Szegő-type Asymptotics of Frobenius–Padé Approximants

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Let $F$ be a function holomorphic at the origin:

$$F(z) = f_0 + f_1 z + f_2 z^2 + \cdots + f_m z^m + \cdots$$

and $P_n$ be its $n$-th Taylor polynomial:

$$P_n(z) = f_0 + f_1 z + f_2 z^2 + \cdots + f_n z^n.$$ 

Then

$$P_n(z) \Rightarrow F(z) \quad \text{as} \quad n \to \infty$$

in the largest disk of holomorphy of $F$. 
Let $P_n$ and $Q_m$ be polynomials of respective degrees at most $n$ and $m$ such that

$$(Q_mF - P_n)(z) = O(z^{m+n+1}) \text{ as } z \to 0.$$ 

The rational function $P_n/Q_m$ is unique and is called $(n,m)$-th Padé approximant at the origin.

**Theorem (de Montessus de Ballore)**

If $D$ is the largest disk centered at the origin where $F$ has exactly $m$ poles counting multiplicities, then

$$(P_n/Q_m)(z) \Rightarrow F(z) \text{ as } n \to \infty$$ 

in the spherical metric.
Let \( f \) be a function holomorphic and vanishing at infinity:

\[
f(z) = \frac{f_1}{z} + \frac{f_2}{z^2} + \cdots + \frac{f_n}{z^n} + \cdots.
\]

Further, let \( p_n, q_n \) be a pair of polynomials of degree at most \( n \) such that

\[
R_n(z) := (q_n f - p_n)(z) = \frac{1}{m_n z^{n+1}} + O(z^{-n-2}) \quad \text{as} \quad z \to \infty.
\]

The rational function \( p_n/q_n \) is always unique and is called the \( n \)-th diagonal Padé approximant to \( f \) at infinity.

**Theorem (Markov)**

If \( \sigma \) is a compactly supported positive Borel measure on the real line, then

\[
(p_n/q_n)(z) \Rightarrow f(z) = \int \frac{d\sigma(t)}{t - z} \quad \text{as} \quad n \to \infty
\]

locally uniformly in \( \overline{\mathbb{C}} \setminus I \), where \( I \) is the smallest interval containing \( \text{supp}(\sigma) \).
It is easy to show that

$$\int x^i q_n(x) d\sigma(x) = 0, \quad i \in \{0, \ldots, n - 1\}.$$ 

Let $w(z) := \sqrt{z^2 - 1}$ and $\rho(z)$ be holomorphic and non-vanishing in a neighborhood of $[-1, 1]$. By assuming

$$d\sigma(x) = \frac{\rho(x)}{2\pi i} \frac{dx}{w^+(x)}$$

and studying (thanks to Fokas, Its, and Kitaev) the matrices

$$Y_n := \begin{pmatrix} q_n & R_n \\ m_{n-1}q_{n-1} & m_{n-1}R_{n-1} \end{pmatrix}$$

via the steepest descent method of Deift and Zhou, one can get very precise asymptotics of $q_n$ and $R_n$. 
In particular, one deduces Szegő’s asymptotics:

\[
\begin{cases}
q_n(z) = (1 + o(1)) \left( \Phi^n S_\rho \right)(z), \\
(wR_n)(z) = (1 + o(1)) \left( \Phi^n S_\rho \right)^{-1}(z),
\end{cases}
\]

where \( \Phi(z) := z + w(z) \) and

\[
S_\rho(z) = \exp \left\{ \frac{w(z)}{2\pi i} \int_{[-1,1]} \frac{\log \rho(x)}{x - z} \frac{dx}{w^+(x)} \right\},
\]

which satisfies \( S_\rho^+ S_\rho^- = \rho \) on \((-1,1)\). We can rewrite the asymptotic formulae as

\[
\begin{cases}
q_n(z) = (1 + o(1)) \left( \Phi^n S \right)^{(0)}(z), \\
(wR_n)(z) = (1 + o(1)) \left( \Phi^n S \right)^{(1)}(z).
\end{cases}
\]
Orthogonality \( \int x' q_n(x) d\sigma(x) = 0 \) tell us that the logarithmic potential

\[
V^\tau(z) := -\int \log |z - x| d\tau(x)
\]

of a weak* limit point of the normalized counting measures of the zeros of \( q_n \) is such that

\[
2V^\tau = \min_{\text{supp} \mu} 2V^\tau \quad \text{on} \quad \text{supp} (\tau) \subseteq \text{supp} (\mu).
\]

