S-Contours and Convergent Interpolation

Maxim L. Yattselev

Department of Mathematical Sciences
Indiana University-Purdue University Indianapolis
402 North Blackford Street, Indianapolis, IN 46202
maxyatts@iupui.edu

Abstract: The notion of a symmetric contour introduced by Stahl and further generalized by Gonchar and Rakhmanov in connection with theory of rational interpolants with free poles is recalled. Refinement of this notion proposed by Baratchart and the author is discussed.

1 Multipoint Padé Approximants

In what follows $f(z)$ will always denote a function holomorphic at infinity and $\mathcal{V} = \{V_n\}_{n=1}^\infty$, $V_n = \{v_{n,i}\}_{i=1}^{2n}$, will stand for an *interpolation scheme*, i.e., a triangular array of not necessarily distinct nor finite points from a domain to which $f(z)$ possesses a single-valued analytic continuation. The *n-th diagonal multipoint Padé approximant to $f(z)$ associated with an interpolation set* V_n is a rational function $[n/n;V_n]_f(z) = p_n(z)/q_n(z)$ such that $\deg(p_n) \leq n$, $\deg(q_n) \leq n$, the function

$$R_n(z) := \frac{q_n(z)f(z) - p_n(z)}{v_n(z)} = O\left(z^{-n-1}\right) \quad \text{as} \quad z \to \infty$$

and has the same region of analyticity as $f(z)$, where $v_n(z) = \prod_{|v_{n,i}|<\infty}(z - v_{n,i})$ is the monic polynomial vanishing at the finite elements of V_n according to their multiplicity. Hence, $[n/n;V_n]_f(z)$ interpolates $f(z)$ at the elements of V_n according to their multiplicity with one additional interpolation condition at infinity. This approximant always exists as the conditions placed on $R_n(z)$ amount to solving a system of $2n+1$ equations with $2n+2$ unknowns. *Classical diagonal Padé approximants*, denoted by $[n/n]_f(z)$, are recovered from this definition by placing all the interpolation points at infinity.

Let Γ be a simple positively oriented rectifiable closed curve that encircles the point at infinity and such that $f(z)$ is analytic across Γ as well as in its exterior, which contains V_n. Then it follows from Cauchy theorem that

$$0 = \int_\Gamma z^k \frac{q_n(z)f(z) - p_n(z)}{v_n(z)}\mathrm{d}z = \int_\Gamma z^k \frac{q_n(z)f(z)}{v_n(z)}\mathrm{d}z, \quad k \in \{0,\dots,n-1\}.$$

The function $f(z)$ can always be written in the form $f(z) = f(\infty) + \int(z-s)^{-1}\mathrm{d}\mu(s)$, where μ is in general a complex-valued Borel measure (if no better choice for μ can be found, it always can be taken in the form $(2\pi\mathrm{i})^{-1}f(s)\mathrm{d}s$ on some curve that lies in the interior of Γ and has the same properties as Γ). Then Fubini-Tonelli theorem yields that

$$\int \frac{s^k q_n(s)}{v_n(s)}\mathrm{d}\mu(s) = 0, \quad k \in \{0,\dots,n-1\}. \tag{1}$$

That is, $q_n(z)$ is a polynomial of the smallest degree (necessarily at most n) orthogonal to all polynomials of degree at most $n-1$ with respect to the measure $v_n^{-1}(s)\mathrm{d}\mu(s)$.

Research Perspectives CRM Barcelona, Fall 2019, vol. 12, in Trends in Mathematics
Springer-Birkhäuser, Basel

2 Stahl-Gonchar-Rakhmanov Theory

Let D be an unbounded domain with non-polar boundary (a compact set K is called *polar* if $\iint \log|z-w| d\sigma(z)\sigma(w) = -\infty$ for every probability Borel measure σ supported on K). Denote by $g_D(z,w)$ the *Green function* for D with pole at $w \in D$. That is, $g_D(z,w)$ is harmonic in $D \setminus \{w\}$, has logarithmic singularity at w ($g(z,w) + \log|z-w|$ is bounded around w if $|w| < \infty$ and $g(z,w) - \log|z|$ is bounded at infinity if $w = \infty$), and its trace is equal to zero quasi everywhere (up to a polar set) on ∂D. The *logarithmic capacity* of a compact set K is given by

$$\operatorname{cap}(K) = \exp\left\{-\lim_{z\to\infty}\left(g_{D_K}(z,\infty) - \log|z|\right)\right\},$$

where D_K is the unbounded component of the complement of K, see [4, 5] for the in depth discussion of the notions of potentials theory.

