 1 \)). Necessarily, such a polynomial has all its roots in \( \mathbb{T} \cup \{0\} \).
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**Conjecture (Lehmer, 1933)**

Is 1 an isolated point of the range of \( M(\cdot) \) on integer polynomials?

Lehmer himself constructed the smallest known example:

\[
M(z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1) \approx 1.18.
\]
Theorem (Chern-Vaaler, 2001)

The number of integer polynomials of height at most $T$ behaves as

$$\text{vol}(B_N) T^{N+1} + \mathcal{O}(T^N), \quad T \to \infty,$$

where $B_N$ is the Mahler measure unit star body. Moreover,

$$\text{vol}(B_N) = \frac{2}{N+1} F_N(N+1),$$

where

$$F_N(s) = C_N \prod_{m=0}^{\lfloor (N-1)/2 \rfloor} \frac{s}{s-(N-2m)}$$

and $C_N$ is an explicit constant.

Notice that Lehmer’s conjecture asks what happens when $T \to 1$.

Observe also that both $(z-1)^N$ and $z^N - 1$ belong to $B_N$ but have drastically different coefficient vectors.
More generally, the \( \lambda \)-homogeneous Mahler measure is given by

\[
M^\lambda(P) = |a|^\lambda \prod_{n=1}^{N} \max \{1, |\alpha_n|\}.
\]

The corresponding unit star body is defined as

\[
B^\lambda_N := \left\{(a_1, \ldots, a_{N+1}) \in \mathbb{R}^{N+1} : M^\lambda \left( \sum_{n=0}^{N} a_{n+1}z^n \right) \leq 1 \right\}.
\]

**Theorem (Chern-Vaaler, 2001)**

\[
\text{vol}(B^\lambda_N) = \frac{2}{N+1} F_N \left( \frac{N+1}{\lambda} \right).
\]
Volumes of Star Bodies

\[ \text{vol}(B_N^\lambda) = \int_{-\infty}^{\infty} \text{vol} \left\{ b : M(\lambda) \left( c z^N + \sum_{n=0}^{N-1} b_{n+1} z^n \right) \leq 1 \right\} \, dc \]

\[ = \int_{-\infty}^{\infty} \text{vol} \left\{ c b : M(\lambda) \left( c z^N + \sum_{n=0}^{N-1} c b_{n+1} z^n \right) \leq 1 \right\} \, dc. \]

Using the \( \lambda \)-homogeneity of \( M^\lambda \) one then gets

\[ \text{vol}(B_N^\lambda) = 2 \int_0^\infty c^N \text{vol} \left\{ b : M(b) \leq c^{-\lambda} \right\} \, dc \]

\[ = \frac{2}{\lambda} \int_0^{\infty} \xi^{-(N+1)/\lambda} \text{vol} \{ b : M(b) \leq \xi \} \frac{d\xi}{\xi}, \]

where \( M(b) \) is the Mahler measure of \( z^N + \sum_{n=0}^{N-1} b_{n+1} z^n \). Integration by parts then gives

\[ \text{vol}(B_N^\lambda) = \frac{2}{N+1} \int_0^{\infty} \xi^{-(N+1)/\lambda} \text{dvol} \{ b : M(b) \leq \xi \} \]

\[ = \frac{2}{N+1} \int_{\mathbb{R}^N} M(b)^{-(N+1)/\lambda} \text{d}\mu_N^\mathbb{R}(b). \]
Making a change of variables from coefficients of polynomials to their roots gives

\[ F_N(s) := \int_{\mathbb{R}^N} M(b)^{-s} d\mu_R^N(b) = \sum_{L+2M=N} \frac{Z_{L,M}(s)}{L!M!}, \]

where \( L \) and \( M \) stand for the number of real and complex roots, and

\[ Z_{L,M}(s) = \int_{\mathbb{R}^L} \int_{\mathbb{C}^M} \prod_{l=1}^L \Phi(\alpha_l)^{-s} \prod_{m=1}^M \Phi(\beta_m)^{-2s} |\Delta(\alpha, \beta)| d\mu_R^L(\alpha) d\mu_C^M(\beta) \]

with \( \Delta(\alpha, \beta) \) being the Vandermonde of \( \alpha_1, \ldots, \alpha_L, \beta_1, \beta_1, \ldots, \beta_M, \overline{\beta}_M \).

