N-TH ROOT OPTIMAL RATIONAL APPROXIMANTS TO FUNCTIONS WITH POLAR SINGULAR SET

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ABSTRACT. Let D be a bounded Jordan domain and A be its complement on the Riemann sphere. We investigate the asymptotic behavior in D of the best rational approximants in the uniform norm on A of functions holomorphic on A that admit a multi-valued continuation to quasi every point of D with finitely many possible branches. More precisely, we study weak* convergence of the normalized counting measures of the poles of such approximants as well as their convergence in capacity. We place best rational approximants into a larger class of n-th root optimal meromorphic approximants whose behavior we investigate using potential-theory on certain compact bordered Riemann surfaces.

CONTENTS

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List of Symbols

General Point Sets:

$\mathbb{T}, \mathbb{D}, \mathbb{C}, \overline{\mathbb{C}}$	unit circle, open unit disk, complex plane, and extended complex plane
$\mathbb{T}_r, \mathbb{D}_r$	circle and open disk of radius r centered at the origin
T, D, A	Jordan curve, its interior domain, and the closure of its exterior domain in $\overline{\mathbb{C}}$

Riemann Surfaces:

$\mathcal{R}_*, p$	compact Riemann surface with natural projection $p : \mathcal{R}_* \to \overline{\mathbb{C}}$
$\mathcal{R}$	$\mathcal{R} := \{z \in \mathcal{R}_* : p(z) \in D\}$
$\mathcal{T}$	a connected component of $p^{-1}(T)$ homeomorphic to T
$\mathbf{rp}(\cdot)$	the set of ramification points of a given Riemann surface
$m(z)$	ramification order of a point z on a Riemann surface
M	total number of sheets of $\mathcal{R}$

Classes of Functions:

$C(E)$	continuous functions on a set E
$\mathcal{H}(A)$	functions analytic in some neighborhood of A
$\mathcal{S}(A)$	subclass of $\mathcal{H}(A)$ of functions multi-valued and quasi everywhere analytic off A
$\mathcal{E}(A)$	subclass of $\mathcal{H}(A)$ of functions single-valued and quasi everywhere analytic off A
$\mathcal{F}(\mathcal{R})$	class of functions quasi everywhere analytic on $\overline{\mathcal{R}}$
$\mathcal{F}(A)$	subclass of approximated functions analytic on A
$\mathcal{P}_n$	algebraic polynomials of degree at most n
$\mathcal{M}_n(D)$	monic algebraic polynomials of degree n with all their zeros in D
$\mathcal{R}_n(D)$	$\mathcal{P}_n\mathcal{M}_n^{-1}(D)$
$H^\infty(D)$	space of bounded holomorphic functions in D
$H_n^\infty(D)$	$H^\infty(D)\mathcal{M}_n^{-1}(D)$
$L^2(\mathbb{T})$	square integrable functions on the unit circle
$L^\infty(\mathbb{T})$	essentially bounded functions on the unit circle
H^2	Hardy space of functions in $L^2(\mathbb{T})$ with vanishing Fourier coefficients of negative index
H^2_-	$L^2(\mathbb{T}) \ominus H^2$

Operators:

$\mathbb{P}_+, \mathbb{P}_-$	orthogonal projections from $L^2(\mathbb{T})$ onto H^2, H^2_-
Γ_f	Hankel operator from H^2 to H^2_-, $h \mapsto \mathbb{P}_-(fh)$
$s_n(\Gamma_f)$	the n-th singular number of Γ_f

Potential Theory:

$\mathrm{cap}(K)$	logarithmic capacity of K
$\mathrm{cap}_\Omega(K)$	Greenian capacity of K relative to Ω
$\mathrm{cap}(E,K)$	capacity of the condenser (E,K)
$g_\Omega(\cdot,w)$	Green function for Ω with pole at w
$g(\sigma,\Omega;z)$	Green potential of the measure σ relative to Ω
$V^\sigma(z)$	logarithmic potential of the measure σ
$\mu_{\Omega,K}$	Green equilibrium distribution on a set $K\subset\Omega$ relative to Ω
$\mathcal{B}_v^E$	balayage function of a superharmonic function v relative to a set E
σ^E	balayage measure of a measure σ onto a set E
$\widehat{\sigma}$	lift of a measure σ on D to $\mathcal{R}$
$p_*(\sigma)$	projection (pushforward) of a measure σ on $\mathcal{R}$ to D
$\partial_{\mathrm{f}}A, \mathrm{clos}_{\mathrm{f}}(E)$	fine boundary and closure of a set E
$b(E), i(E)$	base and the set of finely isolated points of a set E

Various Symbols:

ϕ	a conformal map from $\mathbb{D}$ onto D
$\mathcal{K}_f$	collection of "branch cuts" for f
K_f	"branch cut" of minimal Greenian capacity for $f\in\mathcal{S}(A)$
$\|\cdot\|_K$	essential supremum norm on K
$\rho_n(f,A)$	error of best rational approximation of f analytic on A by functions in $\mathcal{R}_n(D)$

1. Introduction

Rational approximation to holomorphic functions of one complex variable has long been a requisite chapter of classical analysis with notable applications to number theory [30, 55, 33], spectral theory [43, 26, 47] and numerical analysis [25, 32, 24, 17]. Over the last decades it became a cornerstone of modeling in Engineering [64, 65, 50, 1, 27], and it can also be viewed today as a technique to regularize inverse potential problems [22, 31, 4]. Finding best rational approximants of prescribed degree to a specific function, say in the uniform norm on a given set, seems out of reach except in rare, particular cases. Indeed, such approximants depend in a rather convoluted manner, both on the approximated function and on the set where approximation should take place. Accordingly, the constructive side of the theory has focused on estimating optimal convergence rates as the degree grows large and devising approximation schemes coming close to meet them [66, 7, 23, 61], or else studying the behavior of natural, computationally appealing candidate approximants like Padé interpolants and their variants [3, 60, 44, 38, 41].

When a function is holomorphic in some neighborhood of a continuum A, the optimal speed of convergence for rational approximants on A is at least geometric in the degree. Then, a coarse but manageable estimate of this speed proceeds via asymptotics of the n-th root of the error of approximation by rational functions of degree n. For functions continuable analytically off A except over a polar set containing branchpoints (throughout *polar* means of logarithmic capacity zero), and provided that A does not divide the extended complex plane, Gonchar and Rakhmanov constructed, using multipoint Padé interpolants and dwelling on work by the second author, a sequence of rational approximants whose n-th root error on A is asymptotically the smallest possible. They further showed that these interpolants converge in capacity on the complement of a compact set K minimizing the capacity of the condenser (A,K) under the constraint that the function is analytic off K, and proved that the normalized counting measures of their poles converge to the condenser equilibrium distribution on K [23]. It is remarkable that the set K solves a certain geometric extremal problem from logarithmic potential theory, close in spirit to the Lavrentiev type [34], and that it depends merely on the set where approximation takes place, on the branchpoints of the approximated function and its monodromy around them, but nothing else. Such a structure emerges because only the n-th root of the error is considered, rather than the error itself. Since then, it has been an open issue whether any n-th root optimal sequence of approximants – in particular a sequence of best approximants – has the same behavior. The present paper answers this question in the positive, at least when the branchpoints are finite in number and order. In particular, our results connect, apparently for the first time, the singularities of best uniform approximants to those of the approximated function. We also address the case of no branchpoints, when the approximated function is analytic except over a polar set and the speed of best rational approximation on A is known to be faster than geometric with the degree. We prove that n-th root optimal approximants converge in capacity outside the singular set of the function, and that the "most effective" poles converge in a sense to that set.

The gist of the paper becomes more transparent upon observing that the behavior of rational approximants can be surmised when the function to be approximated extends analytically to a multiply-sheeted Riemann surface over the complex plane. As rational functions are single-valued, this topological discrepancy leads the approximants to mark out their domain of approximation by accumulating poles so as to form a cut, thereby preventing single-valued continuation in the limit. In the case of (diagonal) multipoint Padé interpolants, this cut has been characterized as being of smallest weighted capacity in a field that depends on the limiting distribution of the interpolation points, and poles asymptotically distribute according to the weighted equilibrium measure of that cut; moreover, the Padé interpolants converge in capacity on the extremal domain thus demarcated. This was established in [23], dwelling on the works [56, 57, 59, 60, 61] that deal with classical Padé interpolants and correspond to the zero field and unit weight; see also [45] for early developments along these lines, and [6] for applications to L^2-best rational approximation on the circle. Subsequently, by choosing interpolation points adequately and performing surgery to eliminate spurious poles, the authors of [23] construct, on any continuum A not dividing the extended plane and contained in the analyticity domain of a function indefinitely continuable except over a closed polar set containing branchpoints, a sequence of rational approximants converging uniformly to that function on A as their degree n grows large and whose n-th root error has a *liminf* which is smallest possible, as well as a true limit. For this weakly optimal choice of interpolation points (meaning that the choice is optimal in the n-th root sense), the cut K of minimum weighted capacity is also the cut minimizing the condenser capacity of (A, K), as well as the cut of minimum Greenian capacity in the complement of A. The smallest value for the limit of the n-th root error is a simple, explicit function of this Greenian capacity, and the poles of the approximants thus constructed distribute asymptotically according to the Green equilibrium measure of that cut.

Now, assuming in addition that the branchpoints of the continuation off A of the function to be approximated are finite in number and of algebraic type, we shall prove that *any* sequence of rational (or meromorphic) approximants of increasing degree n whose n-th root error on A converges to the smallest possible limit – *a fortiori* every sequence of best approximants – has the same asymptotic distribution of poles as the particular sequence constructed in [23]. More precisely, if a function analytic in a simply connected neighborhood of a continuum A in the extended complex plane is indefinitely continuable off that neighborhood except over a closed polar set containing finitely many branchpoints, all of algebraic type, then *the normalized counting measures of the poles of any sequence of rational approximants of increasing degree n with asymptotically optimal n-th root error on A do converge weak-star, as n grows large, to the Green equilibrium distribution of the compact set of minimum Greenian capacity outside of which the function is single-valued; moreover, convergence holds in Greenian capacity everywhere off that compact set.* Here, Green functions are understood with respect to the complement of the continuum A where approximation takes place. Finally, if there are no branchpoints, that is, if the approximated function is single-valued and analytic on the extended complex plane except possibly over a closed polar set E, then there are rational approximants converging on A faster than geometrically with the degree. We shall prove that such sequences of approximants (as well as their meromorphic analogs) converge in capacity on the extended plane deprived from E, and that retaining the singular part that comes close to E

generates new sequences of approximants, still converging faster than geometrically with the degree, while satisfying in addition that any weak-star limit point of the normalized counting measures of their poles is supported on E.

Previously cited references, which deal with Padé or multipoint Padé interpolants, exploit the connection between denominators thereof and non-Hermitian orthogonal polynomials on a system of arcs encompassing the singular set of the interpolated function. Indeed, the core of the work in [59, 23] is to derive asymptotics of such polynomials on extremal systems of arcs like those constructed in [56, 57], so as to qualify the behavior of the poles of the interpolants when the degree grows large and deduce from it the desired convergence properties. Here, we proceed in the opposite direction: we assume that the optimal rate is met in the n-th root sense and deduce from it the behavior of the poles. For this we cannot make use of orthogonal polynomials, and in fact it is not even known if interpolation takes place at all in the case of best approximants. However, the construction from [56, 57] will still be basic to our purposes.

This work was initiated jointly by the three authors in 2009, but the untimely passing away of the second one on April 22nd, 2013 prevented him from seeing its completion. Still, some fundamental ideas are his.

2. Preliminaries and Main Results

Given a function f holomorphic in a neighborhood of a closed set $A \subset \overline{\mathbb{C}}$, the error of approximation of f on A by rational functions of degree n is

$$\rho_n(f,A) := \inf_{r\in\mathcal{R}_n(\mathbb{C}\setminus A)} \|f-r\|_A, \tag{2.1}$$

where $\|\cdot\|_A$ stands for the supremum norm on A and, for $\Omega \subset \overline{\mathbb{C}}$, we let $\mathcal{R}_n(\Omega)$ be the class of rational functions of type (n,n) with all their poles in Ω. That is, if $\mathcal{P}_n$ denotes the space of algebraic polynomials of degree at most n and $\mathcal{M}_n(\Omega)$ the monic polynomials of degree n with all zeros in Ω, then $\mathcal{R}_n(\Omega) := \mathcal{P}_n\mathcal{M}_n^{-1}(\Omega)$. It was shown by Walsh [66, 2], using interpolation techniques, that

$$\limsup_{n\to\infty} \rho_n^{1/n}(f,A) \leqslant \exp\big\{-1/\mathrm{cap}(A,K)\big\}, \tag{2.2}$$

where K is any closed set disjoint from A in the complement of which f is holomorphic and $\mathrm{cap}(A,K)$ denotes the capacity of the condenser (A,K). A definition of condenser capacity can be found in [54, Chapter VIII, Section 3]; for our purposes, it is enough to know that if $\overline{\mathbb{C}}\setminus A$ is connected, then $\mathrm{cap}(A,K)$ coincides with the Greenian capacity $\mathrm{cap}_{\overline{\mathbb{C}}\setminus A}(K)$ defined in Section A.4, see [54, Chapter VIII, Theorem 2.6 & Corollary 2.7] for this equivalence.

It is known that Walsh's inequality (2.2) cannot be improved [37]. On the other hand, it was conjectured by Gonchar [28] and proven by Parfënov when A is a continuum with connected complement [46] (also later by Prokhorov for any compact set A [51]) that

$$\liminf_{n\to\infty} \rho_n^{1/n}(f,A) \leqslant \exp\big\{-2/\mathrm{cap}(A,K)\big\}. \tag{2.3}$$

Hence, $\rho_n(f,A)$ has no limit in general as $n\to\infty$, and when the limit exists it cannot exceed the right-hand side of (2.3). For certain classes of functions and certain *loci* of approximation A, it was nevertheless shown that $\rho_n(f,A)$ does have a limit, which is equal to the right-hand side of (2.3). More precisely, let $\mathcal{H}(A)$ denote the space of functions holomorphic on a (variable) neighborhood of A, and $\mathcal{S}(A) \subset \mathcal{H}(A)$ those functions continuable analytically into the complement of A along any path that avoids some compact polar[1] subset of $\overline{\mathbb{C}}\setminus A$ (which may depend on the function); we require in addition that this continuation is not single-valued, namely that there are paths with the same endpoints leading to different analytic branches. Now, when A is a continuum that does not

[1]see Section A.5 for a definition and basic properties of polar sets, that may be defined as sets of zero logarithmic capacity.

separate the plane and $f \in \mathcal{S}(A)$, it follows from work by the second author in [56, 57, 58] and it was explicitly stated by Gonchar and Rakhmanov in [23, Theorem 1′] that

$$\lim_{n\to\infty} \rho_n^{1/n}(f,A) = \inf_K \exp\left\{-2/\mathrm{cap}(A,K)\right\}, \tag{2.4}$$

where the infimum is taken over all compact sets K such that f admits a single-valued analytic continuation to $\overline{\mathbb{C}}\setminus K$.

Hereafter, we let $T \subset \mathbb{C}$ be a Jordan curve with interior domain D, and we put $A := \overline{\mathbb{C}}\setminus D$. In this setting $\infty \in A$, which is no loss of generality for a preliminary Möbius transform can ensure this; in contrast, our requirement that D be a Jordan domain is a regularity assumption on the set A where approximation takes place. Given $f \in \mathcal{H}(A)$, let $\mathcal{K}_f$ be the collection of all compact sets $K \subset D$ such that f, initially defined on A, admits a single-valued analytic continuation to $\mathbb{C}\setminus K$. It follows from [56, 57] that there exists $K_f \in \mathcal{K}_f$, unique up to addition and/or removal of a polar set, with

$$\mathrm{cap}_D(K_f) \leqslant \mathrm{cap}_D(K), \quad K \in \mathcal{K}_f. \tag{2.5}$$

We can and will normalize K_f to be the smallest possible, i.e., we make it the intersection of all $K \in \mathcal{K}_f$ for which $\mathrm{cap}_D(K)$ is minimal, see [57]. As $\mathrm{cap}(K,A) = \mathrm{cap}_D(K)$, in light of equation (2.4), our main goal is to investigate the asymptotic behavior of sequences $\{r_n\}$ of rational functions of type (n,n) meeting this optimal n-th root rate:

$$\lim_{n\to\infty} \|f - r_n\|_A^{1/n} = \exp\left\{-2/\mathrm{cap}_D(K_f)\right\}. \tag{2.6}$$

We call any such sequence $\{r_n\}$ a sequence of *n-th root optimal rational approximants to f on A*. In order to study $\{r_n\}$, we are led to consider more general sequences of *meromorphic* approximants of the form $r_n + h_n$, where h_n is holomorphic in D and continuous on $\overline{D}$, see Section 2.3 for the definitions. Even though best meromorphic approximants may look less natural than rational ones, they make contact with both the spectral theory of Hankel operators (through AAK theory) and Green potentials (because they generate errors with constant modulus on T), while remaining essentially equivalent to rational approximants as far as n-th root error rates are concerned [46]. This is why n-th root optimal meromorphic approximants (meeting (2.6) in place of r_n) are of principal importance in our study. Yet, the potential-theoretic tools on Riemann surfaces that we use only allow us to handle compact surfaces so far, and this induces some finiteness conditions on the functions from the class $\mathcal{S}(A)$ that we can deal with. These are set forth in the next section.

2.1. **Class of Approximated Functions.** We consider functions in $\mathcal{H}(A)$ such that

(i) they can be continued into D along any path originating on T that stays in $\overline{D}$ while avoiding a closed polar subset of D (which may depend on the function);
(ii) they are not single-valued, meaning there are continuations along at least two paths as in (i) with the same initial and terminal points that lead to distinct function elements, but still they are finite-valued in that the number of such function elements lying above a point of D is uniformly bounded (the bound may depend on the function);
(iii) their number of *branchpoints* (points in any neighborhood of which some analytic continuation along a closed path in D encircling that point while avoiding the exceptional polar set leads to a different function element) is finite.

In view of (i) and (ii), such functions lie in $\mathcal{S}(A)$. Note that (iii) is not superfluous, for there are functions meeting (i) and (ii) with infinitely many branchpoints. For instance, an open Riemann surface $\mathcal{X}$ made of two copies of $\overline{\mathbb{C}}\setminus\{0\}$, glued along a sequence of disjoint cuts in $\mathbb{D}$ shrinking to the point 0, has projection $p : \mathcal{X} \to \overline{\mathbb{C}}\setminus\{0\}$ a two sheeted covering with infinitely many branchpoints of order 2. As $\mathcal{X}$ carries a holomorphic function f assuming more than one value on $p^{-1}(z)$ for z not a critical value of p [18, Theorem 26.7], we deduce on putting $D = \mathbb{D}$ and $A = \overline{\mathbb{C}}\setminus\mathbb{D}$ that each branch of $f \circ p^{-1}$ is of the announced type.

We formalize (i), (ii) and (iii) as follows. Let $\mathcal{R}_*$ be an auxiliary algebraic Riemann surface, whose set of ramification points $\mathbf{rp}(\mathcal{R}_*)$ lies on top of D. That is, there exists an irreducible polynomial in two variables $P(z,a)$, of degree at least 2 in a, such that $\mathcal{R}_* = \{(z,a) : P(z,a) = 0\}$ and all branchpoints of the algebraic function $a(z)$ lie in D. We denote by $p : \mathcal{R}_* \to \overline{\mathbb{C}}$ the natural projection $p((z,a)) = z$, and we let $\mathcal{R} \subset \mathcal{R}_*$ be the (open) Riemann surface defined as

$$\mathcal{R} := \{z \in \mathcal{R}_* : p(z) \in D\};$$

here and below, whenever it causes no confusion, we use letter z to denote both points in $\mathbb{C}$ and on $\mathcal{R}_*$. Clearly, the ramification points of $\mathcal{R}_*$ and $\mathcal{R}$ are identical: $\mathbf{rp}(\mathcal{R}_*) = \mathbf{rp}(\mathcal{R})$. Let us denote by $\overline{\mathcal{R}}$ the closure of $\mathcal{R}$ in $\mathcal{R}_*$, and define a class of functions $\mathcal{F}(\mathcal{R})$ by

$$\begin{aligned}\mathcal{F}(\mathcal{R}) := \big\{f :\ \ & f \text{ is holomorphic and single-valued on } \overline{\mathcal{R}}\backslash E_f, \\ & E_f \ \text{ is closed, } \ p(E_f) \text{ is polar and contained in } D, \\ & f(z_1) \neq f(z_2) \text{ for some } z_1, z_2 \text{ with } p(z_1) = p(z_2)\big\}.\end{aligned} \tag{2.7}$$

In (2.7), we wrote E_f for the singular set of f on $\mathcal{R}$ but it would have been more appropriate to write $E_f(\mathcal{R})$, as the complete Riemann surface of f could be significantly larger than $\mathcal{R}$ and its singular set bigger than E_f. Since all ramification points of $\mathcal{R}_*$ lie on top of D, the simple-connectedness of A implies that the preimage $p^{-1}(T)$ consists of finitely many homeomorphic copies of T under p^{-1}; we generically denote by $\mathcal{T}$ such a copy, so that $p : \mathcal{T} \to T$ is a homeomorphism. Then, denoting with a subscript $\lfloor E$ the restriction to a set E, the class of functions that we study is defined as

$$\mathcal{F}(A) := \left\{f :\ f \text{ is holomorphic on } A \text{ and } \exists\, \mathcal{R}_*,\, \mathcal{T},\, \widehat{f} \in \mathcal{F}(\mathcal{R}) \text{ with } f_{\lfloor T} = \widehat{f} \circ (p_{\lfloor \mathcal{T}})^{-1}\right\}. \tag{2.8}$$

From (2.7) and (2.8), one sees that $\mathcal{F}(A) \subset \mathcal{S}(A) \subset \mathcal{H}(A)$ and members of $\mathcal{F}(A)$ meet (i), (ii), (iii). Conversely, when $f \in \mathcal{H}(A)$ satisfies (i), (ii) and (iii), one can check that $f \in \mathcal{F}(A)$. Indeed, if B is the closed polar subset of D defined by (i), we get from (ii) because $D\backslash B$ is connected, see Section A.5, that the number of sheets of the Riemann surface of f above $D\backslash B$ is a finite constant, say M. Therefore, since the branchpoints are finitely many by (iii), the algebraic surface $\mathcal{R}_*$ can be constructed by a classical glueing process described in Section 3.2. The fine point, when applying to the present case this familiar procedure based on glueing pairwise in a certain order the banks of M copies of a system of cuts joining the branchpoints, is that any two points of D can be joined by a smooth simple arc entirely contained in $D\backslash B$, except for its endpoints if they lie in B. It is so because $D\backslash B$ is a connected open set and each point of B is the center of a circle of arbitrary small radius contained in $D\backslash B$, as well as the endpoint of a segment contained in $D\backslash B$ (that may even be chosen to have quasi-any direction). These properties hold because B is polar, and therefore thin at each point of $\mathbb{C}$, see Section A.6.

We also consider functions in $\mathcal{H}(A)$ meeting (i) but not (ii). These are analytic and single-valued in $\overline{\mathbb{C}}\backslash E$, where $E \subset D$ is closed and polar, i.e., there are no branchpoints. This case complements the previous one on putting $\mathcal{R}_* = \overline{\mathbb{C}}$ and omitting the last requirement in (2.7); we denote the corresponding class of functions by $\mathcal{E}(A)$. Since $\operatorname{cap}(A,E) = \operatorname{cap}_D(E) = 0$ when $E \subset D$ is polar, see Section A.4, we get from (2.3) and (2.2) that

$$\lim_{n\to\infty} \rho_n^{1/n}(f,A) = 0, \qquad f \in \mathcal{E}(A). \tag{2.9}$$

That is to say, some sequence $\{r_n\}_{n\in\mathbb{N}}$, $r_n \in \mathcal{R}_n$, of rational functions, converges *faster than geometrically* towards f on A as the degree grows large, meaning that

$$\lim_{n\to\infty} \|f - r_n\|_A^{1/n} = 0. \tag{2.10}$$

We call any such sequence a sequence of *n-th root optimal rational approximants to* $f \in \mathcal{E}(A)$ *on* A.

2.2. **Optimal Rational Approximants.** Notions of potential theory in D and $\mathcal{R}$ play a fundamental role in what follows, and the reader might want to consult Appendix A for a comprehensive account thereof. Let us here recall the definition of Green potentials and Green equilibrium distributions.

The *Green function* $g_D(\cdot,w)$ of the domain D with pole at $w\in D$ is the unique non-negative harmonic function in $D\setminus\{w\}$, with logarithmic singularity at w, whose largest harmonic minorant is zero. The *Green potential* of a positive Borel measure ν in D is $g(\nu,D;z):=\int g_D(z,w)d\nu(w)$. Putting $|\nu|$ for the total mass of ν, the *Greenian capacity* (in D) of a Borel set $B\subset D$ is defined by

$$\mathrm{cap}_D(B):=\left(\inf_{\nu\geqslant 0,|\nu|=1,\mathrm{supp}\,\nu\subseteq B} I_D(\nu)\right)^{-1}, \quad I_D(\nu):=\int\int g(z,w)d\nu(w)d\nu(z); \tag{2.11}$$

the infimum above is taken over all probability Borel measures ν supported on B. For any set $S\subset D$, the *outer Greenian capacity* of S in D is defined as $\mathrm{cap}_D(S)=\inf\mathrm{cap}_D(U)$, where the infimum is taken over all open sets $U\supset S$ in D (using again the symbol cap_D causes no confusion for the outer Greenian capacity is known to coincide with the Greenian capacity on Borel sets). Polar subsets of D are those whose outer Greenian capacity is 0. If K is a non-polar compact subset of D, then there exists a unique Borel probability measure $\mu_{D,K}$ supported on K, called the *Green equilibrium distribution on K relative to D*, such that $\mathrm{cap}_D(K)=1/I_D(\mu_{D,K})$. It is characterized by the property that its Green potential is bounded on D and equal to its maximum (which is then necessarily $1/\mathrm{cap}_D(K)$) *quasi everywhere* (that is, up to a polar set) on K. To recap:

$$g(\mu_{D,K},D;z)\begin{cases}\leqslant 1/\mathrm{cap}_D(K), & z\in D,\\ = 1/\mathrm{cap}_D(K), & \text{for q.e. } z\in K,\\ < 1/\mathrm{cap}_D(K), & z\in D\setminus K,\end{cases} \tag{2.12}$$

where the last inequality follows from the (generalized) maximum principle for harmonic functions.

We are concerned with two types of asymptotics for sequences of rational approximants: the weak* behavior of the normalized counting measures of their poles, and the convergence in capacity of the functions themselves. More precisely, given a rational function r of type (n,n), we define

$$\mu(r):=\frac{1}{n}\sum_{z:r(z)=\infty}\delta_z, \tag{2.13}$$

where each pole z appears in the sum as many times as its order. Equivalently $-2\pi\mu(r)$ is the distributional Laplacian $\Delta(\log|q|^{1/n})$, with q the denominator of an irreducible form of r. One says that a sequence $\{\nu_n\}$ of finite Borel measures on D converges *weak** to a measure ν, denoted as

$$\nu_n\stackrel{w*}{\to}\nu \quad \text{as} \quad n\to\infty,$$

if $\int hd\nu_n\to\int hd\nu$ for every continuous compactly supported function h on D. We further say that a sequence of functions h_n converges *in Greenian capacity* to a function h on a set $U\subset D$ if

$$\lim_{n\to\infty}\mathrm{cap}_D\big(\{z\in F:|h_n(z)-h(z)|>\epsilon\}\big)=0 \tag{2.14}$$

for each $\epsilon>0$ and every compact $F\subset U$; we denote this claim by $h_n\stackrel{\mathrm{cap}}{\to}h$ in U. The convergence (2.14) is said to hold *at a geometric rate* if we can replace ϵ in (2.14) by a^n for some positive $a=a(F)<1$; the convergence rate is called *faster than geometric* if (2.14) holds with ϵ replaced by a^n for any $a>0$.

Theorem 2.1. *Let $T\subset\mathbb{C}$ be a Jordan curve, D its interior and A the complement of $D\cup T$ on the Riemann sphere. Given $f\in\mathcal{F}(A)$, let $K_f\in\mathcal{K}_f$ be such that* (2.5) *holds and $\{r_n\}$ be a sequence of n-th root optimal rational approximants to f on A, as defined in* (2.6)*. Then,*

$$\mu(r_n)\stackrel{w*}{\to}\mu_{D,K_f}, \tag{2.15}$$

where $\mu(r_n)$ is the normalized counting measure of the poles of r_n and μ_{D,K_f} is the Green equilibrium distribution on K_f relative to D. Furthermore, it holds that

$$\frac{1}{2n}\log|f-r_n| \overset{\text{cap}}{\to} g(\mu_{D,K_f},D;\cdot) - \frac{1}{\mathrm{cap}_D(K_f)} \quad \text{in} \quad D\backslash K_f \quad \text{as} \quad n\to\infty, \tag{2.16}$$

where $g(\mu_{D,K_f},D;\cdot)$ is the Green potential of μ_{D,K_f} in D.

Theorem 2.1 and (2.12) imply that n-th root optimal rational approximants converge to f in capacity in $D\backslash K_f$ at a geometric rate, given pointwise by $\exp\{g(\mu_{D,K_f},D;z) - 1/\mathrm{cap}_D(K_f)\}$. In fact, convergence in Greenian capacity in $D\backslash K_f$ and uniform convergence on A together with the limiting behaviour (2.15) for the poles imply convergence in *logarithmic capacity* on $\overline{D}\backslash K_f$ at a geometric rate, less than or equal to $\exp\{g(\mu_{D,K_f},D;z) - 1/\mathrm{cap}_D(K_f)\}$ pointwise; see Section A.4 for a definition of logarithmic capacity. This remark is a consequence of the proof of (2.16) given in Section 3.11, and it equally applies to the forthcoming Theorem 2.4, as well as Theorems 2.2 and 2.5 in which K_f gets replaced by a polar set E and the convergence takes place at faster than geometric rate. Moreover, if in Theorem 2.4 the n-th root optimal approximants M_n are their own Nehari modification, then a more precise conclusion holds: $\exp\{g(\mu_{D,K_f},D;z) - 1/\mathrm{cap}_D(K_f)\}$ is the exact pointwise rate of geometric convergence in logarithmic capacity on $\overline{D}\backslash K_f$ like it is in Greenian capacity on $D\backslash K_f$; see Section 3.3 for a definition of Nehari modifications.

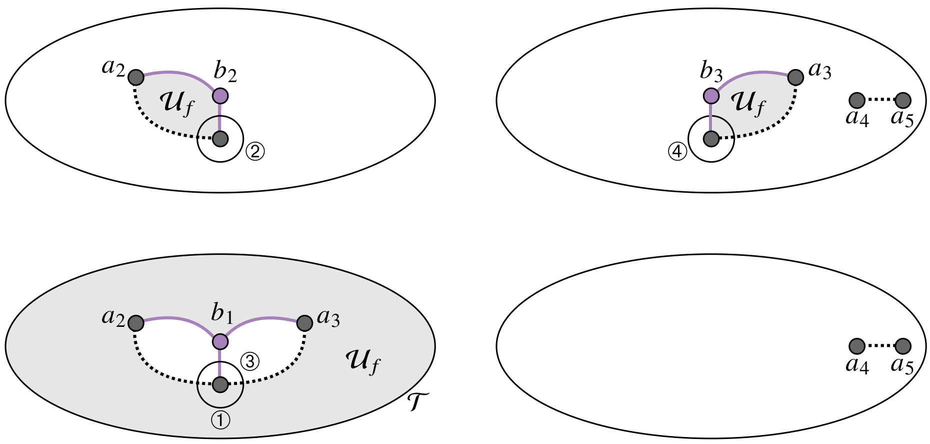

FIGURE 1. Surface $\mathcal{R}$ with five ramification points $\mathbf{rp}(\mathcal{R}) = \{a_1,a_2,a_3,a_4,a_5\}$ (a_1 is not labeled) and four sheets. Ramification point a_1 has order 3 and the remaining points have order 2. The sequence ①-②-③-④-① represents the monodromy around a_1 while encircling it clockwise, where the transitions happen across the dashed curves that stand for the cuts between different sheets of $\mathcal{R}$. The domain $\mathcal{U}_f$ is depicted as a shaded region. In this example $E_0 = \varnothing$, $E_{11} = \{p(a_1),p(a_2),p(a_3)\}$ are the active branch points, and $E_{10} = \{b\}$, where $b = p(b_1) = p(b_2) = p(b_3)$. The set K_f is a threefold equal to the natural projection of the solid (purple) curves. The solid curves also represent a different choice of the transition cuts between different sheets of $\mathcal{R}$, with the latter the domain $\mathcal{U}_f$ will lie entirely on one sheet.