We shall say that a holomorphic at infinity function $f(z)$ belongs to the Stahl's class $\mathcal{S}$ if it has an analytic continuation along any path originating at infinity that belongs to $\mathbb{C} \setminus E_f$ for some compact polar set E_f and there do exist points in $\mathbb{C} \setminus E_f$ that possess distinct continuations. The following two theorems summarize one of the fundamental contributions of Herbert Stahl to complex approximation theory [6, 7, 8, 9], in which he exploited orthogonality relations (1) to obtain results on convergence of Padé approximants for functions in the class $\mathcal{S}$.

Theorem 1. *Given $f(z) \in \mathcal{S}$, there exists an unbounded domain D_f such that the Padé approximants $[n/n]_f(z)$ converge in capacity to $f(z)$ in D_f with the rate function $\exp\{-2ng_{D_f}(z,\infty)\}$. That is, for any compact set $K \subset D_f$ and any $\epsilon > 0$, it holds that*

$$\lim_{n\to\infty} \operatorname{cap}\left\{z \in K : \left||f(z) - [n/n]_f(z)|^{1/2n} - e^{-g_{D_f}(z,\infty)}\right| > \epsilon\right\} = 0. \tag{2}$$

The domain D_f is the largest in the sense that if D is a domain with $\operatorname{cap}(D \setminus D_f) > 0$, then no subsequence of $[n/n]_f(z)$ converges in capacity to $f(z)$ everywhere in D.

The complement $\Delta_f := \overline{\mathbb{C}} \setminus D_f$ has very special structure.

Theorem 2. *The set Δ_f can be decomposed as $\Delta_f = E_0 \cup E_1 \cup \bigcup \Delta_j$, where $E_0 \subseteq E_f$, E_1 consists of isolated points to which $f(z)$ has unrestricted continuations from infinity leading to at least two distinct function elements, and Δ_j are open analytic arcs. The set Δ_f possesses the S-property*

$$\frac{\partial g_{D_f}(z,\infty)}{\partial \boldsymbol{n}_+} = \frac{\partial g_{D_f}(z,\infty)}{\partial \boldsymbol{n}_-} \quad \text{on} \quad \bigcup \Delta_j, \tag{3}$$

where $\partial/\partial\boldsymbol{n}_\pm$ are the one-sided normal derivatives on $\bigcup\Delta_j$. Define $h_{D_f}(z) := \partial_z g_{D_f}(z)$, $2\partial_z := \partial_x - \mathrm{i}\partial_y$. The function $h^2_{D_f}(z)$ is holomorphic in $\overline{\mathbb{C}} \setminus (E_0 \cup E_1)$, has a zero of order 2 at infinity, and the arcs Δ_j are orthogonal critical trajectories of the quadratic differential $h^2_{D_f}(z)\mathrm{d}z^2$. For each point $e \in E_0 \cup E_1$ denote by $i(e)$ the number of different arcs Δ_j incident with e. If E_f is finite, then

$$h^2_{D_f}(z) = \prod_{e\in E_0\cup E_1}(z-e)^{i(e)-2}\prod_{e\in E_2}(z-e)^{2j(e)}, \tag{4}$$

where E_2 is the set of critical points of $g_{D_f}(z)$ with $j(e)$ standing for the order of $e \in E_2$, i.e., $\partial_z^j g_{D_f}(e) = 0$ for $j \in \{1,\dots,j(e)\}$ and $\partial_z^{j(e)+1} g_{D_f}(e) \neq 0$.

Informally, Padé approximants converge to $f(z)$ in the complement of a branch cut Δ_f that can be characterized as the one of smallest logarithmic capacity or equivalently the one with the symmetry property (3). The latter property becomes essential in understanding the extension of Stahl's work to multipoint Padé approximants.