The summands \( Z_{L,M}(s) \) are not simple and Chern-Vaaler went through a dozen pages of rational function identities to show that

\[ F_N(s) = C_N \prod_{m=0}^{[(N-1)/2]} \frac{s}{s - (N - 2m)}. \]
In fact, one could consider polynomials with complex coefficients. Set

\[ B_\lambda^N(\mathbb{C}) := \left\{ (a_1, \ldots, a_{N+1}) \in \mathbb{C}^{N+1} : M^\lambda \left( \sum_{n=0}^{N} a_{n+1}z^n \right) \leq 1 \right\}. \]

Then

\[
\begin{align*}
\text{vol}(B_\lambda^N(\mathbb{C})) &= \int_{\mathbb{C}} \text{vol} \left\{ b : M^\lambda \left( cz^N + \sum_{n=0}^{N-1} b_{n+1}z^n \right) \leq 1 \right\} \, dc \\
&= \int_{\mathbb{C}} |c|^{2N} \text{vol} \left\{ b : M(b) \leq |c|^{-\lambda} \right\} \, dc \\
&= \frac{\pi}{N+1} \int_0^\infty \xi^{-2(N+1)/\lambda} \, d\text{vol}\{b : M(b) \leq 1\} \\
&= \frac{\pi}{N+1} \int_{\mathbb{C}^N} M(b)^{-2(N+1)/\lambda} \mu_N^C(b).
\end{align*}
\]
As before, making a change of variables from the coefficients to the roots gives

\[ G_N(s) := \int_{\mathbb{C}^N} M(b)^{-2s} d\mu_C(b) = \frac{Z_N(s)}{N!}, \]

where

\[ Z_N(s) = \int_{\mathbb{C}^N} \prod_{n=1}^{N} \Phi(\lambda_n)^{-2s} |\Delta(\lambda)|^2 d\mu_C(\lambda) \]

\[ = (2\pi)^N \sum_{\sigma} \left( \prod_{n=1}^{N} \int_{0}^{\infty} \Phi(\rho_n)^{-2s} \rho_n^{2\sigma(n)-1} d\rho_n \right) \]

\[ = (2\pi)^N \sum_{\sigma} \left( \prod_{n=1}^{N} \frac{s}{2\sigma(n)(s-\sigma(n))} \right) = \pi^N \prod_{n=1}^{N} \frac{s}{s-n}. \]

**Theorem (Chern-Vaaler, 2001)**

\[ \text{vol}(B_N^\lambda(\mathbb{C})) = \frac{\pi}{N+1} G_N \left( \frac{N+1}{\lambda} \right). \]
Determinantal Interpretation

**Theorem (Sinclair, 2008)**

Let $\Pi_0, \ldots, \Pi_{N-1}$ be polynomials such that

$$\langle \Pi_n | \Pi_m \rangle = \delta_{n,m},$$

where inner product $\langle \cdot | \cdot \rangle$ is defined by

$$\langle f | g \rangle_C = \int_C f(z) \overline{g(z)} \Phi(z)^{-2s} d\mu_C(z),$$

with $\Phi(z) := \max\{1, |z|\}$. Then

$$G_N(s) = \prod_{n=0}^{N-1} \gamma_n^{-2},$$

where $\Pi_k(z) = \gamma_k z^k + \cdots$. 
Theorem (Sinclair, 2008)

Let $\pi_0, \ldots, \pi_{N-1}$ be polynomials such that

$$\langle \pi_{2n} | \pi_{2m} \rangle = \langle \pi_{2n+1} | \pi_{2m+1} \rangle = 0 \quad \text{and} \quad \langle \pi_{2n} | \pi_{2m+1} \rangle = \delta_{n,m},$$

where skew-symmetric inner product $\langle \cdot | \cdot \rangle = \langle \cdot | \cdot \rangle_R + \langle \cdot | \cdot \rangle_C$ is defined by

$$\langle f | g \rangle_R = \int_R \int_R f(x)g(y) \text{sgn}(y - x) \Phi(x)^{-s} \Phi(y)^{-s} d\mu_R(x) d\mu_R(y),$$