In view of its importance, let us describe in greater detail the set K_f. As mentioned before, the problem of finding elements of minimal capacity in $\mathcal{K}_f$ was extensively studied by the second author [56, 57, 58]. The existence of such sets for $f\in\mathcal{H}(A)$ was proven in [56], and one can choose K_f so that $K_f \subseteq \tilde{K}$ for any $\tilde{K}\in\mathcal{K}_f$ with $\mathrm{cap}_D(\tilde{K}) = \inf_K \mathrm{cap}_D(K)$, which makes K_f unique [57]. The topological structure of K_f was investigated in [58], where it is shown that

$$K_f = E_0 \cup E_1 \cup \bigcup_i J_i \tag{2.17}$$

where the J_i are open analytic arcs, E_1 comprises the endpoints of the arcs J_i, and E_0 is a subset of the singular set of f in D (the singular set consists of those points in D to which some continuation of f from T has a singularity). As soon as $f \in \mathcal{S}(A)$, the set E_0 is polar by definition. To understand this decomposition better when $f \in \mathcal{F}(A)$, let $J_f := \cup_i \overline{J}_i$ so that $E_1 \subset J_f$. Then f possesses a single-valued continuation into $D \backslash J_f$ with singular set E_0, consisting of polar and essential singularities (but no branching singularities). If $\mathcal{T}$, $\widehat{f}$ are as in (2.8) and $\mathcal{U}_f$ is the connected component of $p^{-1}(D \backslash J_f)$ containing $\mathcal{T}$ in its boundary, then we can further decompose:

$$E_0 = p(E_{\widehat{f}} \cap \mathcal{U}_f), \quad E_1 = E_{10} \cup E_{11}, \quad E_{11} := p\big(\mathbf{rp}(\mathcal{R}) \cap \overline{\mathcal{U}}_f\big), \quad E_{10} := E_1 \backslash E_{11}.$$

That is, E_{11} is the set of "active" branchpoints of f, i.e., branchpoints that can be reached by continuation of f from T into $D \backslash J_f$, as opposed to those points in $\mathbf{rp}(\mathcal{R})$ that cannot be so reached. Also, each $e \in E_{10}$ is an endpoint of at least three arcs J_i, and generically f possesses analytic continuations to e from any direction within $D \backslash J_f$ (unless by chance e is a singularity of f as well), see Figure 1. As $\mathbf{rp}(\mathcal{R})$ is finite, so is the collection $\{J_i\}$ in cases that we consider.

For $f \in \mathcal{F}(A)$, Theorem 2.1 asserts two things: (i) the weak* convergence of $\mu(r_n)$ to μ_{D,K_f} whenever $\{r_n\}$ is an n-th root optimal sequence of rational approximants to f on A, and (ii) the convergence in capacity of r_n to f at a geometric rate on $D \backslash K_f$. If now $f \in \mathcal{E}(A)$ and $\{r_n\}$ is a sequence of n-th root optimal rational approximants to f on A, i.e., a sequence meeting (2.10) and thus converging faster than geometrically to f on A, then we shall see that r_n converges in capacity to f at faster than geometric rate in D. However, one can no longer expect a specific behavior of $\mu(r_n)$ in this case, for if R_n is a sequence in $\mathcal{R}_n(D)$ that converges to zero faster than geometrically on T, then $r_{n/2} + R_{n/2}$ is again a sequence of n-th root optimal rational approximants to f on A, whereas the weak* limit points of $\mu(R_n)$ can be arbitrary amongst positive measures of mass at most 1 on D. Thus, faster than geometric convergence does not qualify rational approximants enough to imply much on the behavior of their poles. Still, those poles of r_n that stay away from the singular set of f cannot account for the rate of convergence. This is made precise in the following result, which complements Theorem 2.1 in the case of no branchpoints.

Theorem 2.2. *Let T, A and D be as in Theorem 2.1. Given $f \in \mathcal{E}(A)$, let $\{r_n\}$ be a sequence of rational functions of type (n,n) meeting (2.10). Then, it holds that*

$$r_n \overset{\text{cap}}{\to} f \quad \text{in} \quad D \backslash E$$

at faster than geometric rate, where $E \subset D$ a closed polar set outside of which f is analytic and single-valued. Moreover, for any neighborhood V of E there is a sequence $\{R_{k_n}\}$, $R_{k_n} \in \mathcal{R}_{k_n}(V)$, $k_n \leqslant n$, such that the poles of R_{k_n} are among the poles of r_n lying in V and

$$\lim_{n\to\infty} \|f - R_{k_n}\|_A^{1/n} = 0. \tag{2.18}$$

Using neighborhoods V shrinking to E and a diagonal argument, Theorem 2.2 yields a corollary of independent interest.

Corollary 2.3. *If $f \in \mathcal{E}(A)$ and $E \subset D$ a closed polar set outside of which f is analytic and single-valued, then there is a sequence of rational functions $\{r_n\}$, $r_n \in \mathcal{R}_n(D)$, converging faster than geometrically to f on A, and such that every weak* cluster point of the sequence $\{\mu(r_n)\}$ of normalized counting measures of the poles is supported on E.*

2.3. **Optimal Meromorphic Approximants.** Let $H^\infty(D)$ denote the space of bounded analytic functions on D and $\mathcal{A}(D)$ the subspace of those extending continuously to $\overline{D}$. When T is rectifiable each $h \in H^\infty(D)$ has a non-tangential limit almost everywhere on T with respect to arclength, that we still call h, and putting $\|\cdot\|_T$ for the essential supremum norm on T (with respect to arclength) it holds that $\|h\|_T = \|h\|_D$ [12, Theorems 10.3 & 10.5]. When T is a non-rectifiable Jordan curve, however, limiting values of $H^\infty(D)$-functions on T generally exist at sectorially accessible points only, and such points may reduce to a set of zero linear measure [49, Theorem 6.25]. This will force us into a somewhat careful discussion of meromorphic approximants. Remember the set $\mathcal{M}_n(D)$ of monic polynomials of degree n whose zeros belong to D, and put

$$\begin{cases} H_n^\infty(D) & := \{h/q : \ h \in H^\infty(D), \ q \in \mathcal{M}_n(D)\}, \\ \mathcal{A}_n(D) & := \{h/q : \ h \in \mathcal{A}(D), \ q \in \mathcal{M}_n(D)\}. \end{cases}$$

That is, $H_n^\infty(D)$ is the set of meromorphic function with at most n poles in D that are bounded near T, and $\mathcal{A}_n(D)$ is the subset of those extending continuously to T. We shall say that $\{M_n\}$ is a sequence of *n-th root optimal meromorphic approximants to $f \in \mathcal{S}(A)$ on T* if $M_n \in \mathcal{A}_n(D)$ and

$$\limsup_{n\to\infty} \|f - M_n\|_T^{1/n} \leqslant \exp\left\{ -2/\mathrm{cap}_D(K_f)\right\}. \tag{2.19}$$

Any sequence of n-th root optimal rational approximants is a particular sequence of n-th root optimal meromorphic ones, by (2.6). When T is rectifiable, Corollary 2.6 to come will entail that in (2.19) the condition $M_n \in \mathcal{A}_n(D)$ can be traded for the seemingly weaker requirement $M_n \in H_n^\infty(D)$. However, it is important to require that $M_n \in \mathcal{A}_n(D)$ when T is not rectifiable, for otherwise the left-hand side of (2.19) may no longer make sense for the essential supremum norm (with respect to arclength) and the considerations below would not apply. In his proof of (2.3), Parfënov [46] has shown that the limit inferior of $\rho_n^{1/n}(f, A)$ is the same as the one of the n-th root of the error in best meromorphic approximation, see (2.22) for a definition of best (not just n-th root optimal) meromorphic approximants. Hence, in light of (2.6), replacing rational approximants with meromorphic ones does not improve the rate of decay of the n-th root error, and n-th root optimal meromorphic approximants to functions in $\mathcal{S}(A)$ could be defined with the limit superior and the inequality sign replaced by a full limit and the equality sign in (2.19).

Theorem 2.4. *With the notation of Theorem 2.1, let $\{M_n\}$ be a sequence of n-th root optimal meromorphic approximants to $f \in \mathcal{F}(A)$ on T, see (2.19). Then, the conclusions of Theorem 2.1 hold with r_n replaced by M_n.*

Let us reiterate that n-th root optimal rational approximants are just an instance of n-th root optimal meromorphic approximants, and therefore Theorem 2.1 is a special case of Theorem 2.4.

When $f \in \mathcal{E}(A)$, we define meromorphic approximants in a manner similar to (2.19) but this time with an eye on (2.9). Namely, we say that $\{M_n\}$ is a sequence of *n-th root optimal meromorphic approximants to $f \in \mathcal{E}(A)$ on T* if $M_n \in \mathcal{A}_n(D)$ and

$$\lim_{n\to\infty} \|f - M_n\|_T^{1/n} = 0. \tag{2.20}$$

The following complements Theorem 2.4 in the case of no branchpoints, and subsumes Theorem 2.2.

Theorem 2.5. *With the notation of Theorem 2.2, let $\{M_n\}$ be a sequence of n-th root optimal meromorphic approximants on T to $f \in \mathcal{E}(A)$, see (2.20). Then, the conclusions of Theorem 2.2 hold with r_n replaced by M_n.*

Besides best rational approximants, a noteworthy instance of n-th root optimal meromorphic approximants are the AAK (short for Adamyan-Arov-Krein) approximants. These are best meromorphic approximants with at most n poles. Specifically, consider the following (Nehari-Takagi) problem: given $f \in L^\infty(\mathbb{T})$, find $M_n^\infty \in H_n^\infty(\mathbb{D})$ such that

$$\|f - M_n^\infty\|_{\mathbb{T}} = \inf_{M \in H_n^\infty(\mathbb{D})} \|f - M\|_{\mathbb{T}}. \tag{2.21}$$

If $n = 0$, then (2.21) reduces to the question of best analytic approximation of bounded functions on the unit circle by elements of $H^\infty(\mathbb{D})$, which is the so-called *Nehari problem* named after [42] (that deals with an equivalent issue). It is known that M_n^∞ always exists and that it is unique when f lies in $C(\mathbb{T}) + H^\infty(\mathbb{D})$, see [47, Chapter 4]. Moreover, if f is Dini-continuous on $\mathbb{T}$, then M_n^∞ is continuous on $\mathbb{T}$. Indeed, if we write $M_n^\infty = r_n + g$ where $r_n \in \mathcal{R}_n(\mathbb{D})$ and $g \in H^\infty(\mathbb{D})$, we see that g must be the best Nehari approximant to $f - r_n$, which is Dini-continuous on $\mathbb{T}$, and the claim follows from [9]. When T is a rectifiable Jordan curve, one can readily replace $\mathbb{D}$ by D in (2.21) and carry over to T all the properties of best meromorphic approximants on $\mathbb{T}$ by conformal mapping. When T is non-rectifiable, the very existence of approximants depends on the analyticity of f on T, and follows from the proof of the next corollary to Theorem 2.4.

Corollary 2.6. *Let T, A and D be as in Theorem 2.1. Given $f \in \mathcal{F}(A)$ or $\mathcal{E}(A)$, to each integer n there exists a unique $M_n^\infty \in \mathcal{A}_n(D)$ such that*

$$\|f - M_n^\infty\|_T = \inf_{M \in \mathcal{A}_n(D)} \|f - M\|_T, \tag{2.22}$$

and if T is rectifiable then $\mathcal{A}_n(D)$ can be replaced by $H_n^\infty(D)$ in (2.22) without changing M_n^∞. Of necessity, the conclusions of Theorem 2.4 and 2.5 hold with M_n replaced by M_n^∞.

Much interest in best meromorphic approximants stems from their striking connection to operator theory. Denote by $L^2(\mathbb{T})$ the space of square integrable functions on $\mathbb{T}$, and let $H^2 \subset L^2(\mathbb{T})$ be the Hardy space of functions whose Fourier coefficients with negative index do vanish. It is known that H^2 can be identified with (non-tangential limits on $\mathbb{T}$ of) analytic functions in $\mathbb{D}$ whose L^2-means on circles centered at the origin are uniformly bounded, see [12, Theorem 3.4]. Set $H_-^2 := L^2(\mathbb{T}) \ominus H^2$ to be the orthogonal complement of H^2, which is the Hardy space of L^2-functions whose Fourier coefficients with nonnegative index are equal to zero. The latter can be identified with functions analytic in $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ that vanish at infinity, and whose L^2-means with respect to normalized arclength on circles centered at the origin are uniformly bounded. Let $\mathbb{P}_- : L^2(\mathbb{T}) \to H_-^2$ be the orthogonal projection. Given $f \in L^\infty(\mathbb{T})$, one defines the *Hankel operator with symbol f* to be

$$\Gamma_f : H^2 \to H_-^2, \qquad \Gamma_f(g) := \mathbb{P}_-(gf). \tag{2.23}$$

For n a non-negative integer, let $s_n(\Gamma_f)$ be the $(n+1)$-th singular number of the operator Γ_f, that is $s_n(\Gamma_f) := \inf_{\mathrm{rank}\, R \leqslant n} \|\Gamma_f - R\|$, where the infimum is taken over all operators $R : H^2 \to H_-^2$ of rank at most n and $\|\cdot\|$ stands for the operator norm. Then, one has that

$$\|f - M_n^\infty\|_{\mathbb{T}} = s_n(\Gamma_f). \tag{2.24}$$

If, in addition, $f \in C(\mathbb{T}) + H^\infty(\mathbb{D})$, then Γ_f is compact so that $s_n^2(\Gamma_f)$ is the $(n+1)$-st eigenvalue of $\Gamma_f^*\Gamma_f$ when these are arranged in non-increasing order, and (2.24) becomes a pointwise equality:

$$|(f - M_n^\infty)(z)| = s_n(\Gamma_f) \qquad \text{a.e. on } \mathbb{T}. \tag{2.25}$$

Moreover, if v_n is an $(n+1)$-st singular vector of Γ_f, i.e., an eigenvector of $\Gamma_f^*\Gamma_f$ with eigenvalue $s_n^2(\Gamma_f)$, then M_n^∞ is explicitly given in terms of f and v_n by the formula

$$f - M_n^\infty = \frac{\Gamma_f(v_n)}{v_n}. \tag{2.26}$$

Though not obvious at first glance, the right-hand side of (2.26) is independent of which n-th singular vector v_n is chosen and it has constant modulus on $\mathbb{T}$, in accordance with (2.25). The next corollary, of independent interest, follows from (2.25), (2.19), (2.9), and Theorems 2.4–2.5.

Corollary 2.7. *Let Γ_f be the Hankel operator with symbol $f \in \mathcal{F}\big(\overline{\mathbb{C}}\setminus\mathbb{D}\big)$. Then, it holds that*

$$\lim_{n\to\infty} s_n^{1/n}(\Gamma_f) = \exp\big\{-2/\operatorname{cap}_{\mathbb{D}}(K_f)\big\}.$$

Moreover, if $f \in \mathcal{E}(\overline{\mathbb{C}}\setminus\mathbb{D})$, then the above limit is equal to 0*.*

3. Proof of Theorem 2.4

3.1. **Existence of Best Meromorphic Approximants.** Below, we establish the existence and uniqueness part of Corollary 2.6, along with the assertion that $H_n^\infty(D)$ may replace $\mathcal{A}_n(D)$ when T is rectifiable. The rest of the corollary will follow from Theorem 2.4 upon conclusion of its proof.

Let $\phi : \mathbb{D} \to D$ be a conformal map. As mentioned in the paragraph before (2.5), ϕ extends to a homeomorphism from $\overline{\mathbb{D}}$ to $\overline{D}$. Let $L^2(\mathbb{T})$, H^2, and H^2_- be as defined after Corollary 2.6 and $\mathbb{P}_+ : L^2(\mathbb{T}) \to H^2$, $\mathbb{P}_- : L^2(\mathbb{T}) \to H^2_-$ be the orthogonal projections. If we pick a continuous function f on T, then $f\circ\phi$ is continuous on $\mathbb{T}$ and *a fortiori* it lies in $L^2(\mathbb{T})$. Set $F := \mathbb{P}_-(f\circ\phi)$ and $G := \mathbb{P}_+(f\circ\phi)$, so that $F \in H^2_-$ and $G \in H^2$. Since $\mathbb{P}_+ + \mathbb{P}_-$ is the identity operator, it holds that

$$F(z) = (f\circ\phi)(z) - G(z), \quad \text{a.e. } z \in \mathbb{T}. \tag{3.1}$$

Now, if $f \in \mathcal{H}(A)$, then $f\circ\phi(z)$ is holomorphic in $r < |z| < 1$ and continuous in $r \leqslant |z| \leqslant 1$, for r close enough to 1. Hence, the right-hand side of (3.1) is holomorphic in $r < |z| < 1$ with uniformly bounded L^2-means on circles centered at the origin, while the left-hand side lies in H^2_- and both sides have the same non-tangential limit on $\mathbb{T}$. By an easy variant of Morera's theorem [21, Chapter II, Exercise 12], the function equal to $F(z)$ for $|z| > 1$ and to $(f\circ\phi)(z) - G(z)$ for $r < |z| < 1$ is holomorphic across $\mathbb{T}$, in particular F extends analytically across $\mathbb{T}$ and G extends continuously to $\overline{\mathbb{D}}$. Then, as described after (2.21), the best meromorphic approximant $M_n^\infty \in H_n^\infty(\mathbb{D})$ to $f\circ\phi$ exists and is unique, moreover it is readily checked that M_n^∞ is equal to the sum of G (a member of $\mathcal{A}(D)$) and of the best approximant to F from $H_n^\infty(\mathbb{D})$, which lies in $\mathcal{A}_n(\mathbb{D})$ because F is analytic across $\mathbb{T}$ and so is *a fortiori* Dini-continuous on $\mathbb{T}$; hence, we get that $M_n^\infty \in \mathcal{A}_n(\mathbb{D})$. If now $M \in \mathcal{A}_n(D)$, then $M\circ\phi \in \mathcal{A}_n(\mathbb{D})$ and

$$\|f - M\|_T = \|f\circ\phi - M\circ\phi\|_{\mathbb{T}} \geqslant \|f\circ\phi - M_n^\infty\|_{\mathbb{T}}$$

by definition of M_n^∞. As $M_n^\infty\circ\phi^{-1} \in \mathcal{A}_n(D)$, it is the unique best meromorphic approximant to f we are looking for. The previous argument also shows that, when $f \in \mathcal{H}(A)$, the best meromorphic approximant to $f\circ\phi$ necessarily belongs to $\mathcal{A}_n(\mathbb{D})$. Because composition with ϕ is an isometric isomorphism $L^\infty(T) \to L^\infty(\mathbb{T})$ (understood with respect to arclength measure) when T is rectifiable, one can equivalently use $H_n^\infty(D)$ instead of $\mathcal{A}_n(D)$ in definition (2.22).

3.2. **Reduction to the Unit Disk.** Let K be a compact subset of $\mathbb{D}$. Since Green potentials are non-negative superharmonic functions whose largest harmonic minorant is zero, while a characteristic property of Green equilibrium potentials is to be constant quasi everywhere on the support of their defining measure while being no greater than this constant everywhere in the domain, it holds that

$$g(\mu_{\mathbb{D},K},\mathbb{D};z) = g(\mu_{D,\phi(K)},D;\phi(z)) \quad \text{and} \quad \operatorname{cap}_{\mathbb{D}}(K) = \operatorname{cap}_D(\phi(K))$$

where $\phi : \mathbb{D} \to D$ is a conformal map, see (2.12) as well as Sections A.3 and A.4. One has in this case that $\mu_{D,\phi(K)} = \phi_*(\mu_{\mathbb{D},K})$, the pushforward of $\mu_{\mathbb{D},K}$ under ϕ. Hence, we get that

$$\frac{1}{n}\sum_{i=1}^n \delta_{z_i} \overset{*}{\to} \mu_{\mathbb{D},K} \quad \text{if and only if} \quad \frac{1}{n}\sum_{i=1}^n \delta_{\phi(z_i)} \overset{*}{\to} \mu_{D,\phi(K)},$$

where the weak* convergence is understood for $n \to \infty$. Furthermore, by conformal invariance, a sequence of functions h_n converges in Greenian capacity to a function h in D if and only if the functions $h_n\circ\phi$ converge in Greenian capacity to the function $h\circ\phi$ in $\mathbb{D}$.

Let now $\mathcal{R}_*$ and $\mathcal{R}$ be as in Section 2.1. Denote by M the number of sheets of $\mathcal{R}$, so that a generic point in D has M preimages under the natural projection $p : \mathcal{R} \to D$. Let $J \subset D$ be a smooth oriented Jordan arc joining two points of $p(\mathbf{rp}(\mathcal{R}))$ while passing through all others exactly once. Constructing J is tantamount to enumerating the points of $p(\mathbf{rp}(\mathcal{R}))$. Write $J = \cup_l J_l$ where each J_l connects exactly two points in $p(\mathbf{rp}(\mathcal{R}))$. The surface $\mathcal{R}$ can be realized as M copies $U_1, \dots, U_M$ of $D\backslash J$, suitably glued to each other along the banks of the cuts J_l in each copy U_i (the glueing rule can be encoded, for example, as a collection of 4-tuples (l, i, j, k) telling one that the left and right banks of the cut J_l in U_i need to be glued to the right bank of the cut J_l in U_j and the left bank of the cut J_l in U_k respectively, see [40] for a discussion of Hurwitz's theorem). Using the same gluing rule, we can construct another M-sheeted surface $\mathcal{S}$ out of the domains $\phi^{-1}(U_1), \dots, \phi^{-1}(U_M)$. The map ϕ can then be lifted to a conformal map $\Phi : \mathcal{S} \to \mathcal{R}$ for which $\phi(\pi(z)) = p(\Phi(z))$, $z \in \mathcal{S}$, where $\pi : \mathcal{S} \to \mathbb{D}$ is the natural projection.

Let $f \in \mathcal{F}(A)$ and $\widehat{f} \in \mathcal{F}(\mathcal{R})$ be as in (2.8) and (2.7). Clearly $\widehat{f} \circ \Phi \in \mathcal{F}(\mathcal{S})$, and the surface $\mathcal{S}_*$ can be constructed by gluing M copies of $\overline{\mathbb{C}}\backslash\mathbb{D}$ to $\mathcal{S}$ along the M homeomorphic copies of $\mathbb{T}$ that comprise the boundary of $\mathcal{S}$. Argueing as we did after (3.1), we find that $F := \mathbb{P}_-(f \circ \phi)$ lies in $\mathcal{F}(\overline{\mathbb{C}}\backslash\mathbb{D})$ with the corresponding $\widehat{F} \in \mathcal{F}(\mathcal{S})$ given by $\widehat{f} \circ \Phi - G \circ \pi$, where $G := \mathbb{P}_+(f \circ \phi)$. Note that the conformal equivalence of Greenian capacities implies that $K_F = \phi^{-1}(K_f)$. Hence, if $\{M_n\}$ is an n-th root optimal sequence of meromorphic approximants of f in D as defined in (2.19), then $\{\widetilde{M}_n := M_n \circ \phi - G\}$ is an n-th root optimal sequence of meromorphic approximants to F on $\mathbb{T}$ and

$$(f - M_n)(\phi(z)) = (F - \widetilde{M}_n)(z), \quad z \in \overline{\mathbb{D}}\backslash K_F.$$

Therefore, it is sufficient to study the asymptotic behavior of $F - \widetilde{M}_n$ as well as the limit distribution of poles of $\widetilde{M}_n$. That is, it is enough to prove Theorem 2.4 on the unit disk.

3.3. **Nehari Modifications.** For $f \in \mathcal{H}(\overline{\mathbb{C}}\backslash\mathbb{D})$ and $M_n \in \mathcal{A}_n(\mathbb{D})$, let h_{f-M_n} be the best holomorphic (Nehari) approximant of $f - M_n$ in $H^\infty(\mathbb{D})$. That is, $\|f - M_n - h_{f-M_n}\|_\mathbb{T} = \inf_{h \in H^\infty(\mathbb{D})} \|f - M_n - h\|_T$, and $h_{f-M_n} \in H^\infty(\mathbb{D})$. Let us set

$$N(M_n)(z) := (M_n + h_{f-M_n})(z), \tag{3.2}$$

and call $N(M_n)$ the *Nehari modification* of M_n. The discussion after (2.21) shows that $N(M_n)$ lies in $\mathcal{A}_n(\mathbb{D})$. Indeed, we may write $M_n = g_n + r_n$ with $r_n \in \mathcal{R}_n(\mathbb{D})$ and $g_n \in \mathcal{A}(\mathbb{D})$. Then, one can readily check that $h_{f-M_n} = g_n + h_{f-r_n}$. As $f - r_n$ is analytic across $\mathbb{T}$ and in particular Dini-continuous there, it follows that h_{f-r_n} belongs to $\mathcal{A}(\mathbb{D})$ and so does h_{f-M_n}.

Since $\|f - N(M_n)\|_\mathbb{T} \leqslant \|f - M_n\|_\mathbb{T}$ and $N(M_n)$ lies in $\mathcal{A}_n(\mathbb{D})$, the sequence $\{N(M_n)\}$ is one of n-th root optimal meromorphic approximants to f, whenever $\{M_n\}$ is such a sequence. It is beneficial for us to consider Nehari modifications because they enjoy the additional property that the error they generate has constant modulus on $\mathbb{T}$, i.e., it follows from (2.24) that

$$|(f - N(M_n))(z)| = \|\Gamma_{f-M_n}\| \quad \text{for a.e.} \quad z \in \mathbb{T}. \tag{3.3}$$

We claim that it is enough to prove Theorem 2.4 for Nehari modifications only, as we now show.

Assume that Theorem 2.4 holds for $\{N(M_n)\}$. As the poles of M_n and $N(M_n)$ are the same, this automatically yields the statement about weak* convergence of the counting measures of the poles. Moreover, given $\epsilon > 0$ and $K \subset \mathbb{D}\backslash K_f$ a compact set, let us put

$$E(K, \epsilon, N(M_n)) := \left\{ z \in K : \left| \frac{1}{2n} \log |(f - N(M_n))(z)| - g(\mu_{\mathbb{D},K_f}, \mathbb{D}; z) + \frac{1}{\operatorname{cap}_\mathbb{D}(K_f)} \right| > \epsilon \right\}.$$

Define $E(K, \epsilon, M_n)$ analogously. According to our assumption it holds that

$$\lim_{n\to\infty} \operatorname{cap}_\mathbb{D}\big(E(K, \epsilon, N(M_n))\big) = 0, \tag{3.4}$$

and we need to show that (3.4) holds with $N(M_n)$ replaced by M_n. Given $\varepsilon \in (0,1)$, define

$$F_{n,\varepsilon} := \left\{ z \in K : \frac{1}{2n} \log |(f - N(M_n))(z)| > g\left(\mu_{\mathbb{D},K_f}, \mathbb{D}; \cdot\right) - \frac{1}{\operatorname{cap}_{\mathbb{D}}(K_f)} - \varepsilon m_K \right\},$$

where $m_K := \min_K g\left(\mu_{\mathbb{D},K_f}, \mathbb{D}; \cdot\right) > 0$. Since $N(M_n) - M_n$ is analytic in $\mathbb{D}$, we get from the maximum modulus principle and the triangle inequality that $|N(M_n)(z) - M_n(z)| \leqslant 2\|f - M_n\|_{\mathbb{T}}$ for $z \in \mathbb{D}$. Since $\exp\{(1-\varepsilon)m_K\} > 1$, relation (2.19) implies that

$$2\|f - M_n\|_{\mathbb{T}} < \frac{1}{2} \exp\left\{ (1-\varepsilon)2n\, m_K - \frac{2n}{\operatorname{cap}_{\mathbb{D}}(K_f)} \right\}$$

for all n large enough. Hence, by the previous estimates, it holds for all such n and $z \in F_{n,\varepsilon}$ that

$$\left| \frac{N(M_n)(z) - M_n(z)}{f(z) - N(M_n)(z)} \right| < \frac{1}{2} e^{2n(m_K - g(\mu_{\mathbb{D},K_f}, \mathbb{D}; z))} \leqslant \frac{1}{2}.$$

In particular, for any $0 < \varepsilon < \varepsilon'$, there exists n_0 depending on K and $\varepsilon' - \varepsilon$ such that

$$\frac{1}{2n} \left| \log \left| \frac{f(z) - M_n(z)}{f(z) - N(M_n)(z)} \right| \right| < (\varepsilon' - \varepsilon) m_K \tag{3.5}$$

for all $z \in F_{n,\varepsilon}$ and $n \geqslant n_0$. Therefore, we get from the triangle inequality that

$$E(K, \varepsilon' m_K, M_n) \subseteq (K \backslash F_{n,\varepsilon}) \cup E(K, \varepsilon m_K, N(M_n)) = E(K, \varepsilon m_K, N(M_n))$$

for all $n \geqslant n_0$. The above inclusion clearly yields that (3.4) holds with $N(M_n)$ replaced by M_n for $\epsilon = \varepsilon' m_K$. As ε, ε', and K were arbitrary, *the claim follows*.

3.4. **Notation.** We fix $f \in \mathcal{F}(\overline{\mathbb{C}} \backslash \mathbb{D})$ and, with a slight abuse of notation, we keep denoting by f the corresponding function in $\mathcal{F}(\mathcal{R})$ (that was denoted by $\widehat{f}$ in (2.8)). Take $\{M_n\}$ to be a sequence of n-th root optimal meromorphic approximants of f and let $\{N(M_n)\}$ be the sequence of corresponding Nehari modifications. Since $\{N(M_n)\}$ is also n-th root optimal, it holds that

$$\lim_{n\to\infty} \frac{1}{n} \log \|f - N(M_n)\|_{\mathbb{T}} = -\frac{2}{\operatorname{cap}_{\mathbb{D}}(K_f)}, \tag{3.6}$$

see the discussion after (2.19). We shall need an exhaustion of $\mathcal{R} \backslash E_f$ by open sets with "nice" boundaries. That is, we consider a sequence $\{\Omega_m\}_{m\geqslant 1}$ of open sets such that

$$\Omega_m \subset \mathcal{R} \backslash E_f, \quad \overline{\Omega}_m \subset \Omega_{m+1}, \quad \partial\mathcal{R} \subset \partial\Omega_m, \quad \mathcal{R} \backslash E_f = \cup_m \Omega_m, \tag{3.7}$$

and each Ω_m, when viewed as an open subset with compact closure of $\mathcal{R}_*$, is regular for the Dirichlet problem, see Section A.7. We will require in addition that $\lambda(\partial\Omega_m \backslash \partial\mathcal{R}) = 0$ for some Radon measure λ on $\mathcal{R}$ that will be specified at the beginning of Section 3.5 (a Radon measure is a positive Borel measure which is finite on compact sets). To design regular Ω_m that meet (3.7) is possible because E_f, being compact in $\mathcal{R}$, is a countable intersection of compact sets $K_l \subset \mathcal{R}$ with smooth boundary. Indeed, there is a smooth function $h \geqslant 0$ on $\mathcal{R}$ such that E_f is the zero set of h and $h \geqslant c > 0$ outside a compact neighborhood of E_f, hence we can pick K_l to be the sublevel set $\{z : h(z) \leqslant t_l\}$, where $\{t_l\}$ a sequence of regular values of h tending to 0 (almost every positive number is a regular value by Sard's theorem). In fact, using smooth partitions of unity and local coordinates, existence of such h quickly reduces to the corresponding issue in Euclidean space, where it follows easily from a combination of [62, Chapter VI, Theorem 2] and [67, Theorem I] (this result is named after H. Whitney). Furthermore, since the sets $C_{a,b} := \{z : a \leqslant h(z) \leqslant b\}$ are compact for $0 < a < b < \varepsilon$ and $\varepsilon > 0$ small enough, $\lambda(C_{a,b}) < \infty$. So, $\lambda(\{z : h(z) = t\}) \neq 0$ for at most countably many positive $t < \varepsilon$, and we can assume that t_l chosen above are not such values. The domains Ω_m thus constructed satisfy all our requirements.