Given a positive finite Borel measure ω supported in a domain D, denote by

$$G_D(z,\omega) := \int g_D(z,w)\mathrm{d}\omega(w)$$

the *Green potential of* ω *relative to* D. The potential $G_D(z,\omega)$ is superharmonic in D and, if extended by zero to $\overline{\mathbb{C}}\setminus\overline{D}$, is subharmonic in some neighborhood of $\overline{\mathbb{C}}\setminus D$. Therefore, it possesses a distributional Laplacian in this neighborhood, i.e., $\widehat{\omega} := \Delta G_D(\cdot,\omega)/(2\pi)$, which is a positive Borel measure supported on ∂D and is called the *balayage* of ω out of D. The measure $\widehat{\delta}(\infty)$, the balayage of the delta mass at infinity, is also known as the *logarithmic equilibrium distribution* of ∂D as well as $\mathbb{C}\setminus D$.

Gonchar and Rakhmanov [**3**] have proposed the following generalization of the symmetry property (3) (more generally, they introduced a notion of a contour symmetric in an external field, which specializes to (5) in the case of multipoint Padé approximants). Let Δ be a system of finitely many Jordan arcs that does not separate the plane. Assume that almost every point of Δ belongs to an analytic subarc. It is said that Δ is *symmetric with respect to a positive Borel measure* ω supported in $D := \overline{\mathbb{C}}\setminus\Delta$ (has the S-property with respect to ω) if

$$\frac{\partial G_D(z,\omega)}{\partial \boldsymbol{n}_+} = \frac{\partial G_D(z,\omega)}{\partial \boldsymbol{n}_-} \quad \text{a.e. on} \quad \Delta. \tag{5}$$

To make use of (5), one shall choose an interpolation scheme that is asymptotic to ω. More precisely, given Δ, D, and ω as in (5), a function $f(z)$ analytic in D, and an interpolation scheme $\mathcal{V}$ supported in D, that is, $\cap_n\overline{\cup_{k\geq n}V_k}\subset D$, it is said that $\mathcal{V}$ is *asymptotic to* ω if

$$\omega_n \overset{*}{\to} \omega \quad \text{as} \quad n\to\infty, \quad \omega_n := \frac{1}{2n}\sum_{i=1}^{2n}\delta(v_{n,i}),$$

where $\delta(v)$ is the Dirac's delta distribution supported at v (as usual, $\omega_n \overset{*}{\to} \omega$ as $n\to\infty$ if $\int\phi\mathrm{d}\omega_n \to \int\phi\mathrm{d}\omega$ as $n\to\infty$ for every function ϕ continuous in D and supported on a closed subset of D). The following is an adaptation of [**3**, Lemmas 1 and 2] to the case of multipoint Padé approximants.

Theorem 3. *Let Δ, D, and ω be as in* (5)*. If a function $f(z)$ is holomorphic in D and the jump of $f(z)$ across Δ is non-zero almost everywhere, then the diagonal multipoint Padé approximants $[n/n;V_n]_f(z)$ associated with an interpolation scheme $\mathcal{V}$ asymptotic to ω converge to $f(z)$ in logarithmic capacity in D with the rate function* $\exp\{-2nG_D(z,\omega)\}$ *in the sense of* (2)*. Moreover, the normalized counting measures of the poles of* $[n/n;V_n]_f(z)$ *converge weak** *to* $\widehat{\omega}$.

It should be stressed that the above theorem assumes existence of a symmetric contour while Stahl's theorem proves it but in a very specific case.

3 Szegő-type Convergence

If one wants to strengthen convergence in capacity to uniform convergence, the notion of symmetry needs to be refined. Let us start with a case of a single arc. Let Δ be a rectifiable Jordan arc with endpoints ±1 oriented from -1 to 1. Set $w(z) := \sqrt{z^2-1}$, $w(z)/z \to 1$ as $z\to\infty$, to be the branch holomorphic in $\mathbb{C}\setminus\Delta$. Define $\Phi(z) := z + w(z)$, $z\in D := \overline{\mathbb{C}}\setminus\Delta$, which is a non-vanishing univalent holomorphic function in D except for a simple pole at infinity. Observe that $\Phi_+(s)\Phi_-(s) \equiv 1$ for $s \in \Delta$. Let $v \in D$. Define

$$\Phi(z,v) := \frac{\Phi(z)-\Phi(v)}{1-\Phi(v)\Phi(z)}, \quad |v|<\infty, \quad \text{and} \quad \Phi(z,\infty) := \frac{1}{\Phi(z)}, \quad z \in D.$$