$$\langle f | g \rangle_C = -2i \int_C \overline{f(z)} g(z) \text{sgn}(\text{Im}(z)) \Phi(z)^{-2s} d\mu_C(z),$$

with $\Phi(z) = \max\{1, |z|\}$. Then

$$F_N(s) = \prod_{n=0}^{\lfloor (N-1)/2 \rfloor} \left( \gamma_{2n} \gamma_{2n+1} \right)^{-1},$$

where $\pi_k(z) = \gamma_k z^k + \cdots$. 
Recall that

\[ G_N(s) = \int_{\mathbb{C}^N} M(b)^{-2s} \, d\mu_{\mathbb{C}}^N(b) \quad \text{and} \quad F_N(s) = \int_{\mathbb{R}^N} M(b)^{-s} \, d\mu_{\mathbb{R}}^N(b). \]

Under a **random polynomial** we mean a polynomial chosen with respect to

\[ M(b)^{-2s}/G_N(s), \quad b \in \mathbb{C}^N, \quad \text{or} \quad M(b)^{-s}/F_N(s), \quad b \in \mathbb{R}^N. \]

This is equivalent to choosing polynomials uniformly from \( B_{N}^{(N+1)s^{-1}} \).

We would like to study fine statistics of zeros of such random polynomials.
A simultaneous plot of the roots of 100 random polynomials of degree 28. A ball-walk of 10,000 steps of length .01 starting from $x^{28}$ was performed for each polynomial. The arrows indicate directions of outlying roots.
Let $P$ be a random polynomial. For $C \subset \mathbb{C}$ define $N_C := \# C \cap \{ \text{zeros of } P \}$.

In the case of complex coefficients, a function $R_n : \mathbb{C}^n \to [0, \infty)$ is called $n$-th correlation function if

$$E[N_{C_1} \cdots N_{C_n}] = \int_{C_1} \cdots \int_{C_n} R_n(z) d\mu^n_C(z)$$

for pairwise disjoint sets $C_1, \ldots, C_n$. Since the joint density of the zeros is given by

$$\frac{1}{Z_N(s)} \prod_{m < n} |\lambda_n - \lambda_m|^2 \prod_{n=1}^{N} \Phi(\lambda_n)^{-2s} d\mu^N_C(\lambda),$$

$\Phi(z) = \max\{1, |z|\}$, it is well known in random matrix theory that

$$R_n(\lambda) = \det \left[ K_N(\lambda_i, \lambda_j) \right]_{i,j=1}^n,$$

where

$$K_N(z, w) := \Phi(z)^{-s} \Phi(w)^{-s} \sum_{n=0}^{N-1} \prod_n(z) \overline{\prod_n(w)}$$

and $\prod_n$ are orthonormal polynomials w.r.t. $\Phi^{-2s}(z) d\mu_C(z)$. 
In the case of real coefficients, if there is a function $R_{l,m} : \mathbb{R}^l \times \mathbb{C}_+^m \rightarrow [0, \infty)$ such that

$$E\left[N_{A_1} \cdots N_{A_l} N_{B_1} \cdots N_{B_m}\right] := \int_{A_1} \cdots \int_{A_l} \int_{B_1} \cdots \int_{B_m} R_{l,m}(x, z) d\mu_R^l(x) d\mu_C^m(z)$$

for pairwise disjoint sets $A_1, \ldots, A_l \subset \mathbb{R}$ and $B_1, \ldots, B_m \subset \mathbb{C}_+$, then it is called the $(l, m)$-th correlation function.

When such functions exist, it holds in particular that

$$\deg(P) = \int_{\mathbb{R}} R_{1,0}(x, -) d\mu_R(x) + \int_{\mathbb{C}} R_{0,1}(-, z) d\mu_C(z)$$

and the first integral represents the expected number of real zeros, where we set $R_{l,m}(\cdot, \bar{z}) := R_{l,m}(\cdot, z)$. 
**Theorem (Borodin-Sinclair, 2009)**

There exists a $2 \times 2$ matrix kernel $K_N : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}^{2 \times 2}$ such that

$$R_{l,m}(x, z) = \text{Pf} \begin{bmatrix} [K_N(x_i, x_j)]_{i,j=1}^{l} & [K_N(x_i, z_n)]_{i,n=1}^{l,m} \\ -[K_N^T(z_k, x_j)]_{k,j=1}^{m,l} & [K_N(z_k, z_n)]_{k,n=1}^{m} \end{bmatrix}.$$