Let Ω be a subdomain of $\mathbb{D}$ or $\mathcal{R}$. Given a Borel measure σ on Ω, we denote the Green potential of σ relative to Ω by $g(\sigma, \Omega; \cdot)$, see Section A.2. Hereafter, every measure is Borel unless otherwise stated. If σ is a measure on a Borel set containing Ω, we write for simplicity $g(\sigma,\Omega;\cdot)$ to mean $g(\sigma_{\lfloor\Omega},\Omega;\cdot)$. Conversely, for a measure σ on a Borel set $B_0\subset\Omega$, we still denote by σ the measure on Ω mapping a Borel set B to $\sigma(B\cap B_0)$, and write $g(\sigma,\Omega;\cdot)$ for its potential. When $\sigma=\sum_j\delta_{z_j}$ is a (possibly infinite) sum of Dirac delta measures, we put

$$b(\sigma,\Omega;z)=\exp\big\{-g^*(\sigma,\Omega;z)\big\} \tag{3.8}$$

to stand for the corresponding generalized Blaschke product, where $g^*(\sigma,\Omega;\cdot)$ is a complexified Green potential, i.e., it is locally holomorphic in $\Omega\backslash\mathrm{supp}(\sigma)$ and $\mathrm{Re}\,g^*(\sigma,\Omega;\cdot)=g(\sigma,\Omega;\cdot)$. If $g(\sigma,\Omega;\cdot)\equiv+\infty$, which can happen when the points z_j accumulate in Ω or to the boundary $\partial\Omega$ sufficiently slowly, then $b(\sigma,\Omega;\cdot)$ is identically zero. Otherwise, $b(\sigma,\Omega;\cdot)$ is well defined up to a unimodular constant (because the periods of a conjugate function of $g(\sigma,\Omega;\cdot)$ are integral multiples of 2π by Gauss' theorem), holomorphic in Ω, unimodular quasi everywhere on $\partial\Omega$ (everywhere if the latter is regular and the points z_j are finite in number), and it vanishes only at the points z_j (with multiplicities represented by repetition).

3.5. **Stripped Error of Approximation.** We shall study the asymptotics of the error functions $|f-N(M_n)\circ p|$. In this section, we strip off their poles and zeros to take logarithms and obtain harmonic functions whose limiting behavior we then investigate. To this end, we set

$$\widetilde\mu_n:=\sum\delta_{v_{n,j}},\quad \mu_n:=\widetilde\mu_n/n,\quad\text{and}\quad \widetilde\nu_n:=\sum\delta_{u_{n,j}},\quad \nu_n:=\widetilde\nu_n/n, \tag{3.9}$$

where $\{v_{n,j}\}\subset\mathbb{D}$ are the poles of $N(M_n)$ and $\{u_{n,j}\}\subset\mathcal{R}\backslash E_f$ the zeros of $f-N(M_n)\circ p$, with multiplicities counted by repetition. Before we proceed, let us specify the measure λ appearing in the definition of the exhaustion $\{\Omega_m\}$ in the previous subsection. To this end, recall that on a locally compact space X, a sequence of Radon measures $\{\sigma_n\}$ converges *vaguely* to a Radon measure σ if $\int g d\sigma_n\to\int g d\sigma$ for every g in $C_c(X)$, the space of continuous functions with compact support on X endowed with the sup-norm. Moreover, if the measures σ_n are locally bounded on X, then they do contain a vaguely convergent subsequence, see for example [14, Theorem 1.41] for an argument on $\mathbb{R}^n$ which is applicable to any σ-compact locally convex space[2]. Hence, since the measures μ_n have mass at most 1, we get if ν_n is locally bounded along some sequence of integers that there exists a subsequence $\mathcal{N}\subseteq\mathbb{N}$, a measure ν^* on $\mathcal{R}\backslash E_f$ and a measure μ on $\mathbb{D}$ such that ν_n and μ_n converge vaguely to ν^* and μ, respectively, along $\mathcal{N}$; if the measures ν_n have no locally bounded subsequence, i.e., if there exists a compact set $K\subset\mathcal{R}\backslash E_f$ such that $\nu_n(K)\to\infty$ as $\mathbb{N}\ni n\to\infty$, then we put $\nu^*=0$ and we only require the vague convergence $\mu_n\to\mu$ along $\mathcal{N}$. In any case we take λ to be $\nu^*+\widehat\mu$, where $\widehat\mu$ is the lift of μ to $\mathcal{R}$ defined via (A.33).

Using (3.8), we define Blaschke products vanishing at the poles of $N(M_n)$ and the zeros of $f-N(M_n)\circ p$. Namely, we put

$$b_n^{pole}(z):=b(\widetilde\mu_n,\mathbb{D};z)\quad\text{and}\quad b_{n,m}^{zero}(z):=b(\widetilde\nu_n,\Omega_m;z).$$

These functions are not identically zero, as the number of poles of $N(M_n)$ is at most n while the number of zeros of $f-N(M_n)\circ p$ in each Ω_m is finite. To see the latter point, recall from (3.3) that $|f-N(M_n)|$ is constant on $\mathbb{T}$, hence the error $f-N(M_n)$ can be meromorphically continued across $\mathbb{T}$ by reflection. It implies that $N(M_n)$ can be meromorphically continued across $\mathbb{T}$ and this continuation is necessarily analytic in some neighborhood of $\mathbb{T}$. Thus, $f-N(M_n)\circ p$ is analytic in a neighborhood of $\partial\mathcal{R}$ by the analyticity of f there, so the zeros of $f-N(M_n)\circ p$ can only accumulate on E_f and not on $\partial\mathcal{R}$. Thus, there are at most finitely many of them in each Ω_m.

Using these Blaschke products, we define

$$h_{n,m}(z):=\frac{1}{n}\log\left|\left(f-N(M_n)\circ p\right)(z)(b_n^{pole}\circ p)(z)/b_{n,m}^{zero}(z)\right|,\quad z\in\Omega_m, \tag{3.10}$$

[2]Vague convergence is called weak convergence in [14].

which is harmonic in Ω_m. Recall that superharmonic functions on a hyperbolic Riemann surface are either identically $+\infty$ or finite quasi everywhere, and any two of them that coincide almost everywhere (with respect to Lebesgue measure in local coordinates) are in fact equal (the weak identity principle), see Section A.1.

Lemma 3.1. *There exist a subsequence $\mathcal{N}' \subseteq \mathcal{N}$ and a non-negative superharmonic function $u'(z)$ on $\mathcal{R}$ such that*

$$-h_{n_m,m}(z) \to u'(z) \quad as \quad m \to \infty, \tag{3.11}$$

locally uniformly in $z \in \mathcal{R}\backslash E_f$, for any subsequence $\{n_m\}_{m=1}^{\infty} \subseteq \mathcal{N}'$. If u' is finite quasi everywhere, then one has a decomposition

$$u' = g(\nu', \mathcal{R}; \cdot) + h', \tag{3.12}$$

where ν' is a finite positive Borel measure supported on E_f and h' is a non-negative harmonic function on $\mathcal{R}$.

Proof. The regularity of $\partial\Omega_m$ implies that $|b_{n,m}^{zero}(z)| \equiv 1$ for $z \in \partial\Omega_m$, and likewise $|b_n^{pole}(z)| = 1$ for z on $\mathbb{T}$. In particular we get that

$$\left|N(M_n)(z)b_n^{pole}(z)\right| \leqslant \|N(M_n)\|_{\mathbb{T}}, \quad z \in \mathbb{D},$$

by the maximum modulus principle for H^∞-functions. Set $\Gamma_n := \Gamma_{f-M_n}$ be the Hankel operator with symbol $f - M_n$, see (2.23). It follows from the definition of the singular values, (2.24), and (3.2) that $\|\Gamma_n\| = s_0(\Gamma_n) = \|f - N(M_n)\|_{\mathbb{T}} \leqslant \|f - M_n\|_{\mathbb{T}}$. Since the norms $\|f - M_n\|_{\mathbb{T}}$ tend to zero by assumption, we get that

$$\|N(M_n)\|_{\mathbb{T}} \leqslant \|f\|_{\mathbb{T}} + \|\Gamma_n\| \leqslant C_f \tag{3.13}$$

for some constant C_f that depends only on f and the sequence $\{M_n\}$. Thus, we get from the maximum principle for harmonic functions that

$$h_{n,m}(z) \leqslant \frac{1}{n}\log\left(\|f\|_{\partial\Omega_m} + C_f\right), \qquad z \in \Omega_m. \tag{3.14}$$

Set $\mathcal{N}_0 := \mathcal{N}$. Proceeding inductively on $m \geqslant 1$, we deduce from (3.14) that the sequence $\{h_{n,m}\}_{n\in\mathcal{N}_{m-1}}$ is uniformly bounded above in Ω_m and therefore, by Harnack's theorem, see Section A.1, there exists a subsequence $\mathcal{N}_m \subseteq \mathcal{N}_{m-1}$ such that

$$h_{n,m}(z) \to h_m(z) \quad \text{as} \quad \mathcal{N}_m \ni n \to \infty \tag{3.15}$$

locally uniformly in Ω_m, where $h_m(z)$ is either identically $-\infty$ or a non-positive harmonic function. Define $\mathcal{N}_*$ to be the diagonal of the table $\{\mathcal{N}_m\}_{m=1}^{\infty}$; that is, the m-th element of $\mathcal{N}_*$ is the m-th element of $\mathcal{N}_m$. Then, (3.15) holds along $n \in \mathcal{N}_*$ for each m.

Additionally, it follows from the maximum modulus principle for holomorphic functions that

$$|b_{n,m_1}^{zero}(z)| > |b_{n,m_2}^{zero}(z)|, \quad z \in \Omega_{m_1}, \quad m_1 < m_2.$$

Therefore, we get from (3.10) and (3.14) that $h_{m_1}(z) \leqslant h_{m_2}(z) \leqslant 0$ for $z \in \Omega_{m_1}$ when $m_1 < m_2$. Thus, if a finite limit $h_{m_*}(z)$ exists for some index m_*, then it exists for all $m > m_*$. Hence, either the functions $h_{n,m}$ converge to $-\infty$ as $\mathcal{N}_* \ni n \to \infty$ locally uniformly in each Ω_m, in which case $h_m \equiv -\infty$ for all m, or else the functions h_m are finite and harmonic for all m large enough.

If $h_m \equiv -\infty$ for all m, select for each m some $n_m \in \mathcal{N}_*$ such that $h_{n,m}(z) < -m$ for $z \in \overline{\Omega}_{m-1}$ and all $n \geqslant n_m$; we may require in addition that $n_m > n_{m-1}$ for $m \geqslant 1$. Then (3.11) holds with $\mathcal{N}' := \{n_m\}_{m=1}^{\infty}$ and $u' \equiv +\infty$.

If on the contrary the functions h_m are finite, they form an increasing sequence on Ω_ℓ for $m \geqslant \ell$ and fixed ℓ. As they are non-positive, they converge locally uniformly in $\mathcal{R}\backslash E_f$ to a non-positive harmonic function, say $-u'$, again by Harnack's theorem. Since E_f is a closed polar set and u' is non-negative, it follows from the Removability theorem, see Section A.5, that u' extends to a superharmonic function on $\mathcal{R}$ that we keep denoting by u'. Because u' is superharmonic on $\mathcal{R}$ and harmonic on $\mathcal{R}\backslash E_f$, its Laplacian is a negative Radon measure $-\nu'$ supported on E_f, which is necessarily finite since E_f is compact. Thus, by the Riesz representation theorem, see Section A.3, equation (3.12) takes place with h' the largest harmonic minorant of u'.

Given m, choose $\tilde{n}_m \in \mathcal{N}_*$ such that $|h_{n,m}(z) - h_m(z)| \leqslant 1/m$ for $z \in \overline{\Omega}_{m-1}$ and all $n \geqslant \tilde{n}_m$. Define $\mathcal{N}' := \{\tilde{n}_m\}_{m=1}^\infty$, where we again additionally require that $\tilde{n}_m > \tilde{n}_{m-1}$. Given a compact set $K \subset \mathcal{R}\backslash E_f$ and $\epsilon > 0$, we can pick m large enough that

$$K \subseteq \overline{\Omega}_{m-1}, \quad 1/m \leqslant \epsilon/2, \quad \text{and} \quad |u'(z) + h_m(z)| \leqslant \epsilon/2, \quad z \in K.$$

Then, it follows from the last two inequalities that $|h_{n,m}(z) + u'(z)| \leqslant \epsilon$ for $z \in K$ and any $\mathcal{N}' \ni n \geqslant \tilde{n}_m$. Since ϵ and K were arbitrary, this finishes the proof of (3.11). □

Lemma 3.2. *If $u' \not\equiv +\infty$ in Lemma 3.1, then h' in (3.12) continuously extends to $\partial\mathcal{R}$ and*

$$h'(z) = \begin{cases} \dfrac{2}{\mathrm{cap}_{\mathbb{D}}(K_f)}, & z \in \mathcal{T}, \\ 0, & z \in \partial\mathcal{R}\backslash\mathcal{T}. \end{cases} \tag{3.16}$$

Proof. Let $\Omega_* := p^{-1}(\{z : r < |z| < 1\})$, with $r > 0$ close enough to 1 that $\overline{\Omega}_*\backslash\partial\mathcal{R} \subset \Omega_m$ for each m. It follows from the proof of Lemma 3.1 that $h_{n,m}(z) \to h_m(z)$ as $\mathcal{N}' \ni n \to \infty$, locally uniformly in $\overline{\Omega}_*\backslash\partial\mathcal{R}$. Given a connected component Ω of Ω_*, let $\delta_z^{\mathcal{R}\backslash\Omega}$ be its harmonic measure, see Section A.9. Then

$$\log|p(z) - (1+\epsilon)p(\xi)| = \int \log|p(\zeta) - (1+\epsilon)p(\xi)| d\delta_z^{\mathcal{R}\backslash\Omega}(\zeta), \quad z \in \Omega, \tag{3.17}$$

for any $\xi \in \partial\Omega \cap \partial\mathcal{R}$ and $\epsilon > 0$, see (A.28). It then follows from the monotone convergence theorem that we can take $\epsilon = 0$ in (3.17). Recall that $f - N(M_n) \circ p$ is analytic across $\partial\mathcal{R} \cap \partial\Omega$ and hence is non-vanishing there except perhaps for finitely many zeros counting multiplicities. Since $h_{n,m}$ is harmonic in Ω, is continuous on $\partial\Omega\backslash\partial\mathcal{R}$, and is equal to $\frac{1}{n}\log|f - N(M_n) \circ p|$ on $\partial\mathcal{R} \cap \partial\Omega$ (i.e., it is continuous on $\partial\Omega\backslash\partial\mathcal{R}$ except perhaps for finitely many logarithmic singularities), we get from (3.17) and (A.28) that

$$h_{n,m}(z) = \int_{\partial\Omega\backslash\partial\mathcal{R}} h_{n,m} d\delta_z^{\mathcal{R}\backslash\Omega} + \int_{\partial\mathcal{R}\cap\partial\Omega} \frac{1}{n}\log|f - N(M_n)\circ p| d\delta_z^{\mathcal{R}\backslash\Omega}, \quad z \in \Omega.$$

By (3.6) the pointwise limit of $\frac{1}{n}\log|f - N(M_n)\circ p|$ on $\partial\mathcal{R}$ is minus the right-hand side of (3.16) except perhaps for a finite subset of $\partial\mathcal{R}\backslash\mathcal{T}$, where $|f - N(M_n)\circ p|$ may go to zero, contained in

$$\big\{\zeta \,|\, \exists\, \eta \neq \zeta : \; p(\zeta) = p(\eta), \; f(\zeta) = f(\eta)\big\}\backslash\mathcal{T}.$$

As $\delta_z^{\mathcal{R}\backslash\Omega}$ does not charge polar, thus finite sets, the convergence in fact holds almost everywhere with respect to $\delta_z^{\mathcal{R}\backslash\Omega}$ for each fixed z. So, if we can justify the second equality in the following relation:

$$\begin{aligned} h_m(z) &= \lim_{\mathcal{N}'\ni n\to\infty}\left(\int_{\partial\Omega\backslash\partial\mathcal{R}} h_{n,m} d\delta_z^{\mathcal{R}\backslash\Omega} + \int_{\partial\mathcal{R}\cap\partial\Omega}\frac{1}{n}\log|f - N(M_n)\circ p| d\delta_z^{\mathcal{R}\backslash\Omega}\right) \\ &= \int_{\partial\Omega\backslash\partial\mathcal{R}} h_m d\delta_z^{\mathcal{R}\backslash\Omega} + \int_{\partial\mathcal{R}\cap\partial\Omega}\lim_{\mathcal{N}\ni n\to\infty}\frac{1}{n}\log|f - N(M_n)\circ p| d\delta_z^{\mathcal{R}\backslash\Omega}, \end{aligned} \tag{3.18}$$

we shall get that $h_m(z)$ solves a Dirichlet problem on Ω with boundary data equal to $h_{m\lfloor\partial\Omega\backslash\partial\mathcal{R}}$ on $\partial\Omega\backslash\partial\mathcal{R}$, and to the negative of the right-hand side of (3.16) on $\partial\mathcal{R}\cap\partial\Omega$. As such, h_m must extend continuously to $\partial\Omega$ where it is equal to the boundary data, since $\partial\Omega$ is non-thin at any of its points. Subsequently, as h_m converges to $-u'$ locally uniformly on $\mathcal{R}\backslash E_f$ (see the proof of Lemma 3.1), passing to the limit in the leftmost and rightmost sides of (3.18) when $m \to \infty$ yields that u' extends continuously to $\partial\Omega\cap\partial\mathcal{R}$ with values given there by the right-hand side of (3.16). This will give us the desired conclusion, because ν' is compactly supported in $\mathcal{R}$ and therefore $g(\nu',\mathcal{R};\cdot)$ continuously extends by zero to $\partial\mathcal{R}$ since $\mathcal{R}$ is regular.

Altogether, it only remains to justify the swapping of the limit and integration signs in (3.18). On $\partial\Omega\setminus\partial\mathcal{R}$, one can invoke the dominated convergence theorem. Thus, we only need to consider the integral over $\partial\mathcal{R}\cap\partial\Omega$. According to Vitali's convergence theorem, it is enough to show that for every $\epsilon>0$ there exists $\delta>0$ and $n_\varepsilon\in\mathbb{N}$ for which

$$\int_E\left|\frac{1}{n}\log|f-N(M_n)\circ p|\right|d\delta_z^{\mathcal{R}\setminus\Omega}<\epsilon \quad\text{as soon as}\quad |\delta_z^{\mathcal{R}\setminus\Omega}(E)|<\delta \quad\text{and}\quad n\geqslant n_\varepsilon. \tag{3.19}$$

For this, we first deduce from (3.13) that

$$\int_{\partial\mathcal{R}}\frac{1}{n}\log|f-N(M_n)\circ p|d\delta_z^{\mathcal{R}\setminus\Omega}\leqslant\frac{1}{n}\log\left(\|f\|_{\partial\mathcal{R}}+C_f\right) \tag{3.20}$$

as $\delta_z^{\mathcal{R}\setminus\Omega}$ is a probability measure. Now, if $\partial\mathcal{R}\cap\partial\Omega=\mathcal{T}$, then it follows from (3.6) that

$$\int_E\frac{1}{n}\log|f-N(M_n)\circ p|d\delta_z^{\mathcal{R}\setminus\Omega}>-c\delta_z^{\mathcal{R}\setminus\Omega}(E), \tag{3.21}$$

for any $E\subseteq\mathcal{T}$ and some positive constant c. If $\partial\mathcal{R}\cap\partial\Omega=\mathcal{T}'\neq\mathcal{T}$, set $d(\eta):=f(\eta)-f(\zeta)$ where $\eta\in\mathcal{T}'$, $\zeta\in\mathcal{T}$, and $p(\eta)=p(\zeta)$. Then, we get from (3.13) that

$$\begin{aligned}\log|(f-N(M_n)\circ p)(\eta)|\geqslant\log\big||d(\eta)|-\|\Gamma_n\|\big|&=\log\left|\frac{|d(\eta)|^2-\|\Gamma_n\|^2}{|d(\eta)|+\|\Gamma_n\|}\right|\\&\geqslant\log\big||d(\eta)|^2-\|\Gamma_n\|^2\big|-\log(\|f\|_{\mathcal{T}'}+C_f).\end{aligned} \tag{3.22}$$

Since E_f, the singular set of f, and $\mathbf{rp}(\mathcal{R})$, the ramification set of $\mathcal{R}$, are closed and lie on top of $\mathbb{D}$, $d(\eta)$ extends to a holomorphic function, non-identically zero in a neighborhood of $\mathcal{T}'$. Then,

$$\mathcal{D}(\eta):=d(\eta)\overline{d(p^{-1}(1/\overline{p(\eta)}))}\quad\text{satisfies}\quad\mathcal{D}(\eta)=|d(\eta)|^2,\quad\eta\in\mathcal{T}',$$

and is holomorphic about $\mathcal{T}'$. Pick an open set $W\supset\mathcal{T}'$ such that $\overline{W}\cap\mathbf{rp}(\mathcal{R})=\varnothing$ and $\mathcal{D}$ is holomorphic in $\overline{W}$ with no zero on ∂W; then, so is $\mathcal{D}-\|\Gamma_n\|^2$ for n large as $\|\Gamma_n\|^2\to0$. Let ℓ and ℓ_n be minimal degree polynomials, normalized by imposing $\|\ell\|_{\mathbb{T}}=\|\ell_n\|_{\mathbb{T}}=1$, such that

$$\frac{\mathcal{D}(\eta)}{\ell(p(\eta))}\quad\text{and}\quad\frac{\mathcal{D}(\eta)-\|\Gamma_n\|^2}{\ell_n(p(\eta))}$$

are holomorphic and non-vanishing in W. Since $\|\Gamma_n\|^2\to0$, the zeros of $\mathcal{D}-\|\Gamma_n\|^2$ in W tend to those of $\mathcal{D}$ by Rouché's theorem, and so our normalization implies that $\ell_n\to\ell$ uniformly in $\overline{W}$ as $n\to\infty$. Hence, $(\mathcal{D}-\|\Gamma_n\|^2)/(\ell_n\circ p)$ converges to $\mathcal{D}/(\ell_n\circ p)$ uniformly in $\overline{W}$, in particular it is uniformly bounded away from zero there. Consequently, it follows from (3.22) that

$$\log|(f-N(M_n)\circ p)(\eta)|\geqslant C+\log|\ell_n(p(\eta))|,\quad\eta\in\mathcal{T}', \tag{3.23}$$

for some finite constant C. Note that $\log|\ell_n|\leqslant0$ in $\mathbb{D}$ according to our normalization. Let us write $\ell_n(x)=a_n\prod_i(x-x_{i,n})$ and define the reciprocal polynomial $\tilde{\ell}_n$ of ℓ_n by

$$\tilde{\ell}_n(x):=a_n\prod_i\begin{cases}x-x_{i,n} & \text{if } |x_{i,n}|\geqslant1,\\ 1-x\bar{x}_{i,n} & \text{if } |x_{i,n}|<1.\end{cases}$$

Clearly $|\ell_n(\xi)|=|\tilde{\ell}_n(\xi)|$ for $|\xi|=1$, and the maximum principle for harmonic functions implies that

$$\int_{\mathcal{T}'}\log|\tilde{\ell}_n(p(\eta))|d\delta_z^{\mathcal{R}\setminus\Omega}\geqslant\log\left|\tilde{\ell}_n(p(z))\right|,\quad z\in\Omega, \tag{3.24}$$

since both sides of (3.24) are harmonic in Ω and have the trace $\log|\ell_n(p(\eta))|$ on $\mathcal{T}'$ while the left-hand side has zero trace on $\partial\Omega\setminus\mathcal{T}'$ and the right-hand side satisfies $\log|\ell_n\circ p|\leqslant 0$ there. Thus, we get from (3.23) and (3.24) that

$$\begin{aligned}\int_{\mathcal{T}'}\frac{1}{n}\log|f-N(M_n)\circ p|d\delta_z^{\mathcal{R}\setminus\Omega} &\geqslant \frac{C}{n}+\frac{1}{n}\int_{\mathcal{T}'}\log|\tilde{\ell}_n(p(\eta))|d\delta_z^{\mathcal{R}\setminus\Omega}\\ &\geqslant \frac{C}{n}+\frac{1}{n}\log\left|\tilde{\ell}_n(p(z))\right|.\end{aligned}\tag{3.25}$$

As the last term goes to zero uniformly on Ω and since $\tilde{\ell}_n\to\tilde{\ell}$, where $\tilde{\ell}$ is the reciprocal polynomial of ℓ defined similarly to $\tilde{\ell}_n$, the estimate (3.19) now follows from (3.20), (3.21), and (3.25). □

3.6. **Asymptotic Distributions of Poles and Zeros.** Recall the measures μ_n introduced in (3.9). Since these measures have mass at most 1, it follows from Helly's selection theorem [54, Theorem 0.1.3] that there exists a Borel measure μ', supported in $\overline{\mathbb{D}}$ with mass at most 1, such that

$$\mu_n \overset{w*}{\to} \mu' \quad \text{as} \quad \mathcal{N}'\ni n\to\infty,\tag{3.26}$$

perhaps at a cost of further restricting $\mathcal{N}'$, where $\overset{w*}{\to}$ stands for weak* convergence of finite (signed) measures (a sequence of Borel measures $\{\sigma_n\}$ on a locally compact space X converges weak* to a measure σ if $\int gd\sigma_n\to\int gd\sigma$ for every continuous function g in $C_0(X)$, the completion of $C_c(X)$ in the supremum norm). Observe that $\mu'_{\lfloor\mathbb{D}}=\mu$, where μ was defined as the vague limit μ_n in $\mathbb{D}$ along $\mathcal{N}\supseteq\mathcal{N}'$. In particular, $\mu_n\overset{w*}{\to}\mu$ in $\mathbb{D}$.

Lemma 3.3. *For any subsequence $\{n_m\}_{m=1}^\infty\subseteq\mathcal{N}'$, it holds that*

$$\liminf_{m\to\infty}\frac{1}{n_m}\log\left|b_{n_m}^{pole}(z)\right|^{-1}\left\{\begin{array}{c}\geqslant\\=\end{array}\right\}g(\mu,\mathbb{D};z),\tag{3.27}$$

where the inequality holds for every $z\in\mathbb{D}$ and equality holds for quasi every $z\in\mathbb{D}$.

Proof. Observe that $\frac{1}{n}\log\left|b_n^{pole}(z)\right|^{-1}=g(\mu_n,\mathbb{D};z)$, see (3.8). Since $\mu_n\overset{w*}{\to}\mu$ on $\mathbb{D}$, the conclusion follows from the Principle of Descent and the Lower Envelope Theorem, see Section A.7. □

We cannot immediately get an analog of the previous lemma for the measures $\nu_n=\tilde{\nu}_n/n$, because we do not know if these Radon measures on $\mathcal{R}\setminus E_f$ have uniformly bounded masses. Instead, we shall study the asymptotic behavior of their Green potentials in the style of Lemmas 3.1 and 3.2.

Lemma 3.4. *If the Radon measures σ_n converge vaguely to σ, $K\subset E$ is compact, and $\sigma(\partial K)=0$, then the restrictions $\sigma_{n\lfloor K}$ have uniformly bounded masses, $\sigma_{n\lfloor K}\overset{w*}{\to}\sigma_{\lfloor K}$ on K, and $\sigma_n(\partial K)\to 0$.*

Proof. For each $\varepsilon>0$ there is an open set $V\supset\partial K$ such that $\sigma(V)<\varepsilon$ (by outer regularity of σ), and an open set W with compact closure satisfying $V\supset\overline{W}\supset W\supset\partial K$ together with a continuous function $\varphi\geqslant 0$, supported in V, which is 1 on $\overline{W}$ (by Urysohn's lemma). Thus, $K_\varepsilon:=K\setminus W$ is a compact subset of $\operatorname{int}K$ (the interior of K) such that, for n large enough that $|\int\varphi d\sigma_n-\int\varphi d\sigma|<\varepsilon$,

$$\sigma_n(\partial K)\leqslant\sigma_n(K\setminus K_\varepsilon)\leqslant\sigma_n(W)\leqslant\int\varphi d\sigma_n<\varepsilon+\int\varphi d\sigma\leqslant 2\varepsilon.$$

Therefore $\int gd\sigma_n\to\int gd\sigma$ as $n\to\infty$ for any bounded continuous function g on $\operatorname{int}K$ [11, Proposition 6.18], and since $\sigma_n(\partial K)\to 0$ while $\sigma(\partial K)=0$ it implies the weak* convergence of $\sigma_{n\lfloor K}$ to $\sigma_{\lfloor K}$. The uniform boundedness of the masses $\sigma_n(K)=\int_K 1d\sigma_n$ now follows. □

From now on we employ standard notation $\mathbb{D}_r:=\{z:|z|<r\}$ and $\mathbb{T}_r:=\partial\mathbb{D}_r$.

Lemma 3.5. *There exist a subsequence $\mathcal{N}'' \subseteq \mathcal{N}'$ and a non-negative superharmonic function $u''(z)$ on $\mathcal{R}$ such that*

$$\liminf_{m\to\infty} \frac{1}{n} \log \left|b^{zero}_{n_m,m}(z)\right|^{-1} \left\{ \begin{array}{c} \geqslant \\ = \end{array} \right\} u''(z) \tag{3.28}$$

for any $\{n_m\}_{m=1}^{\infty} \subseteq \mathcal{N}''$, where the inequality in (3.28) holds everywhere on $\mathcal{R}\setminus E_f$ while the equality needs only to hold quasi everywhere. When $u'' \not\equiv +\infty$, it holds that

$$u'' = g(\nu'', \mathcal{R}; \cdot) + h'' \tag{3.29}$$

for some Radon measure ν'' carried by $\mathcal{R}$ and some non-negative function h'' harmonic on $\mathcal{R}$.

Proof. Observe that $\frac{1}{n}\log\left|b^{zero}_{n,m}(z)\right|^{-1} = g(\nu_n, \Omega_m; z)$ according to (3.8). Below, we distinguish two cases: (i) when the measures ν_n possess a subsequence which is locally bounded in $\mathcal{R}\setminus E_f$, i.e., having uniformly bounded masses on each compact subset of $\mathcal{R}\setminus E_f$, and (ii) when there exists a compact set $K \subset \mathcal{R}\setminus E_f$ such that $\nu_n(K) \to \infty$ as $\mathcal{N}' \ni n \to \infty$.