Each $\Phi(z,v)$ is a holomorphic function in D with a simple zero at v and non-vanishing otherwise. Given an interpolation scheme $\mathcal{V}$ in D, let

$$\Phi_n(z) := \prod_{v\in V_n} \Phi(z,v), \quad z \in D.$$

It is holomorphic in D with $2n$ zeros there and its traces on Δ satisfy $\Phi_{n+}(s)\Phi_{n-}(s) \equiv 1$. The following definition has been proposed in [2] by Baratchart and the author. It is said that Δ is *symmetric with respect to an interpolation scheme* $\mathcal{V}$ if the functions $\Phi_n(z)$ satisfy $|\Phi_{n\pm}(s)| = \mathcal{O}(1)$ uniformly on Δ and $\Phi_n(z) = o(1)$ locally uniformly in D as $n \to \infty$. This notion has the following connection to (5), see [2, Theorem 1].

Theorem 4. *Let Δ be a rectifiable Jordan arc such that for the endpoints $x = \pm 1$ and all $s \in \Delta$ sufficiently close to x it holds that $|\Delta_{s,x}| \le c|x-s|^\beta$, $\beta > 1/2$, where $|\Delta_{s,x}|$ is the arclength of the subarc of Δ joining s and x. Then the following are equivalent:*

(a) *there exists $\mathcal{V}$ supported in D such that Δ is symmetric with respect to $\mathcal{V}$;*
(b) *there exists a probability Borel measure ω supported in D such that (5) holds;*
(c) *Δ is an analytic Jordan arc, i.e., there exists a univalent function $\Xi(z)$ holomorphic in some neighborhood of $[-1,1]$ such that $\Delta = \Xi([-1,1])$.*

As expected, the proof shows that if Δ is symmetric with respect to $\mathcal{V}$ and ω is a weak* limit point of the normalized counting measures of the elements of V_n, then Δ is symmetric with respect to ω. Moreover, the following result holds, see [2, Theorem 4].

Theorem 5. *Let Δ be an analytic Jordan arc connecting ±1 symmetric with respect to an interpolation scheme $\mathcal{V}$. Let*

$$f(z) := \frac{1}{2\pi \mathrm{i}}\int_\Delta \frac{\rho(s)}{s-z}\frac{\mathrm{d}s}{w_+(s)}, \quad z \in D, \tag{6}$$

where $\rho(s)$ is a non-vanishing Dini-continuous complex-valued function on Δ. Then

$$f(z) - [n/n; V_n]_f(z) = \frac{1+o(1)}{w(z)} S_\rho^2(z)\Phi_n(z) \tag{7}$$

locally uniformly in D, where $S_\rho(z) = \exp\left\{\frac{w(z)}{2\pi\mathrm{i}}\int_\Delta \frac{\log\rho(s)}{w_+(s)}\frac{\mathrm{d}s}{s-z}\right\}$ is the Szegő function of ρ.

The above results can be generalized to more complicated geometries in the following way, see [10]. Let $E = \{e_0,\dots,e_{2g+1}\}$ be a set of $2g+2$ distinct points in $\mathbb{C}$ and

$$\mathfrak{S} := \left\{(z,w):\ w^2 = (z-e_0)\cdots(z-e_{2g+1}),\ z \in \overline{\mathbb{C}}\right\} \tag{8}$$

be a hyperelliptic Riemann surface, necessarily of genus g. Define the natural projection $\pi : \mathfrak{S} \to \overline{\mathbb{C}}$ by $\pi(z,w) = z$ and denote by $\boldsymbol{E} = \{\boldsymbol{e}_0,\dots,\boldsymbol{e}_{2g+1}\}$ the set of ramification points of $\mathfrak{S}$. We shall use bold lower case letters $\boldsymbol{z}, \boldsymbol{s}$, etc. to denote points on $\mathfrak{S}$ with natural projections z, s, etc. We utilize the symbol $\cdot^*$ for the conformal involution on $\mathfrak{S}$, that is, $\boldsymbol{z}^* = (z,-w)$ if $\boldsymbol{z} = (z,w)$. Given $\boldsymbol{v} \in \mathfrak{S}\setminus\boldsymbol{E}$, denote by $g(\boldsymbol{z},\boldsymbol{v})$ a function that is harmonic in $\mathfrak{S}\setminus\{\boldsymbol{v},\boldsymbol{v}^*\}$, normalized so that $g(\boldsymbol{e}_0,\boldsymbol{v}) = 0$, and such that