When \( \text{supp} (\mu) = [-1, 1] \), the measure \( \tau \) is necessarily the arcsine distribution on \([-1, 1]\). Then we construct a Riemann surface \( \mathcal{R} \) using \( \text{supp} (\tau) = [-1, 1] \) and look for a rational function on \( \mathcal{R} \) with a simple pole at \( \infty^{(0)} \) and a simple zero at \( \infty^{(1)} \) and a function \( S \) that solves a certain boundary value problem.

\[
\Phi^{(0)} = \Phi \quad \Phi^{(1)} = \Phi^{-1} \quad S^{(0)} = S_{\rho} \quad S^{(1)} = S^{-1}_{\rho}
\]
Let $\mu$ be a positive Borel measure on $[a, b]$ and $p_n$ be orthonormal polynomials w.r.t $\mu$, i.e., $\int p_np_md\mu = \delta_{mn}$. Given $f \in L^2(\mu)$, associate

$$f \sim \sum_{i=0}^{\infty} c_i(f)p_i, \quad c_i(f) := \int fp_i d\mu.$$

**Theorem (Freud + Mastroianni & Totik)**

If $f \in \text{Lip}^{1+\epsilon}$ and the Christoffel functions satisfy

$$n\lambda_n(x, \mu) \lesssim 1$$

uniformly on a set $S \subseteq [a, b]$, then

$$\sum_{i=0}^{n-1} c_i(f)p_i \Rightarrow f \quad \text{on} \quad S.$$

The condition on the Christoffel functions is satisfied if $\mu$ is doubling on $[a, b]$. 

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8 / 16
A Frobenius–Padé approximant of type \((m, n)\) to \(f\) is a rational function

\[
\frac{P_{m,n}}{Q_{m,n}}, \quad \deg(P_{m,n}) \leq m, \quad \deg(Q_{m,n}) \leq n,
\]
such that

\[
c_i(Q_{m,n}f - P_{m,n}) = 0, \quad i \in \{0, \ldots, m + n\}.
\]

A Frobenius–Padé approximant always exists as its construction boils down to solving a linear system with one more unknown than equations.

A Frobenius–Padé approximant corresponding to \(Q_{m,n}\) of the smallest degree is unique.

\[
\deg(Q_{m,n}) = n \quad \Rightarrow \quad \text{Uniqueness.}
\]
Let
\[ f(z) = \int \frac{d\sigma(x)}{x - z}. \]

The measures \( \mu \) and \( \sigma \) are such that
\[ \Delta_\mu := \text{supp}(\mu) = [b_\mu, a_\mu], \quad \Delta_\sigma := \text{supp}(\sigma) = [a_\sigma, b_\sigma] \]

and
\[ \Delta_\mu \cap \Delta_\sigma = \emptyset. \]

We shall also assume that
\[ n - 1 \leq m, \quad \frac{n}{n + m} \to c \in (0, 1/2] \quad \text{as} \quad n \to \infty. \]
Assume for now that $\mu$ and $\sigma$ are positive measures. Recall that

$$c_i(Q_{m,n}f - P_{m,n}) = 0, \quad i \in \{0, \ldots, m + n\}.$$  

Write $R_{m,n} := Q_{m,n}f - P_{m,n}$. Then

$$\int x^i R_{m,n}(x) d\mu(x) = 0, \quad i \in \{0, \ldots, m + n\}.$$  

Let $V_{m,n}$ be the polynomial vanishing at the zeros of $R_{m,n}$ on $\Delta_{\mu}$. Cauchy tells us that

$$\int \frac{x^i Q_{m,n}(x)}{V_{m,n}(x)} d\sigma(x) = 0, \quad i \leq \min\{n - 1, m\} = n - 1.$$  

Using Cauchy’s work again, we get that

$$\int \frac{x^i V_{m,n}(x)}{Q_{m,n}(x)} \left( \int \frac{Q_{m,n}^2(t)}{V_{m,n}(t) t - x} d\sigma(t) \right) d\mu(x) = 0, \quad i \in \{0, \ldots, m + n\}.$$
We have that
\[
\int x^i \frac{Q_{m,n}(x)}{V_{m,n}(x)} \, d\sigma(x) = 0, \quad i \in \{0, \ldots, n-1\},
\]
and
\[
\int x^i \frac{V_{m,n}(x)}{Q_{m,n}(x)} \left( \cdot \right) \, d\mu(x) = 0, \quad i \in \{0, \ldots, m+n\}.
\]