In particular, it holds that

$$R_{1,0}(x, -) = \text{Pf} K_N(x, x) \text{ and } R_{0,1}(-, z) = \text{Pf} K_N(z, z).$$

Recall that we set $\langle \cdot | \cdot \rangle = \langle \cdot | \cdot \rangle_{\mathbb{R}} + \langle \cdot | \cdot \rangle_{\mathbb{C}}$, where

$$\langle f | g \rangle_{\mathbb{R}} = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x)g(y)\text{sgn}(y - x)\Phi(x)^{-s}\Phi(y)^{-s}d\mu_{\mathbb{R}}(x)d\mu_{\mathbb{R}}(y)$$

$$\langle f | g \rangle_{\mathbb{C}} = -2i \int_{\mathbb{C}} \overline{f(z)}g(z)\text{sgn}(\text{Im}(z))\Phi(z)^{-2s}d\mu_{\mathbb{C}}(z).$$
Theorem (Borodin-Sinclair, 2009)

Let $N = 2J$ and $\pi_0, \ldots, \pi_{N-1}$ be skew-orthogonal polynomials w.r.t. $\langle \cdot | \cdot \rangle$. Set

$$\kappa_N(u, v) := 2\Phi(u)^{-s}\Phi(v)^{-s} \sum_{n=0}^{J} \left( \pi_{2n}(u)\pi_{2n+1}(v) - \pi_{2n}(v)\pi_{2n+1}(u) \right).$$

Then

$$K_N(u, v) = \begin{bmatrix} \kappa_N(u, v) & \kappa_N \epsilon(u, v) \\ \epsilon \kappa_N(u, v) & \epsilon \kappa_N \epsilon(u, v) + \frac{1}{2} \text{sgn}(u - v) \end{bmatrix},$$

where $\text{sgn}(\cdot) = 0$ for non-real arguments and $\epsilon$ is the operator

$$\epsilon f(u) := \begin{cases} \frac{1}{2} \int_{\mathbb{R}} f(t)\text{sgn}(t - u) d\mu_{\mathbb{R}}(t), & u \in \mathbb{R}, \\ i \cdot \text{sgn}(\text{Im}(u)) f(\overline{u}), & u \in \mathbb{C} \setminus \mathbb{R}, \end{cases}$$

which acts on $u$ when written on the left and on $v$ when written on the right.
The following results are from Sinclair-Ya. 2012 (complex case) and 2015 (real case).

**Theorem**

It holds that

\[
\pi_{2n}(z) = \frac{2}{\pi} \sum_{k=0}^{n} \frac{\Gamma(k + 3/2)\Gamma(n - k + 1/2)}{\Gamma(k + 1)\Gamma(n - k + 1)} z^{2k}
\]

and

\[
\pi_{2n+1}(z) = -\frac{1}{2\pi} \sum_{k=0}^{n} \frac{s - (2k + 2)}{2s} \frac{\Gamma(k + 3/2)\Gamma(n - k - 1/2)}{\Gamma(k + 1)\Gamma(n - k + 1)} z^{2k+1}.
\]

It is also true that

\[
\Pi_n(z) = \sqrt{\frac{n+1}{\pi}} \left(1 - \frac{n+1}{s}\right) z^n.
\]
Write \( \tilde{\pi}_k(z) := \pi_k(z)\Phi(z)^{-s} \). Given \( A \subseteq \mathbb{R} \) and \( N \) even, it holds that

\[
E[N_A] = \int_A \text{Pf}\left[\begin{array}{cc} 0 & \kappa_N \epsilon(x, x) \\ \epsilon \kappa_N(x, x) & 0 \end{array}\right] d\mu_{\mathbb{R}}(x)
\]

\[
= \int_A \text{Pf} \left[ \begin{array}{cc} 0 & \kappa_N \epsilon(x, x) \\ \epsilon \kappa_N(x, x) & 0 \end{array} \right] d\mu_{\mathbb{R}}(x)
\]

\[
= 2 \sum_{n=0}^{N/2} \int_A (\tilde{\pi}_{2n}(x)\epsilon \tilde{\pi}_{2n+1}(x) - \tilde{\pi}_{2n+1}(x)\epsilon \tilde{\pi}_{2n}(x)) d\mu_{\mathbb{R}}(x).
\]
Expected Number of Real Zeros