$$g(\boldsymbol{z},\boldsymbol{v}) + \begin{cases} \log|z-v|, & |v|<\infty, \\ -\log|z|, & v=\infty, \end{cases} \quad \text{and} \quad g(\boldsymbol{z},\boldsymbol{v}) - \begin{cases} \log|z-v|, & |v|<\infty, \\ -\log|z|, & v=\infty, \end{cases}$$

are harmonic functions of z around $\boldsymbol{v}$ and $\boldsymbol{v}^*$, respectively. Such a function always exists as it is simply the real part of an integral of the third kind differential with poles at $\boldsymbol{v}$ and $\boldsymbol{v}^*$ that have residues -1 and 1, respectively, and whose periods are purely imaginary.

Definition 6. Let Δ be a system of open analytic arcs together with the set E of their endpoints and $\mathcal{V}$, $V_n = \{v_{n,i}\}_{i=1}^{2n}$, be an interpolation scheme in $D := \overline{\mathbb{C}} \setminus \Delta$. We say that Δ is *symmetric with respect to* $(\mathfrak{S}, \mathcal{V})$ if

(i) $\mathfrak{S} \setminus \boldsymbol{\Delta}$, $\boldsymbol{\Delta} := \pi^{-1}(\Delta)$, consists of two disjoint domains, say $D^{(0)}$ and $D^{(1)}$, and no closed proper subset of Δ has this property;

(ii) the sums $\sum_{i=0}^{2n} g(z, v_{n,i}^{(0)})$ are uniformly bounded above and below on $\boldsymbol{\Delta}$ and converge to $-\infty$ locally uniformly in $D^{(1)}$, where $z^{(i)} := \pi^{-1}(z) \cap D^{(i)}$, $z \in D$.

The first condition above says that Δ does not separate the plane and serves as a branch cut for $w(z)$ from (8) ($w(z)/z^{g+1} \to 1$ as $z \to \infty$), which has a non-zero jump across every subarc of Δ. The second one is essentially a non-Hermitian Blaschke-type condition. It is in fact true that if Δ is symmetric with respect to $(\mathfrak{S}, \mathcal{V})$ and $\mathcal{V}$ is asymptotic to a measure ω, then Δ is symmetric with respect to ω. The following can be said about the existence of such contours, see [**10**, Theorem 3.2].

Theorem 7. *Given $\mathfrak{S}$ as in (8) and $v \in \overline{\mathbb{C}} \setminus E$, there always exists a contour Δ_v symmetric with respect to $(\mathfrak{S}, \mathcal{V}_v)$, where $\mathcal{V}_v$ consists of sets containing only the point v. Further, let $c > 0$ be a constant such that $L_c := \{s : g_{D_v}(s, \infty) = c\}$ is a smooth Jordan curve, where $D_v := \overline{\mathbb{C}} \setminus \Delta_v$. If $\Xi(z)$ is a univalent function in the interior of L_c such that $\Xi(e) = e$ for every $e \in E$, then there exists an interpolation scheme $\mathcal{V}$ in $\overline{\mathbb{C}} \setminus \Xi(\Delta)$ such that $\Xi(\Delta)$ is symmetric with respect to $(\mathfrak{S}, \mathcal{V})$.*

The right-hand side of (7) is obviously defined explicitly, but also can be characterized as a function with a certain zero/pole divisor that solves a particular boundary value problem. We take this second approach to define functions describing the asymptotic behavior of the multipoint Padé approximants for more general contours Δ.

Proposition 8. *Let Δ be as in Definition 6 and $\rho(s)$ be a Lipschitz continuous and non-vanishing function on Δ. There exists a sectionally meromorphic in $\mathfrak{S} \setminus \boldsymbol{\Delta}$ function $\Psi_n(\boldsymbol{z})$ whose zeros and poles there are described by the divisor*[1]

$$(n-g)\infty^{(1)} + \boldsymbol{z}_{n,1} + \cdots + \boldsymbol{z}_{n,g} - n\infty^{(0)} \tag{9}$$

for some set of g points $\boldsymbol{z}_{n,i}$ on $\mathfrak{S}$, and whose traces on $\boldsymbol{\Delta}$ are continuous and satisfy