Weak* limits of the counting measures of zeros then give us
\[
\left\{ \begin{array}{l}
Q_{m,n} \Rightarrow \tau_{\sigma,c}, \quad |\tau_{\sigma,c}| = c, \quad \text{supp}(\tau_{\sigma,c}) \subseteq \Delta_{\sigma}, \\
V_{m,n} \Rightarrow \tau_{\mu,c}, \quad |\tau_{\mu,c}| = 1, \quad \text{supp}(\tau_{\mu,c}) \subseteq \Delta_{\mu},
\end{array} \right.
\]
and we expect their logarithmic potentials to satisfy
\[
\left\{ \begin{array}{l}
2V^{\tau_{\sigma,c}} - V^{\tau_{\mu,c}} = \min_{\Delta_{\sigma}} (2V^{\tau_{\sigma,c}} - V^{\tau_{\mu,c}}) =: \ell_{\sigma,c} \quad \text{on} \quad \text{supp}(\tau_{\sigma,c}), \\
2V^{\tau_{\mu,c}} - V^{\tau_{\sigma,c}} = \min_{\Delta_{\mu}} (2V^{\tau_{\mu,c}} - V^{\tau_{\sigma,c}}) =: \ell_{\mu,c} \quad \text{on} \quad \text{supp}(\tau_{\mu,c}).
\end{array} \right.
\]
We are looking for measures such that

\[
\begin{align*}
2V^{\tau_\sigma,c} - V^{\tau_\mu,c} &= \ell_{\sigma,c} \quad \text{on} \quad \text{supp}(\tau_{\sigma,c}) \subseteq \Delta_\sigma, \\
2V^{\tau_\mu,c} - V^{\tau_\sigma,c} &= \ell_{\mu,c} \quad \text{on} \quad \text{supp}(\tau_{\mu,c}) \subseteq \Delta_\mu.
\end{align*}
\]

**Proposition (Gonchar, Rakhmanov, & Sorokin)**

Such a pair of measures exists and is unique, \(\text{supp}(\tau_{\mu,c}) = \Delta_\mu\), and \(\text{supp}(\tau_{\sigma,c}) =: \Delta_{\sigma,c}\) is an interval. Set \(D^+_{\sigma,c} := \{2V^{\tau_\sigma,c} - V^{\tau_\mu,c} - \ell_{\sigma,c} < 0\}\). Then it is non-empty, contains \(\Delta_{\sigma,c}\) in its boundary, is bounded when \(c < 1/2\), and is equal to \(\overline{C} \setminus \Delta_\sigma\) when \(c = 1/2\).
Define $\Phi_{m,n}$ on $\mathcal{R}_{\frac{n}{n+m}}$ as having a divisor

$$(n + m)^{\infty (2)} - n^{\infty (0)} - m^{\infty (1)}$$

and normalized so that

$$\Phi^{(0)}_{m,n}(z)\Phi^{(1)}_{m,n}(z)\Phi^{(2)}_{m,n}(z) \equiv 1.$$
Theorem (Aptekarev, Bogolubsky, & Y.)

Let
\[ d\nu(x) = \frac{\rho \nu(x)}{2\pi i} \frac{dx}{w_\nu^+(x)}, \quad w_\nu(z) = \sqrt{(z - a_\nu)(z - b_\nu)}, \quad \nu \in \{\mu, \sigma\}, \]
where \( \rho \nu \) is holomorphic and non-vanishing around \( \Delta_\nu \). Assume \( \mu \) possesses the full system of orthonormal polynomials. Then
\[
\begin{align*}
Q_{m,n}(z) &= (1 + o(1)) \left( \Phi_{m+1,nS_c}^{(0)} \right)(z), \\
(w_{\sigma,c}R_{m,n})(z) &= (1 + o(1)) \left( \Phi_{m+1,nS_c}^{(1)} \right)(z).
\end{align*}
\]
locally uniformly in \( \overline{\mathbb{C}} \setminus \Delta_\sigma \). It holds that \( |\Phi_{m+1,n}^{(1)} / \Phi_{m+1,n}^{(0)}| < 1 \) in \( D_{\sigma,c}^+ \).
Steepest descent is performed on

\[
Y_{m,n} := C_{m,n} \begin{pmatrix}
Q_{m,n} & R_{m,n} & H_{m,n} \\
Q_{m+1,n-1} & R_{m+1,n-1} & H_{m+1,n-1} \\
Q_{m,n-1} & R_{m,n-1} & H_{m,n-1}
\end{pmatrix}
\]

where \( C_{m,n} \) is a diagonal matrix of constants,

\[
R_{m+1,n-1}(z) = (Q_{m+1,n-1}f - P_{m+1,n-1})(z) = O(z^{m+1}) \quad \text{as} \quad z \to \infty,
\]

and

\[
H_{m,n-1}(z) := \int \frac{R_{m,n-1}(x)}{x - z} d\mu(x) = O(z^{-(m+n+1)}) \quad \text{as} \quad z \to \infty.
\]