**Theorem**

Let $N_{\text{in}}$ and $N_{\text{out}}$ be the number of real roots on $[-1, 1]$ and $\mathbb{R} \setminus (-1, 1)$. Then

\[
\begin{align*}
E[N_{\text{in}}] &= \frac{1}{\pi} \log N + O_N(1) \\
E[N_{\text{out}}] &= -\frac{1}{\pi} \frac{\sqrt{N(2s - N)}}{s} \log (1 - Ns^{-1}) + \sqrt{Ns^{-1}} O_N(1),
\end{align*}
\]

where the implicit constants are uniform with respect to $s$.

Observe that

\[
E[N_{\text{out}}] = \begin{cases}
\sqrt{Ns^{-1}} O_N(1), & \limsup_{N \to \infty} Ns^{-1} < 1, \\
\frac{\alpha}{\pi} \log N + O_N(1), & s = N + N^{1-\alpha}, \alpha \in [0, 1], \\
\frac{1}{\pi} \log N + O_N(1), & \limsup_{N \to \infty} (s - N) < \infty.
\end{cases}
\]
Let $\zeta \in \mathbb{T}$ and $\delta$ be small. In the complex case we have that

$$E\left[ N_{\zeta + \delta \mathbb{D}} \right] = \int_{\zeta + \delta \mathbb{D}} K_N(z, z) d\mu_C(z)$$

$$= \int_{\mathbb{D}} \delta^2 K_N(\zeta + \delta z, \zeta + \delta z) d\mu_C(z).$$

Similarly, in the real case we have for $\zeta \in \mathbb{T} \setminus \{\pm 1\}$ that

$$E\left[ N_{\zeta + \delta \mathbb{D}} \right] = \int_{\zeta + \delta \mathbb{D}} \text{Pf} K_N(z, z) d\mu_C(z).$$

As we have $N$ total zeros, the scale should be $\delta = 1/N$. 
**Theorem**

Let $\zeta \in \mathbb{T}$. Assume that $\lambda := \lim_{N \to \infty} Ns^{-1} \in [0, 1]$ exists. Then

\[
\lim_{N \to \infty} \frac{1}{N^2} K_N \left( \zeta + \frac{z}{N}, \zeta + \frac{w}{N} \right) = K_\zeta(z, w),
\]

where $\omega(\tau) := \min \left\{ 1, e^{-\text{Re}(\tau)/\lambda} \right\}$ and

\[
K_\zeta(z, w) = \omega(z\bar{\zeta})\omega(w\bar{\zeta}) \frac{1}{\pi} \int_0^1 x(1 - \lambda x) e^{(z\bar{\zeta} + w\bar{\zeta})x} \, dx.
\]

It holds that $\text{Re}(z\bar{\zeta}) > 0$ iff $z$ points outside $\mathbb{D}$ at $\zeta$. 
**Theorem**

Let $\zeta \in \mathbb{T} \setminus \{\pm 1\}$. Assume that $\lambda := \lim_{N \to \infty} N s^{-1} \in [0, 1]$ exists. Then

$$
\lim_{N \to \infty} \frac{1}{N^2} K_N \left( \frac{\zeta}{N}, \frac{\zeta}{N} \right) = \begin{bmatrix} 0 & K_{\zeta}(z, w) \\ -K_{\zeta}(w, z) & 0 \end{bmatrix},
$$

That is, Pfaffian point process becomes essentially determinantal around $\zeta$. 