In case (ii) relation (3.28) holds with $u'' \equiv \infty$ and $\mathcal{N}'' = \mathcal{N}'$ because $\min_{w\in K} g_{\Omega_\ell}(z,w) > 0$ for $z \in \Omega_\ell$ and every ℓ such that $K \subset \Omega_\ell$, and therefore

$$g(\nu_{n'_m}, \Omega_m; z) \geqslant g(\nu_{n'_m}, \Omega_\ell; z) \geqslant \nu_{n'_m}(K) \min_{w\in K} g_{\Omega_\ell}(z,w) \to \infty$$

as $m \to \infty$ and $\{n'_m\}_{m=1}^{\infty} \subseteq \mathcal{N}'$, where the first inequality holds for $m \geqslant \ell$.

In case (i), the measures ν_n converge vaguely to ν^* in $\mathcal{R}\setminus E_f$ along $\mathcal{N}'$. Let $\{r_l\}_{l=1}^{\infty}$ be a positive real sequence increasing to 1 with r_1 large enough that $p(\partial\Omega_m\setminus\partial\mathcal{R}) \subset \mathbb{D}_{r_1}$ and $\nu^*(p^{-1}(\mathbb{T}_{r_l})) = 0$ for each l. This is possible, because for $0 < a < b < 1$ the set $\cup_{a\leqslant r\leqslant b}\, p^{-1}(\mathbb{T}_r)$ is compact, so there are at most countably many $r \in [a,1)$ with $\nu^*(p^{-1}(\mathbb{T}_r)) \neq 0$.

We now argue by double induction over m and l: the reasoning below should be applied inductively in $m \geqslant 1$, so as to define a sequence of integers $\mathcal{N}_m$ for each m. Let $\Omega_{m,l} := \Omega_m \cap p^{-1}(\mathbb{D}_{r_l})$ and proceed inductively in $l \geqslant 1$, starting with $\mathcal{N}_{m,0} := \mathcal{N}_{m-1}$ where $\mathcal{N}_0 = \mathcal{N}'$. Since $\nu_n(\overline{\Omega}_m) < \infty$ for each n, m by definition of ν_n, we can define

$$h_{n,m,l}(z) := g(\nu_n, \Omega_m; z) - g\left(\nu_{n\lfloor\overline{\Omega}_{m,l}}, \Omega_m; z\right), \quad z \in \Omega_{m,l}, \tag{3.30}$$

which is a non-negative harmonic function in $\Omega_{m,l}$. By Harnack's theorem, either there is a subsequence $\mathcal{N}_{m,l} \subseteq \mathcal{N}_{m,l-1}$ of indices n along which $h_{n,m,l} \to h_{m,l}$, locally uniformly in $\Omega_{m,l}$, for some non-negative harmonic function $h_{m,l}$, or else $h_{n,m,l}$ tends to infinity with $n \in \mathcal{N}_{m,l-1}$, locally uniformly in $\Omega_{m,l}$. In the latter case, we set $\mathcal{N}_{m,l} := \mathcal{N}_{m,l-1}$ and $h_{m,l} \equiv +\infty$. Clearly, $h_{n,m,l} \geqslant h_{n,m,l+1}$ and so, for fixed m, either $h_{m,l} \equiv +\infty$ for all l or the $h_{m,l}$ are finite for l large enough. Let $\mathcal{N}_m$ be the diagonal of the table $\{\mathcal{N}_{m,l}\}_{l=1}^{\infty}$. Since $\mathcal{N}_m$ is eventually a subsequence of every $\mathcal{N}_{m,l}$, it holds that $h_{n,m,l} \to h_{m,l}$ as $\mathcal{N}_m \ni n \to \infty$ for every $l \geqslant 1$, locally uniformly in $\Omega_{m,l}$.

In another connection, since $\nu^*(\partial\Omega_{m,l}) = 0$ by construction and the $\nu_{n\lfloor\overline{\Omega}_{m,l}}$ have uniformly bounded mass over n by the assumptions of the considered case, we deduce from Lemma 3.4 that

$$\nu_{n\lfloor\overline{\Omega}_{m,l}} \overset{w*}{\to} \nu^*_{\lfloor\overline{\Omega}_{m,l}} \quad \text{on} \quad \overline{\Omega}_{m,l} \quad \text{as} \quad \mathcal{N}_0 \ni n \to \infty.$$

Now, $\nu_{n\lfloor\overline{\Omega}_{m,l}}$ defines a measure on Ω_m in a natural way, and the weak* convergence above implies weak* convergence on Ω_m, because every continuous function with compact support in Ω_m restricts to a continuous function on $\Omega_m \cap \overline{p^{-1}(\mathbb{D}_{r_l})}$ which itself extends to $\overline{\Omega}_{m,l} = \overline{\Omega}_m \cap \overline{p^{-1}(\mathbb{D}_{r_l})}$ continuously by zero. As Ω_m is a regular open set with compact closure on the surface $\mathcal{R}_*$, the Principle of Descent and the Lower Envelope Theorem yield for any subsequence $\mathcal{N}^* \subseteq \mathcal{N}_0$ that

$$\liminf_{\mathcal{N}^*\ni n\to\infty} g\left(\nu_{n\lfloor\overline{\Omega}_{m,l}}, \Omega_m; z\right) \left\{ \begin{array}{c} \geqslant \\ = \end{array} \right\} g\left(\nu^*_{\lfloor\overline{\Omega}_{m,l}}, \Omega_m; z\right),$$

where the inequality holds everywhere in Ω_m and the equality may only hold quasi everywhere.

In view of (3.30), the above inequality and the very definition of $\mathcal{N}_m$ imply that

$$\liminf_{\mathcal{N}^*\ni n\to\infty} g\left(\nu_n,\Omega_m;z\right)\left\{\begin{array}{c}\geqslant\\=\end{array}\right\} g\left(\nu^*_{\lfloor\overline{\Omega}_{m,l}},\Omega_m;z\right)+h_{m,l}(z),\qquad z\in\Omega_{m,l}, \tag{3.31}$$

along any subsequence $\mathcal{N}^*\subseteq\mathcal{N}_m$, where the inequality holds everywhere in $\Omega_{m,l}$ while the equality needs only to hold quasi everywhere. As the left-hand side of (3.31) does not depend on l and the right-hand side is superharmonic, we get from the weak identity principle that successive right-hand sides are superharmonic continuations of each other when l increases. Let $u_m(z)$ be the superharmonic function in Ω_m given on each $\Omega_{m,l}$ by the right-hand side of (3.31). Since a smooth function with compact support in Ω_m is eventually supported in $\Omega_{m,l}$ for large l, we get from the definition that either $u_m\equiv+\infty$ or $\Delta u_m=-\nu^*$. In the latter case, the Riesz representation theorem yields that

$$u_m(z)=g(\nu^*,\Omega_m;z)+h_m(z),\qquad z\in\Omega_m, \tag{3.32}$$

for some non-negative harmonic function h_m, which is the largest harmonic minorant of u_m.

Let $\widetilde{\mathcal{N}}$ be the diagonal of the table $\{\mathcal{N}_m\}_{m=1}^\infty$. As $\widetilde{\mathcal{N}}$ is eventually a subsequence of each $\mathcal{N}_m$, we get from (3.31) that for each m and any subsequence $\mathcal{N}^*\subseteq\widetilde{\mathcal{N}}$ it holds that

$$\liminf_{\mathcal{N}^*\ni n\to\infty} g(\nu_n,\Omega_m;z)\left\{\begin{array}{c}\geqslant\\=\end{array}\right\} u_m(z), \tag{3.33}$$

where the inequality takes place everywhere in Ω_m and equality at least quasi everywhere. Because the left-hand side of (3.33) increases with m, we have that $u_m(z)\leqslant u_{m+1}(z)$ for quasi every $z\in\Omega_m$. Thus, either $u_m\equiv+\infty$ for all m large enough or else u_m is finite quasi everywhere on Ω_m for all m. In the latter case, since $\Delta u_m=\Delta u_{m+1\lfloor\Omega_m}$ $(=-\nu^*_{\lfloor\Omega_m})$, we get that $u_{m+1}-u_m$ is harmonic on Ω_m. Hence, $0\leqslant u_m\leqslant u_{m+1}$ everywhere on Ω_m, and so $u'':=\lim_m u_m$ is positive and superharmonic on $\mathcal{R}\setminus E_f$. If $u''\equiv+\infty$ we are done, for we get (3.28) from (3.33) with $\mathcal{N}''=\widetilde{\mathcal{N}}$. Otherwise u'' is locally integrable and therefore (3.32), together with the Riesz representation theorem, imply that

$$u''(z)=g(\nu^*,\mathcal{R}\setminus E_f;z)+\tilde h(z),\quad z\in\mathcal{R}\setminus E_f, \tag{3.34}$$

where $\tilde h(z)$ is a non-negative function that is the largest harmonic minorant of u'' on $\mathcal{R}\setminus E_f$.

As E_f is polar and compact in $\mathcal{R}$, we deduce from the Removability theorem and the Riesz representation theorem that

$$\tilde h(z)=h''(z)+g(\tilde\nu,\mathcal{R};z),\quad z\in\mathcal{R}\setminus E_f, \tag{3.35}$$

where h'' is a non-negative harmonic function on $\mathcal{R}$ and $\tilde\nu$ a finite positive measure supported on E_f. Moreover, since for $z\in\mathcal{R}\setminus E_f$ the function $g_\mathcal{R}(z,\cdot)-g_{\mathcal{R}\setminus E_f}(z,\cdot)$ is non-negative harmonic on $\mathcal{R}\setminus E_f$ and bounded above near E_f, the Removability theorem for harmonic functions yields that $g_\mathcal{R}(z,\cdot)-g_{\mathcal{R}\setminus E_f}(z,\cdot)\equiv 0$ as it extends to a non-negative harmonic minorant of $g_\mathcal{R}(z,\cdot)$ on $\mathcal{R}$. Hence,

$$g(\nu^*,\mathcal{R}\setminus E_f;z)=g(\nu^*,\mathcal{R};z),\quad z\in\mathcal{R}\setminus E_f, \tag{3.36}$$

and equations (3.34)–(3.36) imply that $u''(z)$ extends superharmonically to the entire surface $\mathcal{R}$ by

$$u''(z)=g(\nu'',\mathcal{R};z)+h''(z),\quad \nu''=\nu^*+\tilde\nu. \tag{3.37}$$

Now, for any subsequence $\{\tilde n_m\}_{m=1}^\infty\subseteq\widetilde{\mathcal{N}}$, it holds in view of (3.33) that for each $m_0\in\mathbb{N}$ and $z\in\Omega_{m_0}$

$$\liminf_{m\to\infty} g\left(\nu_{\tilde n_m},\Omega_m;z\right)\geqslant\liminf_{m\to\infty} g\left(\nu_{\tilde n_m},\Omega_{m_0};z\right)\geqslant u_{m_0}(z), \tag{3.38}$$

and we obtain the inequality in (3.28) by letting m_0 tend to infinity.

Thus, it only remains to prove the equality quasi everywhere in (3.28) when $u''(z)$ is quasi everywhere finite. As before, the argument should be applied inductively on m with $\widetilde{\mathcal{N}}_0:=\widetilde{\mathcal{N}}$. The functions $g_{n,l,m}:=g(\nu_n,\Omega_l;\cdot)-g(\nu_n,\Omega_m;\cdot)$ are non-negative and harmonic in Ω_m for $l>m$. Therefore, by Harnack's theorem, there are subsequences $\widetilde{\mathcal{N}}_{m,l}\subseteq\widetilde{\mathcal{N}}_{m,l-1}$, $\widetilde{\mathcal{N}}_{m,0}:=\widetilde{\mathcal{N}}_{m-1}$ such that

$g_{n,l,m}$ converges locally uniformly to some function $H_{l,m}$ harmonic in Ω_m as $\widetilde{\mathcal{N}}_{m,l} \ni n \to \infty$ (note that $g_{n,l,m}$ cannot go to $+\infty$ otherwise so would $g(\nu_n, \Omega_l; \cdot)$, and in view of (3.33) $u_l(z)$ would be infinite for quasi every z, contradicting that $u_l \leqslant u'' < \infty$). Of necessity, $H_{l,m} = u_l - u_m$ by (3.33), and a diagonal argument gives us a single subsequence $\widetilde{\mathcal{N}}_m^* \subseteq \widetilde{\mathcal{N}}$ along which the convergence $g_{n,l,m} \to H_{l,m}$ takes place for any $l > 0$. Now, for fixed m and each $l > m$, select $\tilde{n}_l \in \widetilde{\mathcal{N}}_m^*$ such that

$$\left|g_{n,l,m}(z) - \left(u_l(z) - u_m(z)\right)\right| \leqslant 1/l, \quad z \in \overline{\Omega}_{m-1}, \quad \tilde{n}_l \leqslant n \in \widetilde{\mathcal{N}}_m^*. \tag{3.39}$$

Since the functions $u_l - u_m$ are harmonic in Ω_m and increase with l, they converge locally uniformly to $u'' - u_m$ there by Harnack's theorem and the definition of u''. Thus, taking (3.39) into account, for any $\epsilon > 0$ there exists $L > 0$ such that

$$\left|g_{n,l,m}(z) - \left(u''(z) - u_m(z)\right)\right| \leqslant \epsilon, \quad z \in \overline{\Omega}_{m-1}, \quad l \geqslant L, \quad \tilde{n}_l \leqslant n \in \widetilde{\mathcal{N}}_m^*. \tag{3.40}$$

Define $\widetilde{\mathcal{N}}_m := \{\tilde{n}_l\}_{l=1}^{\infty}$. Then it follows from (3.40) and (3.33) that

$$\liminf_{l\to\infty} g\left(\nu_{n_l}, \Omega_l; z\right) = \liminf_{l\to\infty} g\left(\nu_{n_l}, \Omega_m; z\right) + \lim_{l\to\infty} g_{n_l,l,m}(z) = u''(z) \tag{3.41}$$

for quasi every $z \in \overline{\Omega}_{m-1}$, whenever $\{n_l\}_{l=1}^{\infty} \subseteq \widetilde{\mathcal{N}}_m$. Finally, let $\mathcal{N}''$ be the diagonal sequence of the table $\{\widetilde{\mathcal{N}}_m\}_{m=1}^{\infty}$. Since $\mathcal{N}''$ is eventually a subsequence of every $\widetilde{\mathcal{N}}_m$ it follows from (3.41) that

$$\liminf_{m\to\infty} g\left(\nu_{n_m}, \Omega_m; z\right) = u''(z), \quad \{n_m\}_{m=1}^{\infty} \subseteq \widetilde{\mathcal{N}}'', \quad \text{for q.e.} \quad z \in \mathcal{R}\backslash E_f, \tag{3.42}$$

which is the equality case in (3.28). □

3.7. **Logarithmic Error Function.** Hereafter, we redefine $\mathcal{N}$ to be $\mathcal{N}''$ constructed in Lemma 3.5. By Lemmas 3.1, 3.3 and 3.5, the limits (3.6), (3.11), (3.27), and (3.28) hold along this new sequence.

Since μ is finite, there is a G_δ polar set $\widetilde{N}_0 \subset \mathbb{D}$ such that $g(\mu, \mathbb{D}; x) < +\infty$ for $x \in \mathbb{D}\backslash\widetilde{N}_0$, see Sections A.3 and A.5. Let us put $N_0 := p^{-1}(\widetilde{N}_0)$, which is a G_δ polar subset of $\mathcal{R}$, see Section A.5. We now introduce the function $ler : \mathcal{R}\backslash N_0 \to [-\infty, +\infty)$ ("ler" for "logarithmic error"), by putting

$$ler(z) := g(\mu, \mathbb{D}; p(z)) - u'(z) - u''(z), \qquad z \in \mathcal{R}\backslash N_0, \tag{3.43}$$

where u' and u'' are as in Lemmas 3.1 and 3.5. Clearly, $ler(z)$ is a δ-subharmonic function (the difference of two subharmonic functions), and it is well defined for $z \notin N_0$ since $g(\mu, \mathbb{D}; p(z))$ is finite there. As introduced, ler depends on the choice of the subsequence $\mathcal{N}$, but later we shall see that it is in fact unique.

Lemma 3.6. *There exists a polar set $A_0 \subset \mathcal{R}\backslash N_0$ such that, whenever z_1, z_2 are distinct points in $\mathcal{R}\backslash N_0$ with $p(z_1) = p(z_2)$ and $ler(z_i) < 0$ for $i = 1, 2$, then $z_1, z_2 \in A_0$.*

Proof. By the equality quasi everywhere in (3.27), there exists a polar set $A_1 \subset \mathbb{D}$ such that, for every $x \in \mathbb{D}\backslash A_1$, one can find a sequence $\mathcal{N}_x \subseteq \mathcal{N}$ along which

$$\lim_{\mathcal{N}_x \ni n\to\infty} \frac{1}{n} \log\left|b_n^{pole}(x)\right|^{-1} = g(\mu, \mathbb{D}; x).$$

Together with (3.10), (3.11), the inequality in (3.28) and (3.43), this gives us

$$\limsup_{\mathcal{N}_x \ni n\to\infty} \frac{1}{n} \log\left|\left(f - N(M_n) \circ p\right)(z)\right| \leqslant ler(z) \tag{3.44}$$

for every $z \notin N_0$ such that $z \in p^{-1}(x)$ with $x \notin A_1 \cup p(E_f)$. Assume now that $z_1, z_2 \in \mathcal{R}\backslash N_0$ satisfy $z_1 \neq z_2$ and $p(z_1) = p(z_2)$, as well as $ler(z_i) < 0$ for $i = 1, 2$. Let $x \in \mathbb{D}$ be such that $z_1, z_2 \in p^{-1}(x)$. If $x \notin A_1 \cup p(E_f)$, it follows from (3.44) that

$$f(z_1) = \lim_{\mathcal{N}_x \ni n\to\infty} N(M_n)(p(z_1)) = \lim_{\mathcal{N}_x \ni n\to\infty} N(M_n)(p(z_2)) = f(z_2)$$

and necessarily $z_1, z_2 \in A_2 := \big\{\zeta | \; \exists\, \eta \neq \zeta : \; p(\zeta) = p(\eta), \; f(\zeta) = f(\eta)\big\}$. Now, the conditions placed on the class $\mathcal{F}(\mathcal{R})$ imply that the set A_2 is finite, and therefore the lemma holds with $A_0 := p^{-1}\big(A_1 \cup p(E_f) \cup p(A_2)\big)$ which is polar, as inverse image under p of a polar set. □

Lemma 3.7. *The inequality $ler(z) > -\infty$ holds for quasi every $z \in \mathcal{R}\setminus N_0$. In particular $u' \not\equiv +\infty$ and $u'' \not\equiv +\infty$ in Lemmas 3.1 and 3.5, so h' and h'' are finite non-negative harmonic functions on $\mathcal{R}$.*

Proof. Since $g(\mu; \mathbb{D}; p(\cdot))$ is finite on $\mathcal{R}\setminus N_0$ while u', u'' are non-negative and either identically $+\infty$ or finite quasi everywhere, ler is in turn either identically $-\infty$ or finite quasi everywhere on $\mathcal{R}\setminus N_0$. The former possibility contradicts Lemma 3.6, therefore the latter prevails so that $u' \not\equiv +\infty$ and $u'' \not\equiv +\infty$. Hence, Lemmas 3.1 and 3.5 imply that h' and h'' are finite on $\mathcal{R}$. □

Due to the previous lemma, we can rewrite (3.43) as

$$ler(z) = g(\mu, \mathbb{D}; p(z)) - g(\nu, \mathcal{R}; z) - h_{\mathcal{R}}(z), \quad z \in \mathcal{R}\setminus N_0, \tag{3.45}$$

where we have set $\nu := \nu' + \nu''$, which is a locally finite measure on $\mathcal{R}$ with quasi everywhere finite potential, and $h_{\mathcal{R}} := h' + h''$ which is a positive harmonic function on $\mathcal{R}$.

Lemma 3.8. *It holds that $\lim_{z\to\zeta} h_{\mathcal{R}}(z) = 0$ for every $\zeta \in \partial\mathcal{R}\setminus\mathcal{T}$.*

Proof. Fix $\zeta \in \partial\mathcal{R}\setminus\mathcal{T}$, and let $\xi \in \mathcal{T}$ be such that $p(\zeta) = p(\xi)$. *We claim* that

$$\liminf_{z\to\zeta} h_{\mathcal{R}}(z) = 0. \tag{3.46}$$

Indeed, if $\liminf_{z\to\zeta} h_{\mathcal{R}}(z) = l > 0$, take $0 < 2\epsilon := \min\{l, 2/\mathrm{cap}_{\mathbb{D}}(K_f)\}$. Let $S \subset \mathbb{D}$ be the radial segment $\{z : z = rp(\zeta), \; r \in [1-\delta, 1)\}$ and S_ζ (resp. S_ξ) be the connected component of $p^{-1}(S)$ accumulating on $\partial\mathcal{R}$ to ζ (resp. ξ). If $\delta > 0$ is small enough, then

$$h_{\mathcal{R}}(z) \geqslant \epsilon, \quad z \in S_\zeta \cup S_\xi, \tag{3.47}$$

by Lemma 3.2. Furthermore, Lemma 3.6 yields that either $ler(z_1) \geqslant 0$ or $ler(z_2) \geqslant 0$ if $p(z_1) = p(z_2) \in S\setminus p(A_0 \cup N_0)$. In particular, we get from (3.45) and (3.47) that

$$g(\mu, \mathbb{D}; z) \geqslant \epsilon, \quad z \in S\setminus p(A_0 \cup N_0), \tag{3.48}$$

where we notice that $A_0 \cup N_0$ as well as $p(A_0 \cup N_0)$ are polar. This contradicts Lemma A.5, applied with $g(\sigma, D; \cdot) = g(\mu, \mathbb{D}; \cdot)$ and ξ being $p(\zeta)$, since $R_{\epsilon/2}$ from that lemma would necessarily be a subset of $p(A_0 \cup N_0)$. *This proves our claim* (3.46).

Next, assume for a contradiction that $\limsup_{z\to\zeta} h_{\mathcal{R}}(z) = l' > 0$, and pick $0 < 2\epsilon \leqslant \min\{l', 2/\mathrm{cap}_{\mathbb{D}}(K_f)\}$ such that the level line $L_\epsilon := \{z : h_{\mathcal{R}}(z) = \epsilon\}$ is a smooth 1-dimensional submanifold of $\mathcal{R}$ (this can be achieved according to Sard's theorem). Notice that ζ must be a limit point of L_ϵ because any neighborhood of ζ in $\mathcal{R}_*$ contains a connected open set $U \ni \zeta$ with $U \cap \mathcal{R}$ connected (p is a local homeomorphism at ζ and we may take $p(U)$ to be a disk) in which $h_{\mathcal{R}}$ assumes values arbitrary close to 0 and l' by definition of lim inf and lim sup; hence, as $h_{\mathcal{R}}(U \cap \mathcal{R})$ is connected, it contains the value ϵ.

Let D_0 be a disk centered at $p(\zeta)$ of small enough radius so that D_ζ and D_ξ, the components of $p^{-1}(D_0)$ in $\mathcal{R}_*$ that contain respectively ζ and ξ, are in one-to-one correspondence with D_0 under p. Decreasing the radius of D_0 if necessary, we can assume that $h_{\mathcal{R}}(z) \geqslant \epsilon/2$ for $z \in D_\xi$ by Lemma 3.2. Let us redefine

$$S_\zeta := L_\epsilon \cap \overline{D}_\zeta, \quad S := p(S_\zeta), \quad \text{and} \quad S_\xi := p^{-1}(S) \cap \overline{D}_\xi. \tag{3.49}$$

Observe that $h_{\mathcal{R}}$ cannot be constant in view of (3.46) and (3.16). Therefore, no connected component of S can be a closed curve in D_0 by the maximum principle for harmonic functions. In addition, if S has a connected component, say S_*, accumulating at $z_* \in \mathbb{T}$, then

$$g(\mu, \mathbb{D}; z) \geqslant \epsilon/2, \quad z \in S_*\setminus p(A_0 \cup N_0),$$

exactly as in (3.48). Let $R_{\epsilon/3}$ be as in Lemma A.5, applied with $g(\sigma, D; \cdot) = g(\mu, \mathbb{D}; \cdot)$ and ξ being z_*. Then S_* must intersect every circle $\{z \in \mathbb{D} : |z - z_*| = 1 - r\}$, $r \in R_{\epsilon/3}$, by connectedness. Necessarily, the intersection must be a subset of $p(A_0 \cup N_0)$ and therefore polar. Since contractive maps do not increase the logarithmic capacity [52, Theorem 5.3.1], this means that $R_{\epsilon/3}$ is polar, which contradicts Lemma A.5 (as we explain in Section A.4, polar subsets of $\mathbb{D}$ have Greenian and logarithmic outer capacity zero). Thus, S is a system of smooth curves, each connected component of which has at least one limit point on $\partial D_0 \cap \mathbb{D}$. Consequently, if $T_0 \subset D_0$ is a circle centered at $p(\zeta)$, then any connected component of S intersecting the interior of T_0 must intersect T_0 as well since it accumulates at a point of ∂D_0. That is, S must intersect any such circle and we arrive at a contradiction exactly as above. □

The exceptional set where inequality is strict in Lemmas 3.3 and 3.5 a priori depends on the subsequence $\{n_m\}$ under consideration. The next lemma shows that there exists a polar set outside of which equality holds, both in (3.27) and (3.28), for one and the same subsequence, see (3.50).

Lemma 3.9. *For quasi every $z \in \mathcal{R}$, there is a sequence $\mathcal{N}_z = \{n_m^z\}_{m=1}^\infty \subseteq \mathcal{N}$ such that*

$$\begin{cases} \lim_{m\to\infty} g(\nu_{n_m^z}, \Omega_m; z) & = & u''(z), \\ \lim_{m\to\infty} g(\mu_{n_m^z}, \mathbb{D}; p(z)) & = & g(\mu, \mathbb{D}; p(z)). \end{cases}$$

Proof. Our goal is to show that there exists a subsequence $\{n_m^*\} \subseteq \mathcal{N}$ such that

$$\liminf_{m\to\infty} \big(g(\nu_{n_m^*}, \Omega_m; z) + g(\mu_{n_m^*}, \mathbb{D}; p(z))\big) = g(\mu, \mathbb{D}; p(z)) + u''(z) \tag{3.50}$$

for quasi every $z \in \mathcal{R}$. Since the inequalities in (3.27) and (3.28) hold for every $z \in \mathcal{R}$, this will indeed imply the claim of the lemma.

To prove (3.50), we shall rewrite the sum of two potentials in the left-hand side as a single potential on $\mathcal{R}$. To this end, we lift μ_n and μ to $\mathcal{R}$ via the construction described in (A.33). Specifically, with the notation introduced there, it follows from (A.34) that

$$g(\widehat{\mu}_n, \mathcal{R}; z) = g(\mu_n, \mathbb{D}; p(z)) \quad \text{and} \quad g(\widehat{\mu}, \mathcal{R}; z) = g(\mu, \mathbb{D}; p(z)), \quad z \in \mathcal{R}. \tag{3.51}$$

Now, we can write $g(\nu_n, \Omega_m; z) + g(\mu_n, \mathbb{D}; p(z))$ as a sum of three terms:

$$g(\nu_n + \widehat{\mu}_n, \Omega_m; z) + \big(g(\widehat{\mu}_{n\lfloor\Omega_m}, \mathcal{R}, z) - g(\widehat{\mu}_n, \Omega_m; z)\big) + g(\widehat{\mu}_{n\lfloor\mathcal{R}\setminus\Omega_m}, \mathcal{R}; z), \tag{3.52}$$

and we shall study their behavior separately.

To start, recall that $\mu = \mu'_{\lfloor\mathbb{D}}$, where μ' is the weak* limit of μ_n along $\mathcal{N}$ in $\overline{\mathbb{D}}$. Thus, by the discussion after (A.34), an analogous relation holds between $\widehat{\mu}$ and $\widehat{\mu}_n$. Namely, since μ_n has total mass 1, the total mass of $\widehat{\mu}_n$ is equal to M, the number of sheets of $\overline{\mathcal{R}}$. In particular, the sequence $\widehat{\mu}_n$ converges weak* on $\overline{\mathcal{R}}$ to $\widehat{\mu}'$, and on $\mathcal{R}$ to $\widehat{\mu}'_{\lfloor\mathcal{R}} = \widehat{\mu}$.

Since the sets Ω_m exhaust $\mathcal{R}\setminus E_f$, it holds that $\widehat{\mu}(\mathcal{R}\setminus\Omega_m) - \widehat{\mu}(E_f) \to 0$ as $m \to \infty$. Moreover, as each $\mathcal{R}\setminus\Omega_m$ is compact with boundary of $\widehat{\mu}$-measure zero, it follows from Lemma 3.4 that the measures $\widehat{\mu}_{n\lfloor\mathcal{R}\setminus\Omega_m}$ converge weak* to $\widehat{\mu}_{\lfloor\mathcal{R}\setminus\Omega_m}$ along $\mathcal{N}$. In particular, $\widehat{\mu}_n(\mathcal{R}\setminus\Omega_m) - \widehat{\mu}(\mathcal{R}\setminus\Omega_m) \to 0$ as $n \to \infty$. Hence, for each m there exists $n'_m \in \mathcal{N}$ such that

$$|\widehat{\mu}_n(\mathcal{R}\setminus\Omega_m) - \widehat{\mu}(E_f)| \leqslant 2|\widehat{\mu}(\mathcal{R}\setminus\Omega_m) - \widehat{\mu}(E_f)| \quad \text{as soon as} \quad n \geqslant n'_m.$$

Let σ be a weak* limit point on $\mathcal{R}$ of the family $\{\widehat{\mu}_{n'_m\lfloor\mathcal{R}\setminus\Omega_m}\}_{m\in\mathbb{N}}$. Clearly $\operatorname{supp}(\sigma) \subseteq E_f$ and σ is also a weak* limit point of this family on the fixed compact set $\mathcal{R}\setminus\Omega_1$. Thus, integrating against any function which is identically 1 on $\mathcal{R}\setminus\Omega_1$ and passing to the limit gives us $\sigma(E_f) = \widehat{\mu}(E_f)$ by our very choice of n'_m. In another connection, if $\varphi : \mathcal{R} \to [0, 1]$ is a continuous function with compact support which is 1 on a compact set $K \subseteq E_f$, then

$$\sigma(K) \leqslant \int \varphi d\sigma \leqslant \limsup_m \int \varphi d\widehat{\mu}_{n'_m\lfloor\mathcal{R}\setminus\Omega_m} \leqslant \lim_m \int \varphi d\widehat{\mu}_{n'_m} = \int \varphi d\widehat{\mu}.$$

As the infimum over φ of the rightmost term above is $\widehat{\mu}(K)$ by Urysohn's lemma and the outer regularity of Radon measures, we get that $\sigma \leqslant \widehat{\mu}_{\lfloor E_f}$ by the inner regularity of Radon measures on $\mathcal{R}$, see [53, Theorem 2.18]. Altogether, we deduce that $\sigma = \widehat{\mu}_{\lfloor E_f}$, and consequently the measures $\widehat{\mu}_{n'_m \lfloor \mathcal{R}\setminus\Omega_m}$ converge weak* to $\widehat{\mu}_{\lfloor E_f}$ along $\mathcal{N}' := \{n'_m\}$. In particular,

$$\lim_{m\to\infty} g(\widehat{\mu}_{n'_m\lfloor\mathcal{R}\setminus\Omega_m}, \mathcal{R}; z) = g(\widehat{\mu}_{\lfloor E_f}, \mathcal{R}; z), \quad z \in \mathcal{R}\setminus E_f, \tag{3.53}$$

which settles the asymptotic behavior of the last term in (3.52) along the sequence of indices $\{n'_m\}_m$. Notice that by the definition of n'_m, if $\mathcal{N}'$ is replaced by an arbitrary subsequence $\{n''_m\}_m$ thereof, then (3.53) continues to hold along this new subsequence of indices. This will be used in the forthcoming steps.