$$\Psi_{n-}(\boldsymbol{s}) = (\rho(s)/v_n(s))\Psi_{n+}(\boldsymbol{s}), \quad \boldsymbol{s} \in \boldsymbol{\Delta}, \tag{10}$$

where $\boldsymbol{\Delta}$ is oriented so that $D^{(0)}$ lies to the left of $\boldsymbol{\Delta}$. If two functions $\Psi(\boldsymbol{z}), \Psi_*(\boldsymbol{z})$ satisfy (9) and (10), then $\Psi(\boldsymbol{z})/\Psi_*(\boldsymbol{z}) = R(\pi(\boldsymbol{z}))$ for some rational function $R(z)$ with at most $g/2$ poles. In particular, if the set $\{\boldsymbol{z}_{n,i}\}_{i=1}^g$ does not contain involution-symmetric pairs ($\boldsymbol{z}_{n,i} = \boldsymbol{z}_{n,j}^*$ for some $i \neq j$), then $\Psi_n(\boldsymbol{z})$ is unique up to a multiplicative constant.

The sets $\{\boldsymbol{z}_{n,i}\}_{i=1}^g$ can be independently introduced as solutions of a certain explicitly defined Jacobi inversion problem. The asymptotics of Padé approximants now can be describes as follows [**10**, Theorem 3.7].

[1]A meromorphic function $\Psi(\boldsymbol{z})$ has a zero/pole divisor $\sum_i m_i \boldsymbol{x}_i - \sum n_i \boldsymbol{y}_i$ if $\Psi(\boldsymbol{z})$ has a zero of order m_i at $\boldsymbol{x}_i$, a pole of order n_i at $\boldsymbol{y}_i$, and otherwise is non-vanishing and finite.

Theorem 9. *Given $\mathfrak{S}$ as in (8), let Δ be symmetric with respect to $(\mathfrak{S}, \mathcal{V})$ for some interpolation scheme $\mathcal{V}$ supported in $D = \overline{\mathbb{C}} \setminus \Delta$ and $f(z)$ be given by (6), where $\rho(s)$ is a non-vanishing Lipschitz continuous function on Δ. Further, let $\mathbb{N}_* \subseteq \mathbb{N}$ be a subsequence such that the sets $\{z_{n,i}\}_{i=1}^{g}$ as well as their topological limit points in $\mathfrak{S}^g \setminus \Sigma_g$, the quotient of $\mathfrak{S}^g$ by the symmetric group Σ_g, contain neither involution-symmetric pairs nor $\infty^{(0)}$. Then*

$$f(z) - [n/n; V_n]_f(z) = \frac{v_n(z)}{w(z)} \frac{\Psi_n(z^{(1)})}{\Psi_n(z^{(0)})} \frac{1 + \varepsilon_{n1}(z) + \varepsilon_{n2}(z)\Upsilon_n(z^{(1)})}{1 + \varepsilon_{n1}(z) + \varepsilon_{n2}(z)\Upsilon_n(z^{(0)})}$$

for $n \in \mathbb{N}_*$, where $\varepsilon_{ni}(z) = o(1)$ locally uniformly in D and vanish at infinity and $\Upsilon_n(\boldsymbol{z})$ is a rational function on $\mathfrak{S}$ that vanishes at $\infty^{(0)}$ and whose divisor of poles is equal to $\boldsymbol{z}_{n,1} + \cdots + \boldsymbol{z}_{n,g} + \infty^{(1)}$. Moreover, it holds that

$$\left| \frac{v_n(z)}{w(z)} \frac{\Psi_n(z^{(1)})}{\Psi_n(z^{(0)})} \right| \leq C_K \exp\left\{ \sum_{i=1}^{2n} g\left(z^{(1)}, v_{n,i}^{(0)}\right) \right\} = o(1)$$

for every closed subset $K \subset D$, where the last estimate follows from Definition 6(ii).

It needs to be stressed that this theorem is conditional. Even though Theorem 7 describes some symmetric contours, it is by no means comprehensive. More importantly, Theorem 9 is conditional on existence of a sequence $\mathbb{N}_*$. Existence of such a sequence is known only for the case of classical Padé approximants, i.e., $\mathcal{V} = \mathcal{V}_\infty$, see [**1**, Propositions 2.2 and 2.5] or [**10**, Theorem 3.6].

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