**Theorem**

Let $\xi \in \{\pm 1\}$. Assuming that $\lambda := \lim_{N \to \infty} N s^{-1} \in [0, 1]$ exists, it holds that

$$
\lim_{N \to \infty} \frac{1}{N^2} \kappa_N \left( \frac{\xi}{N}, \xi \right) = \kappa_{\xi}(u, v)
$$

where the convergence is locally uniform in $\mathbb{C} \times \mathbb{C}$, $\kappa_{\xi}(u, v) = \omega(u \xi) \omega(v \xi) \xi^4 \int_0^1 \tau (1 - \lambda \tau) \left( M'(u \xi \tau) M(v \xi \tau) - M(u \xi \tau) M'(v \xi \tau) \right) d\tau$,

and $M(z) = F_1(3/2, 1; z)$, i.e.,

$$
\frac{z M''(z)}{2} + (1 - z) M'(z) - \frac{3}{2} M(z) = 0.
$$
Theorem

Let $\zeta \in \mathbb{T} \setminus \{\pm 1\}$. Assume that $\lambda := \lim_{N \to \infty} N s^{-1} \in [0, 1]$ exists. Then

$$\lim_{N \to \infty} \frac{1}{N^2} K_N \left( \zeta + \frac{z}{N}, \zeta + \frac{w}{N} \right) = \begin{bmatrix} 0 & K_\zeta(z, w) \\ -K_\zeta(w, z) & 0 \end{bmatrix},$$

That is, Pfaffian point process becomes essentially determinantal around $\zeta$.

Theorem

Let $\xi \in \{\pm 1\}$. Assuming that $\lambda := \lim_{N \to \infty} N s^{-1} \in [0, 1]$ exists, it holds that

$$\lim_{N \to \infty} \frac{1}{N^2} \kappa_N \left( x + \frac{u}{N}, \xi + \frac{v}{N} \right) = \kappa_\xi(u, v)$$

where the convergence is locally uniform in $\mathbb{C} \times \mathbb{C}$,

$$\kappa_\xi(u, v) = \omega(u \xi) \omega(v \xi) \frac{\xi}{4} \int_0^1 \tau(1-\lambda \tau) \left( M'(u \xi \tau) M(v \xi \tau) - M(u \xi \tau) M'(v \xi \tau) \right) d\tau,$$

and $M(z) = \, _1F_1(3/2, 1; z)$, i.e., $z M''(z) + (1 - z) M'(z) - \frac{3}{2} M(z) = 0$. 
The scaled intensity of complex roots near 1, for $\lambda = 1$ (left) and $\lambda = 0$ (right). Note how the roots tend to accumulate near the unit disk (the $y$-axis here) and repel from the real axis.
Theorem

Assuming that $\lambda := \lim_{N \to \infty} N s^{-1} \in [0, 1]$ exists, it holds that

$$\lim_{N \to \infty} K_N(z, w) = \frac{1}{\pi} \frac{1}{(1 - zw)^2}$$

and

$$\lim_{N \to \infty} \kappa_N(u, v) = \frac{1}{4\pi} \int_{\mathbb{T}} \frac{(v \sqrt{-\tau} - u \sqrt{-\tau}) |d\tau|}{(1 - u^2 \tau)^{3/2} (1 - v^2 \tau)^{3/2}}$$

locally uniform in $\mathbb{D} \times \mathbb{D}$, where $\sqrt{-\tau}$ is the branch defined by $-\frac{2}{\pi} \sum_{-\infty}^{\infty} \frac{\tau^m}{2m-1}$. 

Assuming that $\lambda := \lim_{N \to \infty} Ns^{-1} \in [0, 1] \text{ and } c := \lim_{N \to \infty} (s - N) \in [0, \infty]$ exist, it holds that

$$
\lim_{N \to \infty} \frac{\left|zw\right|^s}{(zw)^N} \frac{K_N(z, w)}{s - N} = \frac{\lambda}{\pi} \frac{1}{z\overline{w} - 1} \left[ 1 + \frac{c^{-1}}{z\overline{w} - 1} \right]
$$

and

$$
\lim_{N \to \infty} \frac{\left|uv\right|^s}{(uv)^N} \frac{\kappa_N(u, v)}{s - N} = \frac{\lambda}{\pi} \frac{1}{uv - 1} \left[ 1 + \frac{c^{-1}}{uv - 1} \right] \frac{v - u}{\sqrt{u^2 - 1}\sqrt{v^2 - 1}}.
$$
Expected Number of Zeros on Bounded Subsets of $\mathbb{C} \setminus \mathbb{D}$

The limiting intensity of complex roots outside the disk, with a close up view near $z = 1$, for the Mahler measure ($c = 1$) case.