Next, let $f_m(z,w) := g_{\mathcal{R}}(z,w) - g_{\Omega_m}(z,w)$ for $z,w \in \Omega_m$. Clearly, $f_m(z,\cdot)$ is harmonic in Ω_m and continuous on $\overline{\Omega}_m$, by the regularity of Ω_m. Its boundary values are equal to $g_{\mathcal{R}}(z,\cdot)$ on $\partial\Omega_m$, in particular they are identically zero on $\partial\mathcal{R}$. Fix $z \in \mathcal{R}\setminus E_f$, and let k be an integer such that $z \in \Omega_k$. Then, we get for all $m > k$ that

$$0 \leqslant f_m(z,w) \leqslant \max_{w\in\partial\Omega_m\setminus\partial\mathcal{R}} g_{\mathcal{R}}(z,w) \leqslant \max_{w\in\partial\Omega_{k+1}} g_{\mathcal{R}}(z,w) =: C_k, \quad w \in \overline{\Omega}_m, \tag{3.54}$$

where the constant C_k is finite, independent of m, and we used the maximum principle for harmonic functions twice (once for $f_m(z,\cdot)$ on Ω_m and once for $g_{\mathcal{R}}(z,\cdot)$ on $\mathcal{R}\setminus\Omega_{k+1}$). Observe further that the functions $f_m(z,\cdot)$, $m > k$, are not only positive harmonic in each Ω_l for fixed l satisfying $k < l \leqslant m$, but form a decreasing sequence there. Therefore they converge in Ω_l to a non-negative harmonic function, say $f^{\{l\}}(z,\cdot)$, by Harnack's theorem. As this claim is true for all large l, the $f^{\{l\}}(z,\cdot)$ inductively define a harmonic function $f(z,\cdot)$ on $\mathcal{R}\setminus E_f$ that satisfies $0 \leqslant f(z,\cdot) \leqslant C_k$ when $z \in \Omega_k$. Thus, it extends harmonically to the entire surface $\mathcal{R}$ by the Removability theorem for harmonic functions, and as its trace on $\partial\mathcal{R}$ is zero we conclude that $f(z,\cdot) \equiv 0$.

Observe now that $f_m(z,w) = f_m(w,z)$ for m large enough so that $z,w \in \Omega_m$. Thus, it is jointly harmonic in both variables [35, p. 561]. Hence, by Harnack's theorem and the diagonal argument, any subsequence of $\{f_m(\cdot,\cdot)\}$ has a further subsequence converging locally uniformly in $\mathcal{R}\setminus E_f \times \mathcal{R}\setminus E_f$, and we know from what precedes that the limit function can only be zero. In particular, it holds that

$$\eta_{l,m} := \max_{z,w\in\overline{\Omega}_l} f_m(z,w) \to 0 \quad \text{as} \quad m \to \infty, \tag{3.55}$$

where we used that each $f_m(\cdot,\cdot)$ extends continuously by zero to $\partial\mathcal{R}\times\partial\mathcal{R}$, and therefore the maximum principle for harmonic functions can be applied to show that the convergence is indeed uniform on $\overline{\Omega}_l$. Given m, define

$$l_m := \max\left\{l \,:\, 1 \leqslant l \leqslant m,\, \eta_{l,m} \leqslant 1/l\right\}.$$

It follows from (3.55) that $l_m \to \infty$ and $\eta_{l_m,m} \to 0$ when $m \to \infty$, whence for each fixed $z \in \Omega_k$ and all $m > k$ it holds that

$$\begin{aligned} 0 \leqslant \int_{\Omega_m} f_m(z,w)d\widehat{\mu}_n(w) &= \left(\int_{\Omega_m\setminus\Omega_{l_m}} + \int_{\Omega_{l_m}}\right) f_m(z,w)d\widehat{\mu}_n(w) \\ &\leqslant C_k\widehat{\mu}_n(\overline{\Omega}_m\setminus\Omega_{l_m}) + M\eta_{l_m,m}, \end{aligned} \tag{3.56}$$

where we used (3.54) while recalling that M is the number of sheets of $\mathcal{R}$, which is a bound for the total mass of each $\widehat{\mu}_n$. Now, we obtain from Lemma 3.4 and our choice of Ω_m that the measures $\widehat{\mu}_n$ converge weak* to $\widehat{\mu}$ along $\mathcal{N}$, not only on $\mathcal{R}$, but also on every compact set of the form $\overline{\Omega}_m\setminus\Omega_{l_m}$. Hence, taking into account that $\widehat{\mu}\big((\mathcal{R}\setminus E_f)\setminus\Omega_{l_m}\big) \to 0$ as $m \to \infty$, we can associate to each m an integer $n''_m \in \mathcal{N}'$ such that $n''_m \geqslant n'_m$ and

$$\widehat{\mu}_{n''_m}\big(\overline{\Omega}_m\setminus\Omega_{l_m}\big) \to 0 \quad \text{as} \quad m \to \infty.$$

Clearly, the choice of $\mathcal{N}'' := \{n''_m\}$ is independent of z. Thus, we get from (3.56) together with the choices of $\{l_m\}$ and $\mathcal{N}''$ that for every $z \in \mathcal{R}\backslash E_f$ it holds

$$\lim_{m\to\infty} g(\widehat{\mu}_{n''_m \lfloor \Omega_m}, \mathcal{R}; z) - g(\widehat{\mu}_{n''_m}, \Omega_m; z) = \lim_{m\to\infty} \int_{\Omega_m} f_m(z,w) d\widehat{\mu}_{n''_m}(w) = 0, \tag{3.57}$$

which settles the asymptotic behavior of the middle term in (3.52) along $\mathcal{N}''$.

Lastly, to describe asymptotics of the first term in (3.52), we need to repeat some steps of the proof of Lemma 3.5 with ν_n replaced by $\nu_n + \widehat{\mu}_n$. Recall the definition of the sets $\Omega_{m,l}$ given just before (3.30). We may adjust it so that $(\nu^* + \widehat{\mu})(\partial\Omega_{m,l}) = 0$. Indeed, $\partial\Omega_{m,l} = (\partial\Omega_m \backslash \partial\mathcal{R}) \cup p^{-1}(\mathbb{T}_{r_l})$. We already know that $(\nu^* + \widehat{\mu})(\partial\Omega_m\backslash\partial\mathcal{R}) = 0$, so we only need to ensure that $(\nu^* + \widehat{\mu})(p^{-1}(\mathbb{T}_{r_l})) = 0$ for each l. This can be achieved as before since $\widehat{\mu}$ is finite and therefore $\nu^* + \widehat{\mu}$ is still a Radon measure. Since $\widehat{\mu}'(\partial\Omega_{m,l}) = \widehat{\mu}(\partial\Omega_{m,l}) = 0$ by construction, it follows from Lemma 3.4 that

$$\widehat{\mu}_{n\lfloor \overline{\Omega}_m \backslash \Omega_{m,l}} \overset{w*}{\to} \widehat{\mu}'_{\lfloor \overline{\Omega}_m\backslash\Omega_{m,l}} \quad \text{and} \quad \widehat{\mu}_n(p^{-1}(\mathbb{T}_{r_l})) \to 0$$

for all m, l. So, when $z \in \Omega_{m,l}$, we get that

$$\begin{aligned} \lim_{\mathcal{N}''\ni n\to\infty} g\big(\widehat{\mu}_{n\lfloor \Omega_m\backslash\overline{\Omega}_{m,l}}, \Omega_m; z\big) &= \lim_{\mathcal{N}''\ni n\to\infty} g\big(\widehat{\mu}_{n\lfloor \overline{\Omega}_m\backslash\Omega_{m,l}}, \Omega_m; z\big) \\ &= g\big(\widehat{\mu}'_{\lfloor \overline{\Omega}_m\backslash\Omega_{m,l}}, \Omega_m; z\big) = g\big(\widehat{\mu}_{\lfloor \Omega_m\backslash\overline{\Omega}_{m,l}}, \Omega_m; z\big) \end{aligned} \tag{3.58}$$

since $g_{\Omega_m}(\cdot, z)$ is continuous on $\overline{\Omega}_m\backslash\Omega_{m,l}$ and vanishes on $\partial\mathcal{R}$ by regularity of Ω_m. Moreover, the convergence is locally uniform in $z \in \Omega_{m,l}$ by the continuity of Green functions with respect to both variables off the diagonal. From (3.58) and the reasoning used after (3.30), it follows that

$$\lim_{\mathcal{N}''\ni n\to\infty} g\big((\nu_n + \widehat{\mu}_n)_{\lfloor \Omega_m\backslash\overline{\Omega}_{m,l}}, \Omega_m; z\big) = h_{m,l}(z) + g\big(\widehat{\mu}_{\lfloor \Omega_m\backslash\overline{\Omega}_{m,l}}, \Omega_m; z\big),$$

locally uniformly in $\Omega_{m,l}$ for each l and some subsequence $\mathcal{N}''_m$ of $\mathcal{N}''$. Arguing as we did to obtain (3.31), but applying this time the Lower Envelope theorem to $(\nu_n + \widehat{\mu}_n)_{\lfloor \overline{\Omega}_{m,l}}$, we get in view of the above limit that

$$\begin{aligned} \liminf_{\widetilde{\mathcal{N}}''\ni n\to\infty} g(\nu_n + \widehat{\mu}_n, \Omega_m; z) &= g((\nu^* + \widehat{\mu})_{\lfloor \overline{\Omega}_{m,l}}, \Omega_m; z) + h_{m,l}(z) + g\big(\widehat{\mu}_{\lfloor \Omega_m\backslash\overline{\Omega}_{m,l}}, \Omega_m; z\big) \\ &= g(\nu^* + \widehat{\mu}, \Omega_m; z) + h_m(z) \end{aligned}$$

for quasi every $z \in \Omega_m$, where $\widetilde{\mathcal{N}}''$ is the diagonal of the table $\{\mathcal{N}''_m\}_{m=1}^\infty$ and to get the second equality we used (3.32) along with the explanation preceding it on the inductive definition of u_m by the right-hand side of (3.31). The previous equation stands analog to (3.33), and continues to hold if $\widetilde{\mathcal{N}}''$ is replaced by any subsequence thereof. Now, the last part of the proof in Lemma 3.5 was predicated on the limit

$$u_m(z) = g(\nu^*, \Omega_m; z) + h_m(z) \to u''(z) \quad \text{as} \quad m \to \infty, \quad z \in \mathcal{R}\backslash E_f,$$

where u_m were initially defined inductively by the right-hand side of (3.31) and their limit u'' assumed the form (3.34). In the present case, u_m is replaced by $u_m + g(\widehat{\mu}, \Omega_m; \cdot)$ and the monotone convergence theorem together with the polar character of E_f imply that

$$\lim_m g(\widehat{\mu}, \Omega_m; z) = g(\widehat{\mu}, \mathcal{R}\backslash E_f; z) = g(\widehat{\mu}_{\lfloor \mathcal{R}\backslash E_f}, \mathcal{R}; z), \quad z \in \mathcal{R}\backslash E_f.$$

Hence, arguing as we did after (3.38), we obtain similarly to (3.42) that

$$\liminf_{m\to\infty} g(\nu_{n^*_m} + \widehat{\mu}_{n^*_m}, \Omega_m; z) = u''(z) + g(\widehat{\mu}_{\lfloor \mathcal{R}\backslash E_f}, \mathcal{R}; z), \tag{3.59}$$

for quasi every $z \in \mathcal{R}\backslash E_f$ and some subsequence $\{n^*_m\} \subseteq \widetilde{\mathcal{N}}''$. The desired limit (3.50) now follows from (3.51), (3.52), (3.53), (3.57), and (3.59). □

Hereafter we shall deal with the *fine topology*, which is the coarsest topology for which superharmonic and therefore δ-subharmonic functions are continuous. In this connection, the reader may want to consult the definitions and properties collected in Section A.5.

Recall the definition of $N_0 \subset \mathcal{R}$ before (3.43) as being the $+\infty$-set of $g(\mu,\mathbb{D};p(\cdot))$, i.e., $N_0 = p^{-1}(\widetilde{N}_0)$ where $\widetilde{N}_0 \subset \mathbb{D}$ is the $+\infty$-set of $g(\mu,\mathbb{D};\cdot)$. Clearly, N_0 is a finely closed and polar G_δ-set. Let us define

$$G_\pm := \big\{z \in \mathcal{R}\backslash N_0 : \pm ler(z) > 0\big\}. \tag{3.60}$$

It is easily seen from (3.43) that $ler : \mathcal{R}\backslash N_0 \to [-\infty,+\infty)$ is finely continuous and that G_- and G_+ are finely open in $\mathcal{R}\backslash N_0$. Since N_0 is polar, if the complement of either G_+ or G_- in $\mathcal{R}\backslash N_0$ is thin at a point z, then the respective complement in $\mathcal{R}$ is also thin at z. Hence, G_+ and G_- are in fact finely open in $\mathcal{R}$. Hereafter, we put $D_\pm := p(G_\pm)$.

Lemma 3.10. *For $z \in G_+$ and any $\zeta \in \mathcal{R}$ with $p(\zeta) = p(z)$, it holds that $\zeta \in G_+$ and $ler(z) = ler(\zeta)$. In particular $G_+ = p^{-1}(D_+)$. Moreover, D_+ is finely open.*

Proof. By the definition of ler given in (3.45), we get from equation (3.10) together with Lemmas 3.1, 3.7, and 3.9 (recall $\mathcal{N}''$ was renamed as $\mathcal{N}$ at the top of Section 3.7) that there exists a polar set $B \subset \mathcal{R}$ with the following property: for each $z \in G_+\backslash B$ there is a subsequence $\mathcal{N}_z \subset \mathcal{N}$ such that

$$\left\{\begin{array}{lcl} \displaystyle\lim_{\mathcal{N}_z \ni n\to\infty} \frac{1}{n}\log|(f - N(M_n)\circ p)(z)| &=& ler(z), \\ \displaystyle\lim_{\mathcal{N}_z \ni n\to\infty} g(\mu_n,\mathbb{D};p(z)) &=& g(\mu,\mathbb{D};p(z)), \end{array}\right. \tag{3.61}$$

where we note that $g(\mu,\mathbb{D};p(z)) < +\infty$ since $z \notin N_0$. Without loss of generality, we may assume that B contains $p^{-1}(p(E_f))$ and A_0, since both sets are polar, see Lemma 3.6 for the definition of A_0. As $ler(z) > 0$ and $f(z)$ is finite on $G_+\backslash B$, the above limit implies that

$$\lim_{\mathcal{N}_z \ni n\to\infty} \frac{1}{n}\log\big|N(M_n)(p(z))\big| = ler(z).$$

Now, if $\zeta \in \mathcal{R}\backslash B$ is such that $p(\zeta) = p(z)$, obviously $\zeta \notin N_0$ and in addition $\zeta \notin E_f$, by definition of B. Thus, on account of the finiteness of $f(\zeta)$, we get that

$$\lim_{\mathcal{N}_z \ni n\to\infty} \frac{1}{n}\log\big|(f - N(M_n)\circ p)(\zeta)\big| = \lim_{\mathcal{N}_z \ni n\to\infty} \frac{1}{n}\log\big|N(M_n)(p(z))\big| = ler(z). \tag{3.62}$$

On the other hand, from the second equation in (3.61), we deduce as in (3.44) that the first limit in (3.62) is at most $ler(\zeta)$, whence $0 < ler(z) \leqslant ler(\zeta)$. In particular, $\zeta \in G_+$, and reversing the roles of z and ζ gives $ler(z) = ler(\zeta)$. This proves the first assertion of the lemma when $z \in G_+\backslash B$.

To prove it on all of G_+, pick $z,\zeta \in \mathcal{R}$ such that $p(z) = p(\zeta)$ and let $D_0 \subset \mathbb{D}$ be a disk centered at $p(z)$. Denote by D_z, D_ζ the connected components of $p^{-1}(D_0)$ that contain z,ζ respectively, and make D_0 small enough so that $D_z \cap \mathbf{rp}(\mathcal{R}) \subseteq \{z\}$ and $D_\zeta \cap \mathbf{rp}(\mathcal{R}) \subseteq \{\zeta\}$. Let as before $m(\xi)$ be the ramification order of ξ, so that D_z (resp. D_ζ) is (isomorphic to) an $m(z)$-sheeted cyclic covering of D_0. For $x \in D_0\backslash\widetilde{N}_0$, define

$$g(x) := m(\zeta)\sum_{\xi\in p^{-1}(x)\cap D_z} m(\xi)ler(\xi) - m(z)\sum_{\xi\in p^{-1}(x)\cap D_\zeta} m(\xi)ler(\xi).$$

It follows from (3.45) and (A.32) that

$$\sum_{\xi\in p^{-1}(x)\cap D_z} m(\xi)ler(\xi) = m(z)g(\mu,\mathbb{D};x) - g\big(p_*(\nu_{\lfloor D_z}),\mathbb{D};x\big) - \sum_{\xi\in p^{-1}(x)\cap D_z} m(\xi)\left(g\big(\nu_{\lfloor\mathcal{R}\backslash D_z},\mathcal{R};\xi\big) + h_{\mathcal{R}}(\xi)\right).$$

Notice that the last summand above is a continuous, even harmonic function in D_0. Hence, the sum itself is finely continuous in $D_0 \setminus \widetilde{N}_0$. Likewise, the second sum in the definition of g is finely continuous in $D_0 \setminus \widetilde{N}_0$ and so is g. In particular, $U := \{x \in D_0 \setminus \widetilde{N}_0 : g(x) \neq 0\}$ is finely open, and since $p(G_+)$ is finely open by Lemma A.2 the set $U \cap p(G_+)$ is in turn finely open. From the first part of the proof, it follows that if $x \in (D_0 \cap p(G_+)) \setminus p(B)$ then $g(x) = 0$, whence $U \cap p(G_+) \subseteq p(B)$. Thus, $U \cap p(G_+)$ must be empty as $p(B)$ is polar, that is, $g \equiv 0$ on $D_0 \cap p(G_+)$. Now, it can be readily checked that

$$g(p(z)) = m(z)m(\zeta)\big(ler(z) - ler(\zeta)\big),$$

and therefore $ler(z) = ler(\zeta)$ if $z \in G_+$, thereby proving the first assertion of the lemma. The second is then obvious, and the third follows from Lemma A.2. $\square$

Lemma 3.11. *The set G_- lies schlicht over $\mathbb{D}$. That is, $p : G_- \to D_-$ is a bijection. Moreover, $G_- \cap \mathbf{rp}(\mathcal{R}) = \varnothing$, and for each $z \in G_-$ we have that*

$$ler(\zeta) = 0 \quad \textit{for all} \;\; \zeta \in p^{-1}(p(z)) \setminus \{z\}. \tag{3.63}$$

Proof. Pick $z \in G_- \setminus \mathbf{rp}(\mathcal{R})$ and let $D_0 \subset \mathcal{R} \setminus \mathbf{rp}(\mathcal{R})$ be a conformal disk centered at z, homeomorphic under p to a Euclidean disk $p(D_0)$; note that $p^{-1}(D_0) \setminus D_0$ is open. Define $U := G_- \cap D_0$ and $V := \big(p^{-1}(D_0) \setminus D_0\big) \cap p^{-1}(p(U))$. Clearly U and V are disjoint finely open subsets of $\mathcal{R}$, by Lemma A.2. In view of Lemma 3.6, there is a polar set $A_0 \subset \mathcal{R} \setminus N_0$ such that $V \cap G_- \subseteq A_0$. Thus, $V \cap G_-$ must be empty as otherwise it is finely open. This shows that $G_- \setminus \mathbf{rp}(\mathcal{R})$ lies schlicht over $\mathbb{D}$. However, the complement of a schlicht set is always non-thin at any ramification point by Lemma A.4. Hence, G_- cannot be a fine neighborhood of a ramification point, and since it is finely open $G_- \cap \mathbf{rp}(\mathcal{R}) = \varnothing$. Altogether, G_- lies schlicht over $\mathbb{D}$ and (3.63) now follows from this and Lemma 3.10. $\square$

3.8. **Modified Logarithmic Error Function.** In this subsection we modify the function $ler(z)$ by clearing out parts of $\mathbb{D}$ and $\mathcal{R}$ from the support of μ and ν, respectively. We shall accomplish this via the technique of balayage, described in Section A.9. Let us start with some preliminary geometric considerations. Recall that any connected (topological) 1-manifold embedded in $\mathcal{R}$ is a Jordan curve.

Lemma 3.12. *For each $\epsilon > 0$ there exists a Jordan curve $J \subset G_-$ such that $p(J)$ is a Jordan curve included in $\{z : 1 - \epsilon < |z| < 1\}$ and $p(\mathbf{rp}(\mathcal{R}))$ belongs to the interior domain of $p(J)$. Moreover, there exists a finely connected component of G_-, say G_J, such that $J \subset G_J$.*

Proof. We may assume that ϵ is small enough that $p(\mathbf{rp}(\mathcal{R})) \subset \mathbb{D}_{1-\epsilon}$. In particular, p is injective on every conformal disk centered at a point of $\partial\mathcal{R}$ with radius smaller than or equal to ϵ.

Recall that $\mathcal{R}$ is a subset of a Riemann surface $\mathcal{R}_*$ lying over $\overline{\mathbb{C}}$. Define $G_-^* := G_- \cup \mathcal{T} \cup \mathcal{D}$, where $\mathcal{D}$ is the connected component of $p^{-1}(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$ that borders $\mathcal{T}$. Let us show that for each $\eta \in \mathcal{T}$ there is a disk $D_\eta \subset \{z : 1 - \epsilon < |z| < 1 + \epsilon\}$, centered at $p(\eta)$ with radius r_η, such that the circle ∂D_η is included in $p(G_-^*)$. In fact, we can pick r_η so that there exist radii r_η' arbitrarily close but not equal to r_η for which each disk D_η' centered at $p(\eta)$ of radius r_η' also satisfies $\partial D_\eta' \subset p(G_-^*)$. Indeed, like we did to establish Lemma 3.9, let $\widehat{\mu}$ be the lift μ to $\mathcal{R}$, see (A.33). By definition, $ler(z) \geqslant 0$ for $z \in \mathcal{R} \setminus (G_- \cup N_0)$. Therefore, if $\eta \in \mathcal{T}$ is a limit point of $\mathcal{R} \setminus G_-$, then we get from (3.16), (3.45), the definition of $h_{\mathcal{R}}(z)$, and the identity $g(\widehat{\mu}, \mathcal{R}; \zeta) = +\infty$ when $\zeta \in N_0$ that

$$\liminf_{\mathcal{R} \setminus G_- \ni \zeta \to \eta} g(\widehat{\mu}, \mathcal{R}; \zeta) \geqslant 2/\mathrm{cap}_{\mathbb{D}}(K_f).$$

The claim now follows from Lemma A.5 by taking $1 - r_\eta$ to be an accumulation point of $R_{1/\mathrm{cap}_{\mathbb{D}}(K_f)}$ in $(1 - \epsilon, 1)$.

Let U_η be the connected component of $p^{-1}(D_\eta)$ containing η, which is an open subset of $\mathcal{R}_*$ satisfying $\partial U_\eta \subset G_-^*$. Since the collection $\{U_\eta\}_\eta$ covers $\mathcal{T}$, which is compact, it contains a finite subcover, say $\{U_{\eta_i}\}$. Replacing D_{η_i} by some D_{η_i}' as above if needed, we can ensure by (finite) induction on i that ∂U_{η_i} and ∂U_{η_j} may intersect only transversally for $i \neq j$, and that $\partial U_{\eta_i} \cap \partial U_{\eta_j} \cap \partial U_{\eta_k} = \varnothing$ if i, j, k are all distinct. Then $V := \cup_i U_{\eta_i}$ is an open neighborhood of $\mathcal{T}$ with boundary ∂V included in G^*_- that consists of a finite union of disjoint Lipschitz-smooth Jordan curves. In fact, there are exactly two such curves, one in $\mathcal{R}$ and $\mathcal{D}$, because each connected component of ∂V can be continuously deformed into $\mathcal{T}$ via radial retraction within V. We can choose the component within $\mathcal{R}$ to be J since Lipschitz curves are finely connected in $\mathcal{R}$, see Section A.5. □

Recall now the finely open subsets D_+ and D_- of $\mathbb{D}$ introduced before Lemma 3.10. Denote by D_J the union of all finely connected components of $D_+ \cup D_-$ that lie entirely within the interior domain of $p(J)$, where J was introduced in Lemma 3.12. Note that D_J is finely open because so are the fine components of finely open sets, see Section A.5. Define

$$D' := D_J \cup i(\mathbb{D} \backslash D_J) \quad \text{and} \quad V' := p^{-1}(D'), \tag{3.64}$$

where $i(\cdot)$ is the subset of finely isolated points. Thus, it follows from (A.9) and Lemma A.2 that D' and V' are finely open while

$$b\big(\mathbb{D} \backslash D'\big) = \mathbb{D} \backslash D' \quad \text{and} \quad b\big(\mathcal{R} \backslash V'\big) = \mathcal{R} \backslash V', \tag{3.65}$$

where $b(\cdot)$ stands for the base of a set (in particular, $\mathbb{D} \backslash D'$ has no finely isolated points). In other words, D' and V' are regular finely open sets, see Section A.7. Recall from Section A.9 the notation σ^E for the balayage of the measure σ onto the set E.

Lemma 3.13. *Let $N_1 := p^{-1}(\widetilde{N}_1)$, where $\widetilde{N}_1$ is the $+\infty$-set of $g\big(\mu^{(1)}, \mathbb{D}; \cdot\big)$ and we set*

$$\mu^{(1)} := \mu^{\mathbb{D} \backslash D'} \quad \text{and} \quad \nu^{(1)} := \nu^{\mathcal{R} \backslash V'}. \tag{3.66}$$

Then $N_1 \subseteq N_0$, $N_1 \backslash V' = N_0 \backslash V'$, and for every $z \in \mathcal{R} \backslash N_1$ we can define

$$ler^{(1)}(z) := g\big(\mu^{(1)}, \mathbb{D}; p(z)\big) - g\big(\nu^{(1)}, \mathcal{R}; z\big) - h_{\mathcal{R}}(z)$$

with values in $[-\infty, +\infty)$. This function satisfies

$$ler^{(1)}(z) = \begin{cases} ler(z), & z \in \mathcal{R} \backslash (V' \cup N_1), \\ 0, & z \in V' \backslash N_1. \end{cases} \tag{3.67}$$

Proof. Since the Green potential of a measure dominates the Green potential of any balayage of that measure, as explained at the beginning of Section A.9, it holds that

$$g\big(\mu^{(1)}, \mathbb{D}; z\big) \leqslant g\big(\mu, \mathbb{D}; z\big) < +\infty, \quad z \in \mathcal{R} \backslash N_0,$$

by the very definition of N_0. Thus, $N_1 \subseteq N_0$ and the upper equality in (3.67) as well as equality $N_1 \backslash V' = N_0 \backslash V'$ are consequences of (3.65) and (A.22). Since $h_{\mathcal{R}}(z)$ is a harmonic function on $\mathcal{R}$ and $\overline{V'} \cap \partial \mathcal{R} = \varnothing$ by construction (recall that $p(V')$ lies interior to $p(J)$), equation (A.39) yields that

$$\int h_{\mathcal{R}}(x) d\delta_z^{\mathcal{R} \backslash V'}(x) = h_{\mathcal{R}}(z), \quad z \in V'. \tag{3.68}$$

Since V' is a regular finely open set, Lemma A.6 implies that $\delta_z^{\mathcal{R} \backslash V'}$ is carried by $\partial_{\mathsf{f}} V'$ and does not charge polar sets. As $ler(z) = 0$ for quasi every $z \in \partial_{\mathsf{f}} V'$ by the definition of V', we get from the definition of $ler^{(1)}$, (3.68), (A.25), Lemma A.7, and (3.45) that

$$\begin{aligned} ler^{(1)}(z) &= \int \big(g(\mu, \mathbb{D}; p(x)) - g(\nu, \mathcal{R}; x) - h_{\mathcal{R}}(x)\big) d\delta_z^{\mathcal{R} \backslash V'}(x) \\ &= \int ler(x) d\delta_z^{\mathcal{R} \backslash V'}(x) = 0 \end{aligned}$$

for $z \in V' \backslash N_1$, which proves the lower equality in (3.67). □

Let G_J be as in Lemma 3.12 for some small $\epsilon > 0$. Denote by G_J^* the union of G_J and the annular region delimited by J and $\mathcal{T}$; the latter is diffeomorphic under p to $\{z : 1-\epsilon < |z| < 1\}$ if we fix ϵ small enough. Clearly, G_J^* is a fine domain that lies schlicht over $\mathbb{D}$. Let $\mathcal{G}$ be the collection of all fine domains $G \subset \mathcal{R}$ lying schlicht over $\mathbb{D}$ and containing G_J^*. The set $\mathcal{G}$ is partially ordered by inclusion and every chain in it is bounded above by the union of its elements. Therefore, by Zorn's lemma, $\mathcal{G}$ possesses a maximal element, say G_{max}. Note that if $\mathcal{R}\backslash G_{\mathsf{max}}$ is thin at two points $\zeta_1, \zeta_2 \in \mathcal{R}$, then $p(\zeta_1) \neq p(\zeta_2)$ as otherwise G_{max} could not be schlicht over $\mathbb{D}$ by Lemma A.2. Hence, it follows from (A.9) and the maximality of G_{max} that $\mathcal{R}\backslash G_{\mathsf{max}}$ is its own base.

Lemma 3.14. *Any maximal domain $G_{\mathsf{max}} \in \mathcal{G}$ is a Euclidean domain. Moreover, no connected component of $\mathcal{R}\backslash G_{\mathsf{max}}$ (resp. $\mathbb{D}\backslash p(G_{\mathsf{max}})$) consists of a single point.*

Proof. Observe that G_{max} cannot contain a ramification point of $\mathcal{R}$ as it is finely open and lies schlicht over $\mathbb{D}$, see Lemma A.4. Thus, for every $\zeta \in G_{\mathsf{max}}$ there exists $r_0 > 0$ such that each component of

$$p^{-1}\big(\{z \in \mathbb{D} : \ |z - p(\zeta)| < r_0\}\big)$$

is in one-to-one correspondence with $\{z \in \mathbb{D} : \ |z - p(\zeta)| < r_0\}$ under p. Let V be the component containing ζ. Since the intersection $V \cap G_{\mathsf{max}}$ is finely open, $V\backslash G_{\mathsf{max}}$ is thin at ζ and therefore there is $r_1 \in (0, r_0)$ such that

$$V \cap p^{-1}\big(\{z \in \mathbb{D} : \ |z - p(\zeta)| = r_1\}\big) \subset G_{\mathsf{max}},$$

see Section A.6. Now, as G_{max} lies schlicht over $\mathbb{D}$, it holds that

$$G_{\mathsf{max}} \cap \big(p^{-1}\big(\{z \in \mathbb{D} : \ |z - p(\zeta)| = r_1\}\big)\backslash V\big) = \varnothing. \tag{3.69}$$

Further, since G_{max} is finely connected, we necessarily have that

$$G_{\mathsf{max}} \cap \big(p^{-1}\big(\{z \in \mathbb{D} : \ |z - p(\zeta)| < r_1\}\big)\backslash V\big) = \varnothing,$$

otherwise G_{max} would intersect the fine boundary of $p^{-1}\big(\{z \in \mathbb{D} : \ |z - p(\zeta)| < r_1\}\big)\backslash V$ which is contained in $p^{-1}\big(\{z \in \mathbb{D} : \ |z - p(\zeta)| = r_1\}\big)\backslash V$ (in fact, equal to it by regularity), thereby contradicting (3.69). The maximality of G_{max} now yields that

$$p^{-1}\big(\{z \in \mathbb{D} : \ |z - \zeta| < r_1\}\big) \cap V \subset G_{\mathsf{max}},$$

hence ζ belongs to the Euclidean interior of G_{max} and so G_{max} is Euclidean open. Thus, its connected components are open and therefore finely open. Hence, G_{max} is a Euclidean domain since it is finely connected by definition.

To prove the second assertion, assume to the contrary that a point $\zeta \in \mathcal{R}\backslash G_{\mathsf{max}}$ is a connected component of the latter. *We claim* that any open neighborhood W of ζ contains an open set $O \ni \zeta$ whose boundary ∂O is a smooth Jordan curve C contained in G_{max}. To see this, assume with no loss of generality that $\overline{W} \subset \mathcal{R}$, and put $F := \mathcal{R}\backslash G_{\mathsf{max}} \cap \overline{W}$. The latter is closed in $\mathcal{R}$, so there is a smooth function $h : \mathcal{R} \to \mathbb{R}^+$ of which F is the zero set, see Section 3.4. Given a sequence $\{c_n\}$ of regular values of h tending to 0, let O_n be the connected component containing ζ of the open set $\{z : \ h(z) < c_n\}$. The sets O_n form a (strictly) nested sequence of connected open sets containing ζ, whose intersection is connected and contained in F whence reduces to ζ. If $O_{n_k} \cap \partial W \neq \varnothing$ for some increasing subsequence of indices n_k, take $z_{n_k} \in O_{n_k} \cap \partial W$ and extract from $\{z_{n_k}\}$ a subsequence converging to $z \in \partial W$, which is possible due to compactness of the latter. On the other hand, since $\overline{O}_{n_{k+1}} \subset O_{n_k}$, $z \in \cap_k \overline{O}_{n_k} = \{\zeta\} \in \overline{W}\backslash\partial W$, which is a contradiction. Therefore, $O_n \subset W$ for n large enough and $\partial O_n \cap F = \varnothing$ by construction, so we may set $O = O_n$ and $C := \partial O_n$ for any n large enough, because ∂O_n is a connected component of the level set $h^{-1}(c_n)$, and thus it is a smooth Jordan curve. *This proves the claim.*

If $\zeta \notin \mathbf{rp}(\mathcal{R})$, let us pick W so small that p is injective on each connected component of $p^{-1}(p(W))$. We now argue as before, observing that $C \subset G_{\mathsf{max}}$ and $(p^{-1}(p(C))\backslash C) \cap G_{\mathsf{max}} = \varnothing$ (by schlichtness), so that $G_{\mathsf{max}} \cap (p^{-1}(p(O))\backslash O) = \varnothing$ because G_{max} is connected and $(p^{-1}(p(C))\backslash C)$ is the boundary of $p^{-1}(p(O))\backslash O$. Thus, by maximality, $G_{\max}$ should contain O which contradicts the fact that $\zeta \in O$. Finally, if $\zeta \in \mathbf{rp}(\mathcal{R})$ then Lemma A.3 contradicts the existence of C.

This proves that no connected component of $\mathcal{R}\backslash G_{\max}$ consists of a single point. To show the same is true of $\mathbb{D}\backslash p(G_{\max})$, observe that such a component would consist, by the same reasoning as before, of a $z \in \partial p(G_{\max})$ lying interior to Jordan curves of arbitrary small diameter contained in $p(G_{\max})$. For $\gamma : [0, 2\pi] \to p(G_{\max})$ such a parametrized Jordan curve, let $\xi \in G_{\max}$ satisfy $p(\xi) = \gamma(0)$ $(= \gamma(2\pi))$. Let further $\ell : [0, 2\pi] \to \mathcal{R}$ be a continuous lift of γ starting at ξ, i.e., $\ell(0) = \xi$ and $p(\ell(t)) = \gamma(t)$ for $t \in [0, 2\pi]$. Of necessity $\ell([0, 2\pi]) \subset G_{\max}$, because $p(\partial G_{\max}) \cap p(G_{\max}) = \varnothing$ since $G_{\max}$ is open and lies schlicht over $\mathbb{D}$, while p is an open map; then schlichtness again implies that ℓ is a parametrized Jordan curve in $G_{\max}$, and it is the unique lift of γ to $G_{\max}$.

Set $p^{-1}(z) = \{\zeta_1, \dots, \zeta_\ell\}$, and let D_z be an open disk centered at z such that each connected component of $p^{-1}(D_z)$ is (isomorphic to) a $m(\zeta_j)$-sheeted cyclic covering D_{ζ_j} of D_z; in addition, we require that D_z is so small that $D_{\zeta_j} \subset U_{\zeta_j}$ for all j, where U_{ζ_j} is as in Lemma A.3. Let $\gamma : [0, 2\pi] \to D_z$ be a parametrized Jordan curve containing z in its interior, and $\ell : [0, 2\pi] \to G_{\max}$ the associated lift. Necessarily $\ell([0, 2\pi])$ is contained in a single component of $p^{-1}(D_z)$, say D_{ζ_j}, and it must contain ζ_j in its interior (otherwise ℓ would be a unit in $\pi_1(D_{\zeta_j}\backslash\zeta_j)$ and so would be $\gamma = p \circ \ell$ in $\pi_1(D_z\backslash z)$). Now, if $m(\zeta_j) > 1$, then we contradict Lemma A.3. Hence, ℓ is valued in a D_{ζ_j} such that $m(\zeta_j) = 1$, which is thus homeomorphic to D_z under p.

Let γ_n be a sequence of Jordan curves in D_z, containing z in their interior and shrinking to z when $n \to \infty$. Let further ℓ_n be the corresponding sequence of lifts to $G_{\max}$. By what precedes, some subsequence ℓ_{m_n} shrinks to ζ_j in D_{ζ_j}, for some j such that D_{ζ_j} is homeomorphic to D_z under p. Moreover, ℓ_{m_n} contains ζ_j in its interior. We can now argue as we did to show that no connected component of $\mathcal{R}\backslash G_{\max}$ consists of a single point, and contradict the maximal character of $G_{\max}$. This completes the proof of the lemma. □

Define $D_{\max} := p(G_{\max})$ and $V_{\max} := p^{-1}(D_{\max})$. Notice that both sets are open by Lemma 3.14 along with openness and continuity of p.

Lemma 3.15. *Let $N_2 := p^{-1}(\widetilde{N}_2)$, where $\widetilde{N}_2$ is the $+\infty$-set of $g\left(\mu^{(2)}, \mathbb{D}; \cdot\right)$ and we set*

$$\mu^{(2)} := \left(\mu^{(1)}\right)^{\mathbb{D}\backslash D_{\max}} \quad \textit{and} \quad \nu^{(2)} := \left(\nu^{(1)}\right)^{\mathcal{R}\backslash V_{\max}}. \tag{3.70}$$

Then $N_2 = N_1\backslash V_{\max}$, and for $z \in \mathcal{R}\backslash N_2$ we can define

$$ler^{(2)}(z) := g\left(\mu^{(2)}, \mathbb{D}; p(z)\right) - g\left(\nu^{(2)}, \mathcal{R}; z\right) - h_{\mathcal{R}}(z). \tag{3.71}$$

In this case it holds that

$$\begin{cases} \limsup\limits_{\zeta \to z} ler^{(2)}(\zeta) \leqslant -2/\mathrm{cap}_{\mathbb{D}}(K_f), & z \in \mathcal{T}, \\ ler^{(2)}(\zeta) = 0, & z \in \mathcal{R}\backslash(G_{\max} \cup N_2). \end{cases} \tag{3.72}$$

Moreover, we have that

$$\|\mu^{(2)}\| \begin{cases} = \|\mu\| & \textit{if } \mathrm{supp}(\mu^{(1)}) \cap D_{\max} = \varnothing, \\ < \|\mu\| & \textit{otherwise}. \end{cases} \tag{3.73}$$

Proof. By Lemma 3.14 the set $\mathbb{D}\backslash D_{\max}$ is closed in $\mathbb{D}$ and none of its connected components reduces to a point. Hence, it has no finely isolated points (remember that a connected set cannot be thin at an accumulation point, see discussion after (A.8)). Thus, $\mathbb{D}\backslash D_{\max}$ is its own base. Consequently, $\mathcal{R}\backslash V_{\max}$ is also its own base by Lemma A.2. Therefore, we get from (A.22) that

$$\mu^{(2)}(D_{\max}) = \nu^{(2)}(V_{\max}) = 0. \tag{3.74}$$

This implies that the potentials of $\mu^{(2)}$ and $\nu^{(2)}$ are harmonic in $D_{\max}$ and $V_{\max}$, respectively. Moreover, since $\partial D_{\max}\backslash\mathbb{T}$ is separated from $\mathbb{T}$ by the very definition of $G_{\max}$, $\partial V_{\max}\backslash\partial\mathcal{R}$ is necessarily separated from $\partial\mathcal{R}$ and these potentials extend continuously by zero to $\mathbb{T}$ and $\partial\mathcal{R}$ respectively by the regularity of the latter, see the discussion after (A.12). Therefore, $ler^{(2)}$ extends harmonically to the whole of $V_{\max}$ and continuously to $\partial\mathcal{R}\backslash\mathcal{T}$ by Lemma 3.8. So, as in Lemma 3.13, the inclusion $N_2 \subseteq N_1\backslash V_{\max}$ follows directly from (A.21) and the equality $N_2 = N_1\backslash V_{\max}$ is then deduced from (A.22). In addition, the first inequality in (3.72) holds in view of (3.16), the definition of $h_{\mathcal{R}}$ given after (3.45), and the fact that h'' is non-negative, see Lemma 3.5.

Since $N_1\backslash V' = N_0\backslash V'$ by Lemma 3.13 we have that $\mathcal{R}\backslash(V' \cup V_{\max} \cup N_0) = \mathcal{R}\backslash(V' \cup V_{\max} \cup N_1)$. Moreover, $ler(z) = 0$ on this set by the very definitions of V' and $V_{\max}$. Further,

$$\mathcal{R}\backslash(V_{\max} \cup N_2) = \mathcal{R}\backslash(V_{\max} \cup N_1) = (\mathcal{R}\backslash(V' \cup V_{\max} \cup N_1)) \cup (V'\backslash N_1).$$

Since $\mathbb{D}\backslash D_{\max}$ and $\mathcal{R}\backslash V_{\max}$ are their own bases, it therefore follows from (A.22) and (3.67) that

$$ler^{(2)}(z) = ler^{(1)}(z) = 0, \quad z \in \mathcal{R}\backslash(V_{\max} \cup N_2). \tag{3.75}$$

To study the values of $ler^{(2)}$ on $V_{\max}\backslash G_{\max}$, let us show that this is an open set. Indeed, since $p(G_{\max}) = p(V_{\max}) = D_{\max}$, for each $\zeta \in V_{\max}$ there exists a disk $D_{p(\zeta)} \subset D_{\max}$, centered at $p(\zeta)$, and a point $z \in G_{\max}$ with $p(z) = p(\zeta)$ such that $D_\zeta \subset V_{\max}$ and $D_z \subset G_{\max}$, where D_ζ and D_z are the connected components of $p^{-1}(D_{p(\zeta)})$ that contain ζ and z, respectively. If $\zeta \neq z$, then $D_\zeta \cap G_{\max} = \varnothing$ since $G_{\max}$ lies schlicht over $\mathbb{D}$ and therefore $D_z \subset V_{\max}\backslash G_{\max}$ as claimed. Moreover, since $\mathcal{R}\backslash V_{\max}$ is its own base, we get from Lemma A.1 that $\mathcal{R}\backslash(V_{\max}\backslash G_{\max})$ is also its own base.

Pick $z \in V_{\max}\backslash G_{\max}$ and notice that $V_{\max}\backslash G_{\max}$ consists of at most finitely many connected components. Let V_z be the component containing z. As $G_{\max}$ contains the annular region delimited by J and $\mathcal{T}$ and lies schlicht over $\mathbb{D}$, it follows that $\partial(V_{\max}\backslash G_{\max}) \cap \partial\mathcal{R} = \partial\mathcal{R}\backslash\mathcal{T}$. Since $h_{\mathcal{R}} \equiv 0$ on $\partial\mathcal{R}\backslash\mathcal{T}$ by Lemma 3.8, we get from Lemmas A.6 and A.10 that

$$\int h_{\mathcal{R}}(x)d\delta_z^{\mathcal{R}\backslash V_z}(x) = \int h_{\mathcal{R}}(x)d\delta_z^{\mathcal{R}\backslash(V_{\max}\backslash G_{\max})}(x) = h_{\mathcal{R}}(z). \tag{3.76}$$

Since V_z, $V_{\max}\backslash G_{\max}$, and $G_{\max}$ are Euclidean open sets, it holds that $\partial V_z \subseteq \partial(V_{\max}\backslash G_{\max}) \subseteq \partial V_{\max}$, a set on which $ler^{(1)}(z)$ is zero except possibly on N_2 by (3.75). Consequently, in view of (3.71), we see upon using Lemma A.7, (A.25), and (3.76) that

$$ler^{(2)}(z) = \int ler^{(1)}(x)d\delta_z^{\mathcal{R}\backslash V_z}(x) = 0,$$

where the second equality follows from Lemma A.6 as $\partial_{\mathfrak{f}} V_z \subseteq \partial V_z$ and $ler^{(1)}(x) = 0$ quasi and therefore $\delta_z^{\mathcal{R}\backslash V_z}$-almost everywhere on $\partial_{\mathfrak{f}} V_z$.

Finally, we get from (A.24) and (A.38) that $\|\mu^{(2)}\| = \|\mu^{(1)}\|$ when $\operatorname{supp}(\mu^{(1)}) \cap D_{\max} = \varnothing$ and $\|\mu^{(2)}\| < \|\mu^{(1)}\|$ otherwise as well as that $\|\mu^{(1)}\| = \|\mu\|$, which proves (3.73). □

3.9. **Projected Logarithmic Error Function.** Recall that $z \in \partial G_{\max}$ is called accessible if there exists a continuous map ψ on $[0,1]$ such that $\psi(t) \in G_{\max}$ for $t \in [0,1)$ and $\psi(1) = z$.

Lemma 3.16. *It holds that* card $\big(p^{-1}(z) \cap (\partial G_{\max}\backslash E^*)\big) \leqslant 2$ *for all except countably many* $z \in \partial D_{\max}$*, where* $E^* \subset \partial G_{\max}$ *is the subset of non-accessible points.*

Proof. Since $\mathbb{T} \subset \partial D_{\max}$ and $p^{-1}(\mathbb{T}) \cap \partial G_{\max} = \mathcal{T}$, we only need to consider $z \in \partial D_{\max} \cap \mathbb{D}$. The set $(\partial D_{\max} \cap \mathbb{D})\backslash p(\mathbf{rp}(\mathcal{R}))$ can be covered by countably many open sets of the form D_x, where $x \in (\partial D_{\max} \cap \mathbb{D})\backslash p(\mathbf{rp}(\mathcal{R}))$ and $D_x \subset \mathbb{D}$ is a disk centered at x, small enough that each component of $p^{-1}(D_x)$ is homeomorphic to D_x under p. Hence, it is enough to show that for each such D_x and all $z \in \partial D_{\max} \cap D_x$ but countably many, one has card $\big(p^{-1}(z) \cap (\partial G_{\max}\backslash E^*)\big) \leqslant 2$. Fix D_x and let $x_1, \dots, x_N$ denote the preimages of x under p. We write V_{x_ℓ} for the connected component of $p^{-1}(D_x)$ containing x_ℓ, $1 \leqslant \ell \leqslant N$. Then, each $z \in D_x$ has preimages $z_1, \dots, z_N$ under p, with $z_\ell \in V_{x_\ell}$. When $z \in \partial D_{\max} \cap D_x$, it follows from the definition of accessibility that if $z_\ell \in \partial G_{\max}\backslash E^*$, then there is a continuous arc $\psi_{z_\ell}:[0,1]\to\overline{G}_{\mathsf{max}}$ such that $\psi_{z_\ell}(t)\in G_{\mathsf{max}}$ for $t\in[0,1)$ and $\psi_{z_\ell}(1)=z_\ell$. Moreover, on shortening and reparametrizing the arc if necessary, we may assume that ψ_{z_ℓ} is valued in V_{x_ℓ}. Now, if $p^{-1}(z)\cap(\partial G_{\mathsf{max}}\backslash E^*)$ contains 3 distinct points, say z_{ℓ_j} for $j=\{1,2,3\}$, then $p(\psi_{z_{\ell_j}})$ are Jordan arcs having only the point z in common, because G_{max} lies schlicht over $\mathbb{D}$ and the $V_{x_{\ell_j}}$ are disjoint. Thus, $T_z:=\cup_j p(\psi_{z_{\ell_j}})$ is a triod, and if we had uncountably many such z then some triple (ℓ_1,ℓ_2,ℓ_3) would occur uncountably many times. Assigning different colors to $\psi_{z_{\ell_1}},\psi_{z_{\ell_2}}$ and $\psi_{z_{\ell_3}}$, we get and uncountable collection of colored triods T_z whose arcs of different colors never meet, again because G_{max} lies schlicht over $\mathbb{D}$ and the $V_{x_{\ell_j}}$ are disjoint. However, this contradicts the Moore triod theorem [49, Proposition 2.18], thereby finishing the proof. □

Put $K_{\mathsf{max}}:=\mathbb{D}\backslash D_{\mathsf{max}}$. It follows from Lemma 3.14 that f has a single-valued meromorphic continuation throughout $D_{\mathsf{max}}\backslash p(E_f)$ and K_{max} is a compact subset of $\mathbb{D}$. Hence, $K_{\mathsf{max}}\in\mathcal{K}_f$, where $\mathcal{K}_f$ was defined just before (2.5). Clearly, the measure $\mu^{(2)}$ is supported on K_{max}.

Lemma 3.17. *Let m_ζ be the ramification order of $\zeta\in\mathcal{R}$. Define*

$$ler_{\mathsf{pr}}(z):=\sum_{\zeta\in p^{-1}(z)}m_\zeta ler^{(2)}(\zeta),\quad z\in\mathbb{D}\backslash\widetilde{N}_2. \tag{3.77}$$

Then ler_{pr} is a δ-subharmonic function in $\mathbb{D}$ such that $ler_{\mathsf{pr}}(z)=0$ when $z\in K_{\mathsf{max}}\backslash\widetilde{N}_2$ and

$$\limsup_{z\to\zeta} ler_{\mathsf{pr}}(z)\leqslant -2/\mathrm{cap}_{\mathbb{D}}(K_f),\quad \zeta\in\mathbb{T}.$$

Moreover, there exist non-negative measures μ_{pr} and ν_{pr}, supported on $\partial D_{\mathsf{max}}$, and a non-negative function h_{pr}, harmonic in $\mathbb{D}$, such that $\mu_{\mathsf{pr}}\leqslant 2\mu^{(2)}$ and for $z\in\mathbb{D}\backslash\widetilde{N}_2$

$$ler_{\mathsf{pr}}(z)=g(\mu_{\mathsf{pr}},\mathbb{D};z)-g(\nu_{\mathsf{pr}},\mathbb{D},z)-h_{\mathsf{pr}}(z)-2/\mathrm{cap}_{\mathbb{D}}(K_f). \tag{3.78}$$

Proof. The first two claims of the lemma follow readily from (3.72) and the computation in (A.11). Let $\widehat{\mu}^{(2)}$ be the pullback of $\mu^{(2)}$ onto $\mathcal{R}$, see (A.33) and (A.34). Then we can equivalently write

$$\begin{aligned} ler^{(2)}(z)&=g\big(\widehat{\mu}^{(2)},\mathcal{R};z\big)-g\big(\nu^{(2)},\mathcal{R};z\big)-h_{\mathcal{R}}(z)\\ &=g(\mu_{\mathsf{R}},\mathcal{R};z)-g(\nu_{\mathsf{R}},\mathcal{R};z)-h_{\mathcal{R}}(z),\end{aligned}$$

where $\mu_{\mathsf{R}}-\nu_{\mathsf{R}}=\widehat{\mu}^{(2)}-\nu^{(2)}$ and μ_{R}, ν_{R} are positive mutually singular measures on $p^{-1}(K_{\mathsf{max}})$, i.e., $\mu_{\mathsf{R}}-\nu_{\mathsf{R}}$ is the Riesz charge of a δ-subharmonic function $ler^{(2)}$. Clearly, $\nu_{\mathsf{R}}\leqslant\nu^{(2)}$, $\mu_{\mathsf{R}}\leqslant\widehat{\mu}^{(2)}$, and for any Borel set $B\subset p^{-1}(K_{\mathsf{max}})\backslash\mathbf{rp}(\mathcal{R})$ that lies schlicht over $\mathbb{D}$ it holds that

$$\mu_{\mathsf{R}}(B)\leqslant\mu^{(2)}(p(B)). \tag{3.79}$$

Similarly to the computation in (A.11), one can show that $\sum_{\zeta\in p^{-1}(z)}m_\zeta h(\zeta)$ is harmonic in $\mathbb{D}$ when h is harmonic on $\mathcal{R}$. Hence, it follows from the maximum principle for harmonic functions and Lemma 3.2 that

$$\sum_{\zeta\in p^{-1}(z)}m_\zeta h_{\mathcal{R}}(\zeta)=2/\mathrm{cap}_{\mathbb{D}}(K_f)+h_{\mathsf{pr}}(z),\quad h_{\mathsf{pr}}(z):=\sum_{\zeta\in p^{-1}(z)}m_\zeta h''(\zeta).$$

Thus, if we set $\mu_{\mathsf{pr}}=p_*(\mu_{\mathsf{R}})$ and $\nu_{\mathsf{pr}}=p_*(\nu_{\mathsf{R}})$, see (A.32), we get (3.78).

As explained in Lemma 3.15, $\mathcal{R}\backslash V_{\mathsf{max}}$ is its own base and $V_{\mathsf{max}}\backslash G_{\mathsf{max}}$, G_{max} are disjoint open sets. Hence, $\mathcal{R}\backslash G_{\mathsf{max}}$ is its own base by Lemma A.1. Moreover, the function $ler^{(2)}$ is equal to zero on this set by (3.72). Hence, we get from the proof of [13, Theorem 2] (that carries over *mutatis mutandis* to any hyperbolic surface) the implication:

$$\delta_z^{\mathcal{R}\backslash G_{\mathsf{max}}}(F)=0\quad\Rightarrow\quad(\mu_{\mathsf{R}}+\nu_{\mathsf{R}})(F)=0 \tag{3.80}$$

for any $F\subset\mathcal{R}\backslash G_{\mathsf{max}}$ and some (therefore any) $z\in G_{\mathsf{max}}$. In particular, it follows from Lemma A.6 that $\mathrm{supp}(\mu_{\mathsf{R}}+\nu_{\mathsf{R}})\subseteq\partial G_{\mathsf{max}}\backslash\mathcal{T}$ and that this measure does not charge polar sets. Hence, since G_{max}

lies schlicht over $D_{\max}$, it holds that $\operatorname{supp}(\mu_{\rm pr}+\nu_{\rm pr})\subseteq\partial D_{\max}\backslash\mathbb{T}$. Moreover, let $\tilde{E}$ be the set of points $z\in\partial D_{\max}$ such that $\operatorname{card}\big(p^{-1}(z)\cap(\partial G_{\max}\backslash E^*)\big)>2$, where E^* is the set of non-accessible points of $\partial G_{\max}$. It follows from Lemma 3.16 that $\tilde{E}$ is at most countable and therefore is not charged by $\mu_{\rm pr}$. Of course, the same is true of $p(\mathbf{rp}(\mathcal{R}))$, as it is a finite set. For any Borel set $B\subseteq\partial D_{\max}$, let $\tilde{B}:=B\backslash(\tilde{E}\cup p(\mathbf{rp}(\mathcal{R})))$. Then, we obtain

$$\begin{aligned}\mu_{\rm pr}(B)=\mu_{\rm pr}(\tilde{B})&=\mu_{\rm R}\big(p^{-1}(\tilde{B})\cap\partial G_{\max}\big)\\&=\mu_{\rm R}\big(p^{-1}(\tilde{B})\cap(\partial G_{\max}\backslash E^*)\big)\leqslant 2\mu^{(2)}(B),\end{aligned}$$

where the third equality follows from [13, Corollary 2], which says that the Riesz charge of a δ-subharmonic function cannot charge points that are non-accessible from the complement of the base of its zero set, and the last inequality is due to Lemma 3.16 and (3.79). □

3.10. **Computation of the logarithmic error function.** Lemma 3.17 implies that $ler_{\rm pr}(z)=0$ for $z\in K_{\max}\backslash\widetilde{N}_2$. Hence, it follows from (3.78) and properties of the Green equilibrium potential that

$$g\big(\mu_{\rm pr},\mathbb{D};z\big)-g\big(\nu_{\rm pr},\mathbb{D};z\big)-h_{\rm pr}(z)=2\frac{\operatorname{cap}_{\mathbb{D}}(K_{\max})}{\operatorname{cap}_{\mathbb{D}}(K_f)}g\big(\mu_{\mathbb{D},K_{\max}},\mathbb{D};z\big)\tag{3.81}$$

for $z\in K_{\max}\backslash\widetilde{N}_2$ (since $K_{\max}$ is equal to its own base, $g\big(\mu_{\mathbb{D},K_{\max}},\mathbb{D};\cdot\big)\equiv 1/\operatorname{cap}_{\mathbb{D}}(K_{\max})$ on $K_{\max}$). Assume that either $\nu_{\rm pr}$ is a non-zero measure or $h_{\rm pr}$ is strictly positive (harmonic) function in $\mathbb{D}$. Then, since $K_{\max}$ is separated from $\mathbb{T}$, it would hold that

$$g\big(\mu_{\rm pr},\mathbb{D};z\big)>2\frac{\operatorname{cap}_{\mathbb{D}}(K_{\max})}{\operatorname{cap}_{\mathbb{D}}(K_f)}g\big(\mu_{\mathbb{D},K_{\max}},\mathbb{D};z\big)$$

for $z\in K_{\max}\backslash\widetilde{N}_2$. Since $\operatorname{supp}(\mu_{\mathbb{D},K_{\max}})\subseteq K_{\max}$ and $K_{\max}$ is its own base, the Strong Domination Principle, see Section A.6, implies that the above inequality holds everywhere in $\mathbb{D}$. Thus, integrating both sides of this inequality against $\mu_{\mathbb{D},K_{\max}}$ which is a probability measure, we get on using Tonelli's theorem on the left-hand side and multiplying by $\operatorname{cap}_{\mathbb{D}}(K_{\max})$ that

$$\|\mu_{\rm pr}\|>2\frac{\operatorname{cap}_{\mathbb{D}}(K_{\max})}{\operatorname{cap}_{\mathbb{D}}(K_f)}.\tag{3.82}$$

On the one hand, by Lemma 3.17 and equation (3.73) together with the very construction of μ, we have that $\|\mu_{\rm pr}\|\leqslant 2\|\mu^{(2)}\|\leqslant 2\|\mu\|\leqslant 2$. On the other hand holds $\operatorname{cap}_{\mathbb{D}}(K_{\max})\geqslant\operatorname{cap}_{\mathbb{D}}(K_f)$, see (2.5). These observations clearly show that (3.82) is impossible. Hence, it is necessarily the case that

$$\nu_{\rm pr}=0\quad\text{and}\quad h_{\rm pr}(z)\equiv 0,\quad z\in\mathbb{D}.\tag{3.83}$$

Then, one can rewrite (3.81) as

$$g\big(\mu_{\rm pr},\mathbb{D};z\big)=2\frac{\operatorname{cap}_{\mathbb{D}}(K_{\max})}{\operatorname{cap}_{\mathbb{D}}(K_f)}g\big(\mu_{\mathbb{D},K_{\max}},\mathbb{D};z\big)\tag{3.84}$$

for $z\in K_{\max}\backslash\widetilde{N}_2$. Using now the Strong Domination Principle in both directions, we get that (3.84) holds for every $z\in\mathbb{D}$. Therefore,

$$\mu^{(2)}\geqslant\frac{1}{2}\mu_{\rm pr}=\frac{\operatorname{cap}_{\mathbb{D}}(K_{\max})}{\operatorname{cap}_{\mathbb{D}}(K_f)}\mu_{\mathbb{D},K_{\max}}\tag{3.85}$$

which, upon recalling once again (2.5) and the fact that $\mu_{\mathbb{D},K_{\max}}$ is a probability measure, gives us

$$\operatorname{cap}_{\mathbb{D}}(K_{\max})=\operatorname{cap}_{\mathbb{D}}(K_f)\quad\text{and}\quad\mu^{(2)}=\mu_{\mathbb{D},K_{\max}}.\tag{3.86}$$

In particular $\widetilde{N}_2=\varnothing$ and the first equality in (3.86), combined with the minimality and uniqueness of K_f, yields that $K_{\max}=K_f$. In addition, as $\|\mu^{(2)}\|=1$ by (3.86), we get from (3.73) that $\operatorname{supp}(\mu^{(1)})\cap D_{\max}=\varnothing$ and therefore $\mu^{(1)}=\mu^{(2)}$, by (A.22), (3.70), and the fact that $\mathbb{D}\backslash D_{\max}$ is its own base (because so is $\mathcal{R}\backslash G_{\max}$ and we can use Lemma A.2). Moreover, we also get from (3.73) that

$\mathrm{supp}_{\mathsf{f}}(\mu^{(1)}) \cap D_{\max} = \varnothing$ (note that $\mathrm{supp}_{\mathsf{f}}(\mu^{(1)})$ exists because $\mu^{(1)}$ is admissible, see Section A.9). Remembering that the fine open set D' is regular by (3.65), we get from (A.24) and Lemma A.6 that $\mathrm{supp}_{\mathsf{f}}(\mu^{(1)}) \supset \cup_t \partial_{\mathsf{f}} D'_t$ where D'_t are the finely connected components of D' such that $\mu(D_t) \neq 0$. However, since $K_{\max}(= K_f)$ has no fine interior by inspection of (2.17), each $\partial_{\mathsf{f}} D'_t$ must intersect $D_{\max}$ whenever D'_t is nonempty. Hence, μ cannot charge D' as otherwise it would contradict that $\mathrm{supp}_{\mathsf{f}}(\mu^{(1)}) \cap D_{\max} = \varnothing$. Consequently, since $\mathbb{D}\backslash D'$ is its own base, we conclude in view of (A.22) and (3.66) that $\mu^{(1)} = \mu$. Thus, we obtain altogether that

$$\mu_{\mathbb{D},K_f} = \mu^{(2)} = \mu^{(1)} = \mu = \mu', \tag{3.87}$$

where the last equality comes from the inequalities $\mu \leqslant \mu'$ and $\|\mu'\| \leqslant 1$, see (3.26). In addition, since $\mu_{\mathbb{D},K_f}$ has finite potential everywhere, we get that $\widetilde{N}_0 = N_0 = \varnothing$ whence also $\widetilde{N}_1 = N_1 = \varnothing$.

From (3.87) one sees that μ does not charge $D_{\max}$, implying in view of (3.43) that ler is subharmonic on $V_{\max} = p^{-1}(D_{\max})$. In particular, since $g(\mu_{\mathbb{D},K_{\max}}, \mathbb{D}; \cdot)$ extends continuously by zero on $\mathbb{T}$ and h' by $2/\mathrm{cap}_{\mathbb{D}}(K_f)$ on $\mathcal{T}$, see Lemma 3.2), while fine $\lim_{z\to\zeta} ler(z) = ler(\zeta) < 0$ when $\zeta \in J$ by the fine continuity of ler and the fact that $J \subset G_-$, it follows from the relative fine boundary maximum principle [20, Theorem 10.8] that $ler < 0$ in the annular region $\mathcal{A}(\mathcal{T}, J)$ bounded by $\mathcal{T}$ and J, since it is bounded above by the fine potential $g(\widehat{\mu}_{\mathbb{D},K_{\max}}, \mathcal{R}; \cdot)$ there, see (A.34). Let now G' be a finely connected component of G_+. Since $G^+ = p^{-1}(p(G^+))$ by Lemma 3.10 and $\mathcal{A}(\mathcal{T}, J) \subset G_-$, $p(G')$ lies in the interior of $p(J)$. Hence, $D' = p(G') \in D_J$ and by what precedes $D' \cap K_f = \varnothing$ as otherwise $\mu = \mu_{\mathbb{D},K_f}$ would charge D' because it is carried by the whole set K_f according to (A.30), i.e., it cannot be carried by $K_f\backslash D'$ (which is finely closed). Consequently, $G' \cap p^{-1}(K_f) = \varnothing$ which implies that $ler \leqslant 0$ on $p^{-1}(K_f)$, and the relative fine boundary maximum principle in turn implies that $ler \leqslant 0$ on $V_{\max}$. Moreover, $ler < 0$ on $G_{\max}$ as it is strictly negative on $\mathcal{A}(\mathcal{T}, J)$. Immediately we deduce that $G_+ = \varnothing$ and $G_{\max} \subseteq G_-$. Furthermore, since G_- lies schlicht over $\mathbb{D}$ by Lemma 3.11 while K_f has no fine interior, maximality of $G_{\max}$ implies that $G_{\max} = G_-$. Altogether, we obtain that $D' = V' = \varnothing$ and $G_- = G_{\max}$. In particular, the step of Lemma 3.13 is vacuous and $\nu^{(1)} = \nu$ as well as $ler = ler^{(1)}$. Moreover, $ler(z) = 0$ holds for $z \in \mathcal{R}\backslash G_{\max}$.

Next, by (3.85), (3.87) and the construction of the measures μ_{pr}, μ_{R} in Lemma 3.17, along with the discussion after (3.80), one has

$$2\mu_{\mathbb{D},K_f} = 2\mu^{(2)} = \mu_{\mathsf{pr}} = p_*(\mu_{\mathsf{R}}) \quad \text{and} \quad \mu_{\mathsf{R}} \leqslant (\widehat{\mu}_{\mathbb{D},K_f})_{\lfloor \partial G_{\max}}.$$

Lemma 3.16 now yields that this last inequality is in fact an equality. As $\mu_{\mathsf{R}} - \nu_{\mathsf{R}}$ is the Riesz charge of $ler^{(2)}$ that vanishes on $\mathcal{R}\backslash G_{\max}$ by (3.72), the discussion after (3.80) implies $\nu_{\mathsf{R}} = \nu^{(2)}_{\lfloor \partial G_{\max}}$, and since $\nu_{\mathsf{pr}} = p_*(\nu_{\mathsf{R}})$ we get from (3.83) and (3.70) that

$$0 = \nu^{(2)}(\overline{G}_{\max}) = \nu^{(1)}(\overline{G}_{\max}) = \nu(\overline{G}_{\max}), \tag{3.88}$$

where the middle equality holds by (A.24) and Lemma A.6 (because $\delta_z^{\mathcal{R}\backslash G_{\max}}$ is a strictly positive function of z in the regular open set $G_{\max}$), while the last equality comes from $ler = ler^{(1)}$. Recall that $\nu = \nu'' + \nu' = \nu^* + \tilde{\nu} + \nu'$, see (3.45) and (3.37), where ν^* the vague limit point of $\{\nu_n\}$ in $\mathcal{R}$, see Lemma 3.5. As $ler(z) = 0$ for $z \in \mathcal{R}\backslash G_{\max}$ and ler is a δ-subharmonic function, we get from [13, Theorem 2] that its Riesz charge is supported on $\partial G_{\max}$. In view of (3.88), it entails that

$$(\widehat{\mu}_{\mathbb{D},K_f})_{\lfloor \overline{p^{-1}(K_f)\backslash \partial G_{\max}}} = \nu = \nu'' = \nu^*, \tag{3.89}$$

where $\tilde{\nu} + \nu' = 0$ since it is a measure supported on E_f, see Lemma 3.1 and (3.35), while ν does not charge polar sets by the first equality above. Since $\nu = \nu^{(1)}$ and $\nu^{(1)}$ has no mass in $V_{\max}$ by (3.89), the step of Lemma 3.15 was also vacuous. We thus get that $ler = ler^{(2)}$, and it follows from (3.83) together with the construction of h_{pr} in Lemma 3.17 that $h_{\mathcal{R}} = h'$. Therefore, the inequality in (3.72) is in fact an equality by Lemma 3.2, and consequently

$$
ler(z) = \begin{cases} 2g(\mu_{\mathbb{D},K_f},, \mathbb{D}; p(z)) - 2/\mathrm{cap}_{\mathbb{D}}(K_f), & z \in G_{\max}, \\ 0, & z \in \mathcal{R}\backslash G_{\max}, \end{cases} \tag{3.90}
$$

as both sides of (3.90) are continuous on $\overline{\mathcal{R}}$, equal to zero on $\mathcal{R}\backslash G_{\max}$, equal to $-2/\mathrm{cap}_{\mathbb{D}}(K_f)$ on $\mathcal{T}$, and harmonic in $G_{\max}$ so that the equality in $G_{\max}$ is consequence of the maximum principle for harmonic functions.

Notice that we started our proof with the limit (3.6) taking place along the full sequence $\mathbb{N}$ of integers, that was later refined into a subsequence in Section 3.5, and refined still further in Section 3.7 to a subsequence $\mathcal{N}$ along which all the above results hold. However, we could have initiated our argument using any subsequence $\mathbb{N}_0 \subset \mathbb{N}$ in (3.6) with the same conclusions holding along some $\mathcal{N} \subseteq \mathbb{N}_0$. Hence, if there existed a subsequence along which either $\mu' \neq \mu_{\mathbb{D},K_f}$, $\nu' \neq 0$, $\tilde{\nu} \neq 0$, or ν^* are not as in the left-hand side of (3.89), or if h'' were not equal to zero, then we could use it as $\mathbb{N}_0$ in (3.6) to arrive at a contradiction. Hence, all lemmas of this section hold along the full sequence $\mathbb{N}$.

3.11. **Convergence in Capacity.** Our next task is to establish that

$$
\frac{1}{2n}\log|(f - N(M_n))(z)| \overset{\mathrm{cap}}{\to} g\big(\mu_{\mathbb{D},K_f}, \mathbb{D}; z\big) - \frac{1}{\mathrm{cap}_{\mathbb{D}}(K_f)}, \quad z \in \mathbb{D}\backslash K_f, \tag{3.91}
$$

as $n \to \infty$. Let V be an open neighborhood of $p^{-1}(K_f)$ on $\mathcal{R}$ whose closure is disjoint from K, and U be an open subset of $\mathcal{R}$ containing $K \cap G_{\max}$ whose closure is disjoint from the closure of V. In particular, $\mathcal{T} \subset \partial U$ while $\mathcal{T}$ and $\partial U\backslash\mathcal{T}$ are disjoint compact sets. For convenience, we also assume that $V = p^{-1}(p(V))$. Recall from (3.8)–(3.10) the relation

$$
\frac{1}{2n}\log|(f - N(M_n)\circ p)(z)| = \frac{1}{2}\big(h_{n,n}(z) + g(\mu_n, \mathbb{D}; p(z)) - g(\nu_n, \Omega_n; z)\big), \quad z \in \Omega_n.
$$

We have established in the previous section that the functions $h_{n,n}$ converge to $-h_{\mathcal{R}}$ locally uniformly in $\mathcal{R}\backslash E_f$, see Lemma 3.1 (notice that $\mathcal{N}'$ may now be replaced with $\mathbb{N}$ by the last remark in Section 3.10). As in Lemma 3.2, we can write

$$
h_{n,n}(z) = \int_{\partial U\backslash\mathcal{T}} h_{n,n} d\delta_z^{\mathcal{R}\backslash U} + \int_{\mathcal{T}} \frac{1}{n}\log|f - N(M_n)\circ p| d\delta_z^{\mathcal{R}\backslash U}, \quad z \in U,
$$

where $\delta_z^{\mathcal{R}\backslash U}$ is the harmonic measure of U. The above formula, the circularity of the error (3.3) and the optimality of the approximants (3.6), (3.16) together with the locally uniform convergence just mentioned show that the functions $h_{n,n}$ converge uniformly to $-h_{\mathcal{R}}$ on $\overline{U}$ and therefore on K. Since

$$
\frac{1}{2}\big(-h_{\mathcal{R}}(z) + g(\mu_{\mathbb{D},K_f}, \mathbb{D}; p(z)) - g(\nu, \mathcal{R}; z)\big) = \frac{1}{2}ler(z) = g(\mu_{\mathbb{D},K_f},, \mathbb{D}; p(z)) - \frac{1}{\mathrm{cap}_{\mathbb{D}}(K_f)}
$$

for $z \in G_{\max}$ by (3.45) and (3.90), we only need to establish that

$$
\begin{cases} \lim\limits_{n\to\infty} \mathrm{cap}\big(\{z \in K : |g(\mu_n, \mathbb{D}; p(z)) - g(\mu_{\mathbb{D},K_f}, \mathbb{D}; p(z))| > a\}\big) = 0, \\ \lim\limits_{n\to\infty} \mathrm{cap}\big(\{z \in K : |g(\nu_n, \Omega_n; z) - g(\nu, \mathcal{R}; z)| > a\}\big) = 0, \end{cases} \tag{3.92}
$$

for any $a > 0$, where $\mathrm{cap}(\cdot)$ is the logarithmic capacity, see (A.6). This will simultaneously prove the claim made after Theorem 2.1 as well as (3.91) because $t \mapsto g_D(t,z) + \log|z - t|$ is bounded uniformly for $z \in F$ for any compact $F \subset \mathbb{D}$, so that (3.93) yields an analogous claim for the Greenian capacity on any compact subset of $G_{\max}$.

Write $\mu_n = \mu_{n,1} + \mu_{n,2}$, where $\mu_{n,1} := \mu_{n\lfloor p(V)}$. Notice that $\mu_{n,1} \overset{*}{\to} \mu_{\mathbb{D},K_f}$. Since these measures have at most unit mass, the differences $g(\mu_{n,1}, \mathbb{D}; p(\cdot)) - g(\mu_{\mathbb{D},K_f}, \mathbb{D}; p(\cdot))$ converge uniformly to zero on $\partial U\backslash\mathcal{T}$. As they are identically equal to zero on $\mathcal{T}$, they converge to zero uniformly on U and hence on K by the maximum principle for harmonic functions. Recall that any Green potential of a measure supported in the unit disk can be written as a difference of the logarithmic potentials of the measure and of its balayage onto $\mathbb{T}$, see (A.29). It was shown in [6, Lemma 21] that if compactly supported measures converge weak* to the zero measure, then their logarithmic potentials converge to zero in capacity in $\mathbb{C}$. Hence, the potentials $g(\mu_{n,2},\mathbb{D};\cdot)$ converge to zero in capacity on K, which finishes the proof of the first limit in (3.93).

The proof of the second limit in (3.93) proceeds similarly, but requires more detailed analysis. As we have shown in the previous subsection, the measures $\nu_{n\lfloor\Omega_n}$ converge vaguely to ν^* on $\mathcal{R}$, where ν^* stands for the left-hand side of (3.89). Hence,

$$\nu_{n,1} := \nu_{n\lfloor\overline{V}\cap\Omega_n} \overset{w*}{\to} \nu^* \tag{3.93}$$

on $\mathcal{R}$. The functions $g_{\mathcal{R}}(z,w)-g_{\Omega_n}(z,w)$ uniformly converge to zero for $w\in V$ and $z\in K$, see (3.55) (we continue $g_{\Omega_n}(z,w)$ by zero to $\mathcal{R}\backslash\Omega_n$). Therefore, as the measures $\nu_{n,1}$ have uniformly bounded masses (they converge weak* to a finite measure), we get that

$$\lim_{n\to\infty}\big(g(\nu_{n,1},\mathcal{R};z)-g(\nu_{n,1},\Omega_n;z)\big)=0 \tag{3.94}$$

uniformly on K. Moreover, since the potentials $g(\nu_{n,1},\mathcal{R};\cdot)$ are positive harmonic functions on U for all n large, they converge uniformly to $g(\nu^*,\mathcal{R};\cdot)$ on K (they converge pointwise on $\overline{U}$ by (3.94), then uniformly on $\partial U\backslash\mathcal{T}$ by Harnack's theorem applied in a neighborhood of $\partial U\backslash\mathcal{T}$, and thus uniformly on $\overline{U}$ by the maximum principle for they are identically zero on $\mathcal{T}$). That is, we get from (3.95) that

$$\lim_{n\to\infty} g(\nu_{n,1},\Omega_n;z)=g(\nu^*,\mathcal{R},z) \tag{3.95}$$

uniformly on K. Next, let $\nu_{n,2}:=\nu_{n\lfloor G_{\max}\backslash\overline{V}}$. As the measures ν_n converge vaguely to ν^*, the measures $\nu_{n,2}$ converge vaguely to the zero measure. *We claim* that these measures have uniformly bounded masses with respect to n. To see this, pick $\varepsilon>0$ and let $0<\eta<\delta$ be such that

$$h_{n,n}(z)+g(\mu_{\mathbb{D},K_f},\mathbb{D};p(z))<-\varepsilon,\quad z\in R_\eta:=\{z\in G_{\mathsf{max}}:1-\eta\leqslant|p(z)|\leqslant 1\}$$

for all n large enough. This is possible since $g(\mu_{\mathbb{D},K_f},\mathbb{D};p(\cdot))$ is a continuous function in $\overline{U}$ and is equal to 0 on $\mathcal{T}$, while $h_{n,n}$ converge uniformly to $h_{\mathcal{R}}$, which is equal to $-2/\mathrm{cap}_{\mathbb{D}}(K_f)$ on $\mathcal{T}$. We may assume that f does not vanish in R_η, for we may add a constant to it while adding the same constant to the approximants M_n. If each circle $\mathbb{T}_r$ for $1-2\eta/3<r<1-\eta/3$ contained a z with $|g(\mu_n,\mathbb{D},z)-g(\mu_{\mathbb{D},K_f},\mathbb{D},z)|\geqslant\varepsilon/2$ for infinitely many n, then it would imply that

$$\mathrm{cap}\big(\{z\in R_r:|g(\mu_n,\mathbb{D},z)-g(\mu_{\mathbb{D},K_f},\mathbb{D},z)|>\varepsilon/2\}\big)\geqslant\eta/12>0,$$

because contractive maps, in particular, circular projection, do not increase the logarithmic capacity [52, Theorem 5.3.1] and (remember that $\mathrm{cap}([a,b])=|b-a|/4$). Clearly, this would contradict the first limit in (3.93). Hence, for each n large enough, there is $r_n\in[1-2\eta/3,1-\eta/3]$ such that

$$g(\mu_n,\mathbb{D},p(z))+h_{n,n}(z)-g(\nu_n,\mathcal{R},z)<g(\mu_n,\mathbb{D},p(z))+h_{n,n}(z)<-\varepsilon/2$$

for $z\in p^{-1}(\mathbb{T}_{r_n})\cap R_\eta$. Thus, since f does not vanish in R_η, we get that $|f-N(M_n)|<|f|$ on $\mathbb{T}_{r_n}$ for all n large enough. Besides, the same inequality certainly holds on $\mathbb{T}$ and therefore, by Rouché's theorem, the number of zeros of $f-N(M_n)$ in R_η is less that the number of its poles (at most n) plus the degree of f on $\mathbb{T}\cup\mathbb{T}_{r_n}$ (which is bounded by a constant independently of n). Hence ν_n has bounded mass in R_η, *which proves the claim.* Thus, the measures $\nu_{n,2}$ and respectively the measures $p_*(\nu_{n,2})$ strongly converge to zero. As above, this means that the potentials $g(p_*(\nu_{n,2}),\mathbb{D};p(\cdot))$ and, by (A.32), the potentials $g(\nu_{n,2},\mathcal{R};\cdot)$ converge to zero in capacity on K. As the latter potentials majorize $g(\nu_{n,2},\Omega_n;\cdot)$, we get that the potentials $g(\nu_{n,2},\Omega_n;\cdot)$ converge to zero in capacity on K. Finally, define $\nu_{n,3}:=\nu_n-\nu_{n,1}-\nu_{n,2}$. The potentials $g(\nu_{n,3},\Omega_n;\cdot)$ form a sequence of positive harmonic functions in $G_{\mathsf{max}}\backslash\overline{V}$. By Harnack's theorem there exists a subsequence of indices, say $\mathcal{N}_0$, along which these potentials converge locally uniformly on U to some non-negative harmonic function, say h_0. Now, we can initiate the proof Theorem 2.4 in Section 3.5 with $\mathcal{N}_0$ instead of the whole sequence $\mathbb{N}$. Then it would follow from (3.89) and (3.96) that the function h'' from Lemma 3.5 must coincide with h_0 in U. Proceeding with the remainder of the proof we again inevitably arrive at the conclusion that $h'' \equiv 0$. Hence, the potentials $g(\nu_{n,3}, \Omega_n; \cdot)$ converge to 0 locally uniformly on $G_{\mathsf{max}} \backslash \overline{V}$, and, in particular, on $\partial U \backslash \mathcal{T}$. As these potential are identically zero on $\mathcal{T}$, they converge to zero uniformly on U and hence on K by the maximum principle for harmonic functions. This finishes the proof of the second limit in (3.93) and, respectively, of Theorem 2.4.

4. Proof of Theorem 2.5

Since the considerations of Sections 3.2 and 3.3 still apply, we can assume that $D = \mathbb{D}$, i.e., that f is analytic in $\overline{\mathbb{C}} \backslash E$, where $E \subset \mathbb{D}$ is closed polar, and we may replace $\{M_n\}$ by the sequence $\{N(M_n)\}$ of its Nehari modifications. Write $N(M_n) = h_n / b_n$, where $h_n \in \mathcal{A}(\mathbb{D})$ and b_n is a Blaschke product vanishing at the poles of $N(M_n)$ according to their multiplicities. That is, $b_n = q_n / \widetilde{q}_n$, where $q_n \in \mathcal{M}_n(\mathbb{D})$ is a polynomial such that $q_n M_n \in \mathcal{A}(\mathbb{D})$ and $\widetilde{q}_n(z) := z^n \overline{q_n(1/\bar{z})}$ is the reciprocal polynomial. We have that

$$|f(z) - (h_n/b_n)(z)| = |(b_n f)(z) - h_n(z)|, \quad z \in \mathbb{T},$$

since Blaschke products are unimodular on the unit circle. Thus, h_n is in fact the best Nehari approximant of $b_n f$ in $\mathcal{A}(\mathbb{D})$. Recall that the above expressions converge to zero faster than geometrically to zero by the very choice of $\{M_n\}$.

Our goal is to show that h_n/b_n converge in logarithmic capacity to f in $\overline{\mathbb{D}} \backslash E$ at faster than geometric rate. That is, we fix a compact set $K \subset \overline{\mathbb{D}}$ disjoint from E and we will prove that

$$\lim_{n \to \infty} \operatorname{cap}\big(\{z \in K : |e_n(z)| > a^n\}\big) = 0 \quad \text{for any} \quad a > 0, \tag{4.1}$$

where $e_n(z) := f(z) - (h_n/b_n)(z)$ is the approximation error. This will establish convergence in logarithmic capacity on compact subsets of $\overline{\mathbb{D}} \backslash E$. Subsequently, as we pointed out in Section 3.11, (4.1) yields an analogous claim for the Greenian capacity on any compact subset of $\mathbb{D} \backslash E$.

Since $\operatorname{cap}(E) = 0$, it follows from [52, Theorems 5.5.2 & 5.5.4] that for each $\eta > 0$ there is $k \in \mathbb{N}$ and $p_k \in \mathcal{M}_k(E)$ (we can take p_k to be the k-th Fekete polynomial for E) such that

$$E \subset L_\eta := \big\{\zeta \in \mathbb{C} : |p_k(\zeta)| < \eta^k\big\}. \tag{4.2}$$

Pick $\eta < \min\{\operatorname{dist}(K, E), 1\}$ to be adjusted later. Of necessity $K \cap L_\eta = \varnothing$, because $|p_k(z)| \geqslant \operatorname{dist}(K, E)^k$ for $z \in K$. Let $\gamma \subset L_\eta$ be a system of closed curves encompassing each point of E exactly once, and such that $\operatorname{dist}(\gamma, E) < \operatorname{dist}(\mathbb{T}, E)/4$. Then, one has that

$$f(z) = \int_\gamma \frac{f(\xi)}{z - \xi} \frac{d\xi}{2\pi \mathrm{i}}, \quad z \in \overline{\mathbb{C}} \backslash \overline{\operatorname{int} \gamma}, \tag{4.3}$$

by the Cauchy formula, where $\operatorname{int} \gamma$ is the union of the bounded components of the complement of γ. Since $\mathbb{P}_-$ evaluated at $|z| > 1$ coincides with the Cauchy projection having kernel $(2\pi \mathrm{i}(z - \zeta))^{-1} d\zeta$ on $\mathbb{T}$, the Hankel operator $\Gamma_{b_n f}$ acts on $v \in H^2$ by

$$\Gamma_{b_n f}(v)(z) = \int_{\mathbb{T}} \frac{b_n(\zeta) f(\zeta) v(\zeta)}{z - \zeta} \frac{d\zeta}{2\pi \mathrm{i}}, \quad |z| > 1. \tag{4.4}$$

Inserting (4.3) into (4.4) yields by Fubini's theorem and the Cauchy formula for H^2-functions that

$$\Gamma_{b_n f}(v)(z) = \int_\gamma \frac{b_n(\xi) f(\xi) v(\xi)}{z - \xi} \frac{d\xi}{2\pi \mathrm{i}}, \quad z \in \overline{\mathbb{C}} \backslash \overline{\operatorname{int} \gamma}. \tag{4.5}$$

Observe that (4.5), initially proven for $|z| > 1$, actually defines an analytic extension to the exterior of γ of $\Gamma_{b_n f}(v) = \mathbb{P}_-(b_n f v) \in H^2_-$ (and therefore an extension to $\overline{\mathbb{C}} \backslash E$ since γ could be taken arbitrary close to E). Next, recall that a first singular vector of the Hankel operator $\Gamma_{b_n f}$ is an element $v_0 \in H^2$ of unit norm that maximizes $\|\Gamma_{b_n f}(v)\|$ over all $v \in H^2$ with $\|v\|_2 = 1$, and that it always can be chosen to be outer[3], see for instance [5, p. 62]. Then, we get from (4.5) and (2.26) (applied with $n = 0$ and f replaced by fb_n) that

$$(e_n b_n)(z) = (b_n f - h_n)(z) = \frac{1}{v_0(z)} \int_\gamma \frac{b_n(\xi) f(\xi) v_0(\xi)}{z - \xi} \frac{d\xi}{2\pi \mathrm{i}}, \quad z \in \mathbb{D} \setminus \overline{\mathrm{int}\,\gamma}. \tag{4.6}$$

Note that the right-hand side of (4.6) is analytic in $\mathbb{D} \setminus \overline{\mathrm{int}\,\gamma}$, since v_0 is outer and thus has no zeros in $\mathbb{D}$. Let $B_k := p_k / \widetilde{p}_k$. Similarly to (4.3)–(4.6), it holds that

$$\mathbb{P}_- \left(e_n b_n B_k^\ell \right)(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{b_n(\xi) f(\xi) B_k^\ell(\xi)}{z - \xi} d\xi, \quad z \in \overline{\mathbb{D}} \setminus \overline{\mathrm{int}\,\gamma}, \tag{4.7}$$

where $\ell \in \mathbb{N}$ is such that $\ell k \leqslant n < (\ell + 1)k$ and the right-hand side again defines an analytic extension of the left-hand side into the exterior of γ. Indeed, we can express the left-hand side of (4.7) for $|z| > 1$ as the Cauchy integral of $e_n b_n B_k^\ell$ on $\mathbb{T}$ like we did in (4.4) for $\mathbb{P}_-(b_n f v)$. Since $e_n b_n$ is analytic across $\mathbb{T}$ and $B_k \in \mathcal{H}(\overline{\mathbb{D}})$, we then deform the contour of integration into a circle of radius slightly smaller than 1, which can be done without changing the value of the integral by Cauchy's theorem. Subsequently, we insert (4.6) in this integral and use Fubini's theorem and the residue formula as before to get (4.7).

Recall that $\mathrm{dist}(\mathbb{T}, E) \geqslant \mathrm{dist}(K, E) > \eta$ by construction. Observe also that $|p_k| \leqslant \eta^k$ on γ by (4.2) and that $|\widetilde{p}_k| \geqslant \mathrm{dist}(\mathbb{T}, E)^k$ in $\mathbb{D}$. Since $|b_n| \leqslant 1$, we get from our choice of ℓ that

$$\left| \mathbb{P}_- \left(e_n b_n B_k^\ell \right)(z) \right| \leqslant \left(\frac{\eta}{\mathrm{dist}(\mathbb{T}, E)} \right)^{n-k} \frac{|\gamma| \|f\|_\gamma}{\mathrm{dist}(z, \gamma)}, \quad z \in \overline{\mathbb{D}} \setminus \overline{\mathrm{int}\,\gamma}, \tag{4.8}$$

where $|\gamma|$ stands for the arclength of γ. In another connection, it follows from [48, Lemma] that for any $\varepsilon \in (0, 1/3)$ there exists $W_n \subset \mathbb{D}$ such that $\mathrm{cap}(W_n) \leqslant 3\varepsilon$ and

$$|p_k^\ell(\zeta) q_n(\zeta)| > \varepsilon^{n + \ell k} \|p_k^\ell q_n\|_{\mathbb{T}}, \quad \zeta \in \mathbb{D} \setminus W_n.$$

As $|\widetilde{p}_k^\ell \widetilde{q}_n|(z) \leqslant \|\widetilde{p}_k^\ell \widetilde{q}_n\|_{\mathbb{T}} = \|p_k^\ell q_n\|_{\mathbb{T}}$ for $z \in \mathbb{D}$ by the maximum principle and the definition of the reciprocal polynomial, we get that $|(b_n B_k^\ell)(z)| \geqslant \varepsilon^{n + k\ell}$ for $z \in \mathbb{D} \setminus W_n$. Since $\mathbb{P}_+ + \mathbb{P}_-$ is the identity operator, we can use the analytic continuation provided by (4.7) to write

$$e_n(z) = \frac{\mathbb{P}_+(e_n b_n B_k^\ell)(z) + \mathbb{P}_-(e_n b_n B_k^\ell)(z)}{b_n(z) B_k^\ell(z)}, \quad z \in \overline{\mathbb{D}} \setminus \overline{\mathrm{int}\,\gamma}. \tag{4.9}$$

The estimate $|(b_n B_k^\ell)(z)| \geqslant \varepsilon^{n + k\ell}$ for $z \in \mathbb{D} \setminus W_n$, (4.8), and the definition of ℓ give us that

$$\left| \frac{\mathbb{P}_-(e_n b_n B_k^\ell)(z)}{b_n(z) B_k^\ell(z)} \right| \leqslant \frac{1}{\varepsilon^{2n}} \left(\frac{\eta}{\mathrm{dist}(\mathbb{T}, E)} \right)^{n-k} \frac{|\gamma| \|f\|_\gamma}{\mathrm{dist}(z, \gamma)}, \quad z \in \overline{\mathbb{D}} \setminus (\overline{\mathrm{int}\,\gamma} \cup W_n). \tag{4.10}$$

Now, given $0 < \varepsilon < 1$ and $0 < a < 1$, choose η in (4.2) so that $0 < \eta < a\varepsilon^2 \mathrm{dist}(\mathbb{T}, E)$. Choice of η of course fixes k in (4.2). Then, since K lies exterior to γ because $K \cap L_\eta = \varnothing$, we get from (4.10) that there exists a natural number $n_0 = n_0(f, K, \varepsilon, \eta, \gamma)$ for which

$$\left| \frac{\mathbb{P}_-(e_n b_n B_k^\ell)(z)}{b_n(z) B_k^\ell(z)} \right| < a^n, \quad n \geqslant n_0, \quad z \in K \setminus W_n. \tag{4.11}$$

Next, as $\|e_n b_n B_k^\ell\|_{\mathbb{T}} = \|e_n\|_{\mathbb{T}} \to 0$ faster than geometrically with n by hypothesis, (4.8) and the triangle inequality yield that

$$\left\| \mathbb{P}_+(e_n b_n B_k^\ell) \right\|_{\mathbb{T}} = \left\| e_n b_n B_k^\ell - \mathbb{P}_-(e_n b_n B_k^\ell) \right\|_{\mathbb{T}} \leqslant C \left(\frac{\eta}{\mathrm{dist}(\mathbb{T}, E)} \right)^n, \quad n \geqslant n_0', \tag{4.12}$$

[3]An outer function $w \in H^2$ is of the form $w(z) = \alpha \exp\left\{ \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} \log |w(\xi)| \frac{|d\xi|}{2\pi} \right\}$, with $w_{|\mathbb{T}} \in L^2(\mathbb{T})$ and $|\alpha| = 1$.

for some constant $C = C(f, E, \gamma)$ and some n_0' depending on C and the speed of approximation of f by h_n/b_n. Subsequently, as in (4.11), we get from the estimate $|(b_nB_k^\ell)(z)| \geqslant \varepsilon^{n+k\ell}$ for $z \in \mathbb{D}\setminus W_n$, (4.12), and the maximum modulus principle that

$$\left|\frac{\mathbb{P}_+(e_nb_nB_k^\ell)(z)}{b_n(z)B_k^\ell(z)}\right| < a^n, \quad n \geqslant n_0'', \quad z \in K\setminus W_n, \tag{4.13}$$

for some natural number $n_0'' = n_0''(f, K, \varepsilon, \eta, \gamma)$. Because ε and a can be arbitrarily small and $\mathrm{cap}(W_n) \leqslant 3\varepsilon$, (4.1) now follows from (4.9), (4.11) and (4.13).

Having proven that $M_n \overset{\mathrm{cap}}{\to} f$ in $D\setminus E$ at faster than geometric rate whenever it is a sequence of n-th root optimal meromorphic approximants to f, we turn to the construction of rational functions $R_{k_n} \in \mathcal{R}_{k_n}(D)$ such that the poles of R_{k_n} are among the poles of M_n lying in V and (2.18) holds, where V any open set such that $E \subset V \subset \overline{V} \subset D$. Let $B \supset E$ be a closed set contained in V which is regular for the Dirichlet problem, see Section A.7. Such a B is easily constructed as a sublevel set, for some small regular value, of a smooth non-negative function whose zero set is E, see, for example, discussion after (3.7). Then the Green equilibrium potential $G(z) := g(\mu_{D,B}, D; z)$ is harmonic in $D\setminus B$, continuous on D, strictly less than the constant $1/\mathrm{cap}_D(B)$ on $D\setminus B$ and equal to that constant on B, see (2.12) or Section A.4 for a more detailed discussion. Since $\partial_z G(z)$ is holomorphic in $D\setminus B$, the critical points of G are isolated and cannot accumulate in $D\setminus B$, so we can find an interval $[t_1, t_2] \subset (0, 1/\mathrm{cap}_D(B))$ that is free of critical values and such that $G^{-1}([t_1, t_2]) \subset V$. For any $t \in [t_1, t_2]$, $\gamma(t) := G^{-1}(t)$ is a 1-dimensional compact manifold, i.e., a finite union of disjoint real analytic closed curves $\gamma_{1,t}, \dots, \gamma_{N,t}$, none of which lies interior to another (by the maximum principle), and such that $B \subset \mathrm{int}\,\gamma(t) \subset \overline{\mathrm{int}\,\gamma(t)} \subset V$. Note that N is independent of $t \in [t_1, t_2]$ since any such t is a regular value; note also that the total length $|\gamma(t)| = \sum_{j=1}^N |\gamma_{j,t}|$ is bounded above independently of t, say by a constant L, because the gradient ∇G is normal to $\gamma_{j,t}$ at its every point and therefore the divergence formula implies for any $t \in [t_1, t_2]$:

$$t|\gamma(t)| \min_{z \in G^{-1}(t)} \|\nabla G(z)\| \leqslant t_2|\gamma(t_2)| \max_{z \in G^{-1}(t_2)} \|\nabla G(z)\| - \int_{G^{-1}([t,t_2])} \|\nabla G\|^2 dxdy.$$

Pick $a > 0$, set $K := \max\{\|\nabla G(z)\| : z \in G^{-1}([t_1, t_2])\}$ and let $n_a \in \mathbb{N}$ be so large that

$$\mathrm{cap}\big(\{z \in G^{-1}([t_1, t_2]) : |f(z) - M_n(z)| > a^n\}\big) < \frac{t_2 - t_1}{4K}, \quad n \geqslant n_a. \tag{4.14}$$

Such a n_a exists by the first part of the proof. Let A_n be the set whose capacity is estimated in (4.14). Assume for the moment that for each $t \in [t_1, t_2]$ there exists $z \in G^{-1}(t) \cap A_n$. Then the image of A_n under $G/K : G^{-1}([t_1, t_2]) \to \mathbb{R}$ is equal to the interval $[t_1/K, t_2/K]$ whose capacity is $(t_2 - t_1)/(4K)$. However, since contractive maps do not increase the logarithmic capacity [52, Theorem 5.3.1], the capacity of $G(A_n)/K$ should be strictly smaller than $(t_2 - t_1)/(4K)$ by (4.14). Hence, for each $n \geqslant n_a$ there is $t_n \in [t_1, t_2]$ for which

$$\left|\int_{\gamma(t_n)} \frac{f(\xi) - M_n(\xi)}{z - \xi}\frac{d\xi}{2\pi\mathrm{i}}\right| \leqslant \frac{La^n}{2\pi\mathrm{dist}(T, \overline{V})}, \quad z \in T. \tag{4.15}$$

Pick a positive sequence $\{a_k\}$ converging to 0 and, without loss of generality, arrange things so that $n_{a_k} < n_{a_{k+1}}$. Define

$$J_n(z) := \int_{\gamma(t_{n_{a_k}})} \frac{M_n(\xi)}{z - \xi}\frac{d\xi}{2\pi\mathrm{i}}, \quad z \in D\setminus\overline{V}, \quad n_{a_k} \leqslant n < n_{a_{k+1}}.$$

Clearly, J_n is a rational function retaining the singular part of M_n inside the system of arcs $\gamma(t_{n_{a_k}})$, and it is of type $(k_n - 1, k_n)$ where $k_n \leqslant n$ is the number of the poles of M_n inside this system of arcs, counting multiplicities. If we put $R_{k_n} := J_n$, then since $a_k \to 0$ we get from (4.15) and the Cauchy formula that (2.18) holds, as desired.

Appendix A. Potential Theory on a Riemann Surface

Even though the proof of Theorem 2.4 in Sections 3.3–3.10 was carried out for $D = \mathbb{D}$, this appendix is written for a general Jordan domain D since specializing D to the unit disk would only shorten the proofs of Lemmas A.5 and A.10 by a couple of paragraphs but otherwise would not lead to any further simplifications.

A.1. **Subharmonic Functions.** Let d be the differential and $*$ the conjugation operators on a connected Riemann surface. The Laplacian $\Delta := d^*d$ takes smooth functions to 2-forms. If U is an open subset of the surface and $u : U \to \mathbb{R}$ a locally integrable function (against the area-form $*\mathbf{1}$, where $\mathbf{1}$ is the constant unit function, or equivalently against the Lebesgue measure $(\mathrm{i}/2)dz \wedge d\bar{z}$ in any system of local coordinates $z, \bar{z}$), the distributional Laplacian Δu is the 0-current acting on a smooth compactly supported functions φ on U by $\int u\Delta\varphi$. When $\Delta u = 0$, one says that u is harmonic on U, and such functions are in fact smooth (even real analytic) by Weyl's lemma [18, Theorem 24.9]. Subharmonic functions on U are defined as upper-semicontinuous functions $u : U \to [-\infty, \infty)$ such that, if V is open in U and $h : V \to \mathbb{R}$ is harmonic, then $u - h$ is either constant or fails to have a maximum in V. On open subsets of $\mathbb{C}$, this definition coincides with the usual one; see [52, Definition 2.2.1 & Theorem 2.4.1]. A superharmonic function is the negative of a subharmonic function. A difference of two subharmonic functions is sometimes called a δ-subharmonic function.

Harmonicity and subharmonicity are local properties: u is harmonic (resp. subharmonic) on U if and only if its restriction to every open subset is, or equivalently if and only if $u \circ \varphi^{-1}$ is harmonic (resp. subharmonic) on the open set $\varphi(V \cap U) \subset \mathbb{C}$ whenever (V, φ) is a local chart. Thus, standard facts regarding such functions on open subsets of a Riemann surface follow from their planar counterparts, using charts. In particular, the *integrability theorem* [52, Theorem 2.5.1] states that a subharmonic function which is not identically $-\infty$ is locally integrable, and therefore it has a distributional Laplacian. Hence, two subharmonic functions that coincide almost everywhere (with respect to area measure) are in fact equal, for either they are both identically $-\infty$ or they have the same distributional Laplacian, and so their difference is harmonic; this is the *weak identity principle*. The following is a variant of Harnack's theorem [52, Theorem 1.3.10] and of [52, Theorem 2.4.6].

Harnack's Theorem. *A sequence of harmonic functions on U that is bounded below has a subsequence that converges locally uniformly on U, either to $+\infty$ or to a harmonic function. For an increasing sequence, convergence holds along the full sequence. A decreasing sequence of subharmonic functions converges pointwise to a subharmonic function.*

A locally integrable function u is subharmonic if and only if Δu is a Radon measure on the surface; that is, Δu is a positive linear form on continuous functions with compact support. Indeed, as this statement is local, it reduces to its planar analog. The "only if" part follows from [52, Section 3.7]. As to the "if" part, let $W \subset \mathbb{C}$ be open and ν be a finite positive Borel measure carried by W. The logarithmic potential of ν, i.e., $V^\nu(z) := \int \log|z - t|^{-1} d\nu(t)$, is superharmonic on $\mathbb{C}$ with distributional Laplacian $-\nu$, so if u is locally integrable on W with $\Delta u = \nu$ there, then $h := u + V^\nu_{\lfloor W}$ is harmonic and therefore $u = -V^\nu_{\lfloor W} + h$ is subharmonic in W (as in the main text, for a set E (that may require further qualification), a subscript $\lfloor E$ indicates "restriction to E").

A.2. **Green Functions.** Throughout, Ω will be a subdomain of some ambient algebraic Riemann surface $\mathcal{R}_*$ such that $p(\Omega)$ is a bounded domain in $\mathbb{C}$, where p stands for the canonical projection; in particular the results apply to $\Omega = \mathcal{R}$ with $p(\Omega) = D$, see beginning of Section 2.1. Since the lift to Ω of a positive non-constant superharmonic function on $p(\Omega)$ is again positive, non-constant, and superharmonic, Ω is hyperbolic and as such possesses Green functions [15, Theorem IV.3.7]. Notice also that there exists a subdomain $\Omega' \subset \mathcal{R}_*$ with $p(\Omega')$ bounded such that $\overline{\Omega} \subset \Omega'$, where an overline (as in $\overline{\Omega}$) always denotes the closure in $\mathcal{R}_*$.

Recall that the Green function for Ω with pole at w, denoted by $g_\Omega(\cdot, w)$, is the unique function that is harmonic and positive in $\Omega\setminus\{w\}$ with a logarithmic singularity at w and whose largest harmonic minorant is identically zero. By a logarithmic singularity at w, it is meant that if (V, φ) is a coordinate chart on Ω such that $w \in V$ and $\varphi(w) = 0$, then $g_\Omega(\varphi^{-1}(\cdot), w) + \log|\cdot|$ is harmonic on $\varphi(V)$. Obviously $g_\Omega(\cdot, w) > 0$ everywhere on Ω, by the minimum principle for harmonic functions. Recall also that if a superharmonic function on Ω is not identically $+\infty$ and has a harmonic minorant, then it has the largest one whose construction can be carried out as in the Euclidean case [29, Theorem 4.3.5], because Poisson modifications can be performed locally.

Clearly, $g_\Omega(\cdot, w)$ is superharmonic and $\Delta g_\Omega(\cdot, w) = -\delta_w$, where δ_w is the Dirac mass at w. Moreover, g_Ω is symmetric in that $g_\Omega(z, w) = g_\Omega(w, z)$ [15, Theorem IV.3.10]. Symmetry entails that $g_\Omega(z, w)$ is separately harmonic in z and w, and therefore jointly harmonic on $\{(z, w) \in \Omega\times\Omega : z \neq w\}$ [36, p. 561]; in particular, $(z, w) \mapsto g_\Omega(z, w)$ is continuous off the diagonal. Note that

$$g_\Omega(z, w) \leqslant g_{\Omega'}(z, w), \quad z, w \in \Omega \subseteq \Omega', \tag{A.1}$$

because $g_\Omega(\cdot, w) - g_{\Omega'}(\cdot, w)$ is a harmonic minorant of $g_\Omega(\cdot, w)$ and therefore must be non-positive. Thus, if F_1, F_2 are relatively closed subsets of Ω with $\overline{F}_1 \cap \overline{F}_2 = \varnothing$, we deduce that $(z, w) \mapsto g_\Omega(z, w)$ is bounded on $F_1 \times F_2$ because $g_{\Omega'}(z, w)$ is continuous on the compact set $\overline{F}_1 \times \overline{F}_2$ whenever $\overline{\Omega} \subset \Omega'$. We also remark that to each $w \in \Omega$ there is an open set $V \ni w$ and a constant $C = C(V)$ such that

$$\int_V g_\Omega(z, w')\, {}^*\mathbf{1}(z) < C, \qquad w' \in V, \tag{A.2}$$

a result that follows by uniformization from the corresponding fact on the disk [29, Theorem 4.4.12]. Moreover, if a sequence of open sets Ω_n increases to Ω as $n \to \infty$, it follows from (A.1) that $g_\Omega(\cdot, w)_{\lfloor\Omega_n} - g_{\Omega_n}(\cdot, w)$ is a decreasing sequence of positive harmonic functions that must converge locally uniformly in Ω, by Harnack's theorem; as the limit is necessarily a non-negative harmonic minorant of $g_\Omega(\cdot, w)$, it must be identically zero.

A.3. **Green Potentials.** A Green potential in Ω is a non-negative superharmonic function whose largest harmonic minorant is identically zero. Given σ, a Radon measure in Ω, let us put

$$g(\sigma, \Omega; z) := \int g_\Omega(z, w)\, d\sigma(w). \tag{A.3}$$

This is a superharmonic function of $z \in \Omega$ which is either identically $+\infty$, or locally integrable with distributional Laplacian $-\sigma$ by Fubini's theorem. If $g(\sigma, \Omega; \cdot)$ is not identically $+\infty$, using the monotone convergence and Fubini's theorem, the proof of [29, Lemma 4.3.6] carries over to integrals instead of sums to show that the largest harmonic minorant of $g(\sigma, \Omega; \cdot)$ is the integral against $d\sigma(w)$ of the largest harmonic minorants of the $g_\Omega(\cdot, w)$, namely zero. Thus, $g(\sigma, \Omega; \cdot)$ is a Green potential. Conversely, it follows from the Riesz representation theorem stated below that every Green potential has the form (A.3). Notice that if $\sigma(\Omega) < \infty$, then $g(\sigma, \Omega; \cdot) \not\equiv +\infty$, for if $V \subset \Omega$ is as in (A.2) and W is a nonempty open set with $\overline{W} \subset V$, then

$$\int_W g(\sigma, \Omega; z)^*\mathbf{1}(z) < C\sigma(V) + C'\sigma(\Omega\setminus V) \int_W {}^*\mathbf{1}$$

by Fubini's theorem, where C' is an upper bound for $g_\Omega(z, w)$ on $(\Omega\setminus V) \times \overline{W}$.

Riesz Representation Theorem. *Let $u \not\equiv +\infty$ be a superharmonic function on Ω that has a harmonic minorant. Then $u = g(\sigma, \Omega; \cdot) + h$, where h is the largest harmonic minorant of u and $\sigma := -\Delta u$.*

Proof. Assume first that $\sigma(\Omega) < \infty$. Then, $g(\sigma, \Omega; \cdot) \not\equiv +\infty$ and therefore $h := u - g(\sigma, \Omega; \cdot)$ is harmonic on Ω. Clearly, h is a minorant of u. Since the largest harmonic minorant of $g(\sigma, \Omega; \cdot)$ is zero, h is the largest harmonic minorant of u. If $\sigma(\Omega) = \infty$, pick Ω_m to be an increasing exhaustion of Ω by relatively compact open sets. Put $\sigma_m := \sigma_{\lfloor \Omega_m}$, which are finite measures on Ω_m because σ is a Radon measure. By what precedes, $u_{\lfloor \Omega_m} = g(\sigma_m, \Omega_m; \cdot) + h_m$, where h_m is the largest harmonic minorant of $u_{\lfloor \Omega_m}$. As the functions $g_{\Omega_m}(\cdot, w)$ increase locally uniformly to $g_\Omega(\cdot, w)$ while h_m decrease and are bounded below by any harmonic minorant of u, we get by monotone convergence and Harnack's theorem that $u = g(\sigma, \Omega; \cdot) + h$, where h is harmonic and necessarily $g(\sigma, \Omega; \cdot) \not\equiv +\infty$. We now conclude the proof as in the first case. □

This version of the Riesz representation theorem featuring the weak-Laplacian may be compared to the more abstract formulation for Green spaces (of which Ω is a special case) in [8, Section VI.7], that does not refer to the Laplacian; see also the planar statement of [52, Theorem 4.5.4].

The previous considerations allow us to simplify in our case the notion of admissibility of a measure given in [19, Section 1] and [20, Section I.3]. According to that definition, a measure σ is *admissible* if it is integrable against continuous Green potentials with compactly supported Laplacian. Fubini's theorem immediately implies that $g(\sigma, \Omega; \cdot) \not\equiv +\infty$ if σ is admissible. In the present Greenian context the condition $g(\sigma, \Omega; \cdot) \not\equiv +\infty$ is also sufficient for (and therefore equivalent to) admissibility of σ. Indeed, if ν is compactly supported in Ω with continuous potential, let V be an open set such that $\operatorname{supp}\nu \subset V \subset \overline{V} \subset \Omega$. By continuity, there exists C such that $g(\nu, \Omega; z) \leqslant C$, $z \in \Omega$. Since $g(\sigma, \Omega; \cdot) \not\equiv +\infty$, the same is true for the potential of $\sigma_{\lfloor \Omega \setminus V}$. As this potential is harmonic in V and is not equal identically to $+\infty$ there, it is finite and locally bounded in V. Hence, $g(\sigma_{\lfloor \Omega \setminus V}, \Omega; z) \leqslant C'$, $z \in \operatorname{supp}\nu$. Therefore, it follows from Fubini's theorem that

$$\int g(\nu, \Omega; z) d\sigma(z) \leqslant C\sigma(V) + C'\nu(\Omega).$$

A.4. **Capacities.** Given two Radon measures σ_1 and σ_2 on Ω, we put

$$(\sigma_1, \sigma_2)_\Omega := \int g(\sigma_1, \Omega; z) d\sigma_2(z) = \int g(\sigma_2, \Omega; z) d\sigma_1(z),$$

which is either a non-negative number or $+\infty$. The *Green energy* of σ is defined as $I_\Omega(\sigma) := (\sigma, \sigma)_\Omega$. The *Greenian capacity* relative to Ω of a compact set $K \subset \Omega$ is a non-negative number

$$\operatorname{cap}_\Omega(K) := \frac{1}{\inf_{\mu \in \mathcal{P}(K)} I_\Omega(\mu)}, \tag{A.4}$$

where $\mathcal{P}(K)$ is the set of Borel probability measures on K. The *Greenian capacity* of a Borel set B is given by

$$\operatorname{cap}_\Omega(B) := \sup_{K \subset B} \operatorname{cap}_\Omega(K) = \inf_{U \supset B} \operatorname{cap}_\Omega(U), \tag{A.5}$$

where the supremum is taken over all compact subsets of B, the infimum is taken over all open sets containing B, and the equality is due to a theorem by Choquet [8, Section VIII.4]. When K is compact and $\operatorname{cap}_\Omega(K) > 0$, there exists a unique $\mu_{\Omega,K} \in \mathcal{P}(K)$, called the *Green equilibrium measure* of K in Ω, to meet the infimum in (A.4). It is characterized by the fact that for some constant $C (= 1/\operatorname{cap}_\Omega(K))$, the *Green equilibrium potential* $g(\mu_{\Omega,K}, \Omega; z)$ satisfies $g(\mu_{\Omega,K}, \Omega; z) \leqslant C$ for $z \in \Omega$ with $g(\mu_{\Omega,K}, \Omega; z) = C$ for $z \in K \setminus E$, where E has Greenian capacity zero; this can be shown as in the Euclidean case [54, Theorems II.5.11 & II.5.12].

In the case of an arbitrary set B, the infimum in (A.5) introduces the *outer Greenian capacity* of B and will serve as a definition of $\operatorname{cap}_\Omega(B)$. However, it may no longer match the supremum (the latter defines the *inner Greenian capacity* of B).

When $\Omega \subset \mathbb{C}$, another notion of capacity is instrumental in this paper, namely the *logarithmic capacity* defined for a compact set $K \subset \mathbb{C}$ as

$$\operatorname{cap}(K) := \exp\left\{ - \inf_{\mu \in \mathcal{P}(K)} \int V^\mu(z) d\mu(z) \right\}, \tag{A.6}$$

where $V^{\mu}(z)$ is the logarithmic potential of μ defined earlier in Section A.1. The logarithmic capacity of a Borel subset of $\mathbb{C}$ and the outer logarithmic capacity of an arbitrary subset are defined via the same process as for Greenian capacity, based on the analog of (A.5), see [52, 54] (note that in [52], $\mathrm{cap}(E)$ denotes the inner logarithmic capacity and potentials carry a sign opposite to the current one). If K is compact and $\mathrm{cap}(K) > 0$, then there is a unique $\mu_K \in \mathcal{P}(K)$, called the *logarithmic equilibrium measure* of K, that realizes the infimum in (A.6). It is characterized by the fact that for some constant $C(= -\log \mathrm{cap}(K))$, the *logarithmic equilibrium potential* $V^{\mu_K}(z)$ is at most C for $z \in K$ and in fact equal to C on K except possibly for a subset of logarithmic capacity zero.

Both the Greenian and logarithmic capacities are right continuous on compact sets, meaning that

$$\mathrm{cap}(\cap_{j=1}^{\infty} K_j) = \lim_n \mathrm{cap}(\cap_{j=1}^{n} K_j) \quad \text{and} \quad \mathrm{cap}_\Omega(\cap_{j=1}^{\infty} K_j) = \lim_n \mathrm{cap}_\Omega(\cap_{j=1}^{n} K_j)$$

if the K_j are compact; see [52, Theorem 5.1.3(a)] for the logarithmic case, the Greenian one being argued the same way with an obvious adaptation of [52, Lemma 3.3.3]. In addition, the (outer) Greenian and logarithmic capacities are left continuous:

$$\mathrm{cap}(\cup_{j=1}^{\infty} E_j) = \lim_n \mathrm{cap}(\cup_{j=1}^{n} E_j) \quad \text{and} \quad \mathrm{cap}_\Omega(\cup_{j=1}^{\infty} E_j) = \lim_n \mathrm{cap}_\Omega(\cup_{j=1}^{n} E_j);$$

for the logarithmic capacity this follows from [52, Theorem 5.1.3(b)] combined with Choquet's theorem, and the Greenian case can be handled similarly, compare to [8, Section VIII.4].

One form of the *domination principle* for Green potentials says that if $g(\sigma, \Omega; \cdot) \leqslant v$ on $\mathrm{supp}\,\sigma$ (the support of σ) for some superharmonic function v, then $g(\sigma, \Omega; \cdot) \leqslant v$ everywhere on Ω; in fact, we shall state a stronger version in Section A.6. It implies the *continuity theorem*, saying that if the restriction of $g(\sigma, \Omega; \cdot)$ to $\mathrm{supp}\,\sigma$ is continuous at $z_0 \in \mathrm{supp}\,\sigma$ then $g(\sigma, \Omega; \cdot)$ is continuous at z_0. When σ is a positive Borel measure with compact support such that $g(\sigma, \Omega; z) < +\infty$ for σ-a.e. z, there is an increasing sequence of measures σ_k supported on $\mathrm{supp}\,\sigma$, having continuous Green potentials and converging to σ in the strong (total variation) sense, such that $g(\sigma_k, \Omega; \cdot)$ increases pointwise to $g(\sigma, \Omega; \cdot)$ on Ω. The proof is *mutatis mutandis* the same as for logarithmic potentials [54, Lemma I.6.10], using the continuity theorem for Green potentials. In particular, if K is compact with $\mathrm{cap}_\Omega(K) > 0$, we find upon letting σ be the Green equilibrium distribution that there exist nonzero positive measures supported on K whose Green potentials are continuous.

When $\Omega \subset \mathbb{C}$, a subset of Ω has (outer) Greenian capacity zero if and only if it has (outer) logarithmic capacity zero. Indeed, it is enough to verify this claim on compact sets since capacity is left continuous and a set of outer (Greenian or logarithmic) capacity zero is contained in a Borel (even G_δ) set of capacity zero, by definition. Moreover, by the increasing character of $g_\Omega(z, w)$ with Ω, we may assume that Ω is simply connected. Then the result follows by comparing the logarithmic kernel $\log(1/|z - w|)$ with the Green kernel $g_\Omega(z, w) = \log|(1 - \varphi(z)\overline{\varphi(w)})/(\varphi(z) - \varphi(w))|$, where φ is a conformal map $\Omega \to \mathbb{D}$. A property holding pointwise except on a set of outer Greenian capacity zero (equivalently: logarithmic capacity zero if $\Omega \subset \mathbb{C}$) is said to hold *quasi everywhere*.

A.5. **Fine Topology.** A basis for the fine topology on Ω is given by all sets of the form

$$\cap_{i=1}^{m} \left\{ z \in B : v_i(z) < \alpha_i \right\}, \tag{A.7}$$

where $B \subseteq \Omega$ is open, v_i are superharmonic functions on B, and α_i are constants. Consequently, all superharmonic functions $\Omega \to (-\infty, +\infty]$ are finely continuous (equivalently: all subharmonic functions $\Omega \to [-\infty, +\infty)$ are finely continuous), and the fine topology is the coarsest with this property because, by the Riesz representation theorem and the monotonicity of Green functions with respect to the domain, each set of the form (A.7) contains one for which v_i are Green potentials. In particular, we may as well require in (A.7) that v_i be defined and superharmonic on the whole of Ω. Hence, the present definition modeled after [29, Definition 6.5.1] (which deals with the Euclidean case) is equivalent to [8, Definition I.1]. It is known that the fine topology on Ω is locally connected [20, Corollary to Theorem 9.11], and that the fine connected components of a finely open set are finely open [19, Corollary 1]. Moreover, Lipschitz curves are finely connected [19, Theorem 7], so that Euclidean domains are fine domains as well.

As a general convention, we use the prefix "fine" to signify that a notion is understood with respect to the fine topology. This way we distinguish the latter from the classical, Euclidean topology (more precisely: the one induced on Ω by the Euclidean topology of charts). The fine boundary of a set S is denoted by $\partial_{\mathfrak f} S$, and its fine closure by $\mathrm{clos}_{\mathfrak f}$.

A set $E \subset \Omega$ is called *polar* (in Ω) if there is a superharmonic function $u \not\equiv +\infty$ on Ω such that $u(z) = +\infty$ for $z \in E$. Superharmonic functions not identically $+\infty$ are locally integrable, therefore polar sets have area measure zero, see Section A.1. By definition a polar set is contained in a G_δ polar set, and every G_δ polar set arises as the $+\infty$-set of a superharmonic function [8, Section VI.9]. If U is a fine domain and E is polar, then $U \backslash E$ is again a fine domain [19, Theorem 6]. In fact, polar sets are exactly the sets of zero outer Greenian capacity (equivalently: of zero outer logarithmic capacity if $\Omega \subset \mathbb{C}$) defined in Section A.4; this is justified in Section A.9, but we take it presently for granted (we stress that [52] defines polar sets as having inner capacity zero, thereby making for a larger class of non-Borel polar sets). One consequence is: if $\Omega' \supset \Omega$ is hyperbolic and E is polar in Ω, then it is polar in Ω' as well; indeed, since $g_{\Omega'}(\cdot, w) \geqslant g_\Omega(\cdot, w)$, it is clear that E has zero outer Greenian capacity in Ω' if it does in Ω. Conversely, it is obvious from the definition that E is polar in Ω if it is polar in $\Omega' \supset \Omega$, therefore we may speak of a polar set without specifying a hyperbolic subset of $\mathcal{R}_*$ in which E is contained.

A countable union of polar sets is polar, for if E_k is included in the $+\infty$-set of a superharmonic function $u_k \not\equiv +\infty$ while $K \subset \Omega$ is compact and of positive Lebesgue measure, then there are $t_k > 0$ such that $u := \sum_k t_k u_k$ is summable on K (therefore $u \not\equiv +\infty$) and is superharmonic with value $+\infty$ at each point of $\cup_k E_k$. So, if $E \subset \Omega$ is polar and $p : \mathcal{R}_* \to \mathbb{C}$ is the natural projection, then $p(E)$ is polar. Indeed, for $V \subset \Omega$ a domain such that $p : V \to p(V)$ is a homeomorphism, $v \circ p^{-1}$ is superharmonic on $p(V)$ when v is superharmonic on V, and $\Omega \backslash \mathbf{rp}(\mathcal{R}_*)$ can be covered with countably many such domains while $\mathbf{rp}(\mathcal{R}_*)$ is finite. Conversely, if $V \subset \mathbb{C}$ is a bounded open set and $E \subset V$ is polar, then $p^{-1}(E)$ is polar because $v \circ p$ is superharmonic as soon as v is superharmonic on V.

If $u \not\equiv +\infty$ is superharmonic and finite on a polar set E, then E has outer Δu-measure zero [8, Section VI.9, item β)]. Consequently a Radon measure σ of finite Green energy cannot charge a polar set E, for we may assume σ has compact support (since Ω is σ-compact) and as in Section A.4 there is an increasing sequence of measures σ_k converging strongly to σ with $g(\sigma_k, \Omega; \cdot)$ continuous, whence $\sigma(E) = \lim_k \sigma_k(E) = 0$.

Removability Theorem. *If $E \subset \Omega$ is a (relatively) closed polar set while u is superharmonic on $\Omega \backslash E$ and locally bounded below in a neighborhood of E, then u extends in a unique manner to a superharmonic function on Ω. Moreover, if u is harmonic in $\Omega \backslash E$ and locally bounded in a neighborhood of E, then u extends harmonically to Ω.*

Proof. The proof of the first statement carries over to hyperbolic Riemann surfaces from its planar version, see [52, Theorem 3.6.1]. When u is harmonic in $\Omega \backslash E$ and locally bounded in a neighborhood of E, it extends both to a subharmonic and a superharmonic function on Ω by the first part. Since Δu does not depend on the extension because E has Lebesgue measure zero, we deduce that $\Delta u = 0$. □

The removability theorem implies the following result.

Generalized Minimum Principle. *If u is superharmonic and bounded below on some open set $U \subset \overline{U} \subset \Omega$, and if moreover $\liminf_{U \ni z \to \xi} u(z) \geqslant 0$ for quasi every $\xi \in \partial U$, then $u \geqslant 0$ in U.*

Proof. Given $\varepsilon > 0$, let $E_\varepsilon := \{\zeta \in \partial U : \liminf_{z \to \xi} u(z) \leqslant -\varepsilon\}$. Then E_ε is a closed polar set and the function $w : \Omega \to (-\infty, +\infty]$ given by $\min(u, -\varepsilon)$ on U and $-\varepsilon$ on $\Omega \backslash (U \cup E_\varepsilon)$ is superharmonic on $\Omega \backslash E_\varepsilon$, by the glueing theorem, see [52, Theorem 2.4.5] for a planar version of this local result. As w is bounded below, it extends to a superharmonic function on Ω. Because

$U \cup E_\varepsilon \subset \overline{U} \subset \Omega$, it holds that $\liminf_{z\to\xi} w(z) = -\varepsilon$ for any $\xi \in \partial\Omega$. Thus, $w \geqslant -\varepsilon$ in Ω by the classical minimum principle contained in the very definition of superharmonic functions. □

The hypothesis "u is bounded below" can be relaxed somewhat: it is enough to assume that $u \geqslant -g$ where g is a *semi-bounded potential*, meaning that it is the increasing pointwise limit of a sequence of locally bounded potentials, see [20, Theorem 9.1]. Note that $g(\sigma, \Omega; \cdot)$ is semi-bounded when it is finite σ-a.e., for we may assume σ has compact support (as Ω is a countable union of compact sets) and then appeal to properties of the measures σ_k in Section A.4. In fact, $g(\sigma, \Omega; \cdot)$ is semi-bounded if and only if it is finite σ-a.e. in Ω, which is also if and only if σ does not charge polar sets, see [20, Section I.2.6, Theorem].

A.6. **Thinness.** Fine topology can also be introduced via the notion of thinness. A set $E \subset \Omega$ is said to be *thin* at $\zeta \in \Omega$ if ζ is not a fine limit point of E. Equivalently, E is thin at ζ if and only if either $\zeta \notin \overline{E}$ or there exists a function v, superharmonic in a neighborhood of ζ, such that

$$\liminf_{E \ni z \to \zeta, z \neq \zeta} v(z) > v(\zeta); \tag{A.8}$$

see [29, Theorem 6.6.3] for a proof of this equivalence in the Euclidean setting, which applies to hyperbolic Riemann surfaces as well and also shows that v in (A.8) may be taken superharmonic on the whole of Ω. Hence, the above definition of thinness (which is local) matches [8, Definition I.2] (whose local character is not immediate, see [8, Theorem VII.1]). Setting $\liminf$ over the empty set to $+\infty$ by convention, (A.8) may still be regarded as characterizing thinness at $\zeta \notin \overline{E}$, upon letting $v \equiv 0$. Note that when the limit inferior in (A.8) is taken over a full Euclidean neighborhood of ζ, superharmonicity of v implies that the inequality gets replaced by an equality. Clearly, E is thin at ζ if and only if for some (hence any) chart (V, φ) with $\zeta \in V$, the planar set $\varphi(V \cap E)$ is thin at $\varphi(\zeta)$, and a countable union of thin sets at ζ is again thin at ζ.

A set V is a fine neighborhood of $\zeta \in V$ if and only if the complement of V is thin at ζ, see [8, Theorem I.3]. In particular, if V is finely open and Z is polar, then $V\backslash Z$ is finely open. The points of $E \subset \Omega$ at which E is thin form a polar set, and E is thin at each of its points if and only if it is polar [8, Theorem VII.7 & Corollary]. One consequence of E being thin at ζ is that, locally in a chart (V, φ) with $\zeta \in V$, there are arbitrary small circles centered at $\varphi(\zeta)$ which do not meet $\varphi(E \cap V)$, see [29, Theorem 6.7.9]; in the same vein, $\varphi(V\backslash E)$ contains a segment of the form $[\varphi(\zeta), \varphi(\zeta) + re^{i\theta}]$ with $r = r(\theta) > 0$ for quasi-every direction $e^{i\theta}$ in $\mathbb{T}$, see [52, Theorem 5.4.3]. In particular, a connected set cannot be thin at an accumulation point and therefore polar sets are totally disconnected.

The *base* $b(E)$ of a set E is the set of points in Ω at which E is non-thin, and E is called a base if $b(E) = E$. It is known that $b(E)$ is a finely closed G_δ set, see [8, Proposition VII.8]. We record the following, elementary fact.

Lemma A.1. *Let $E, F \subset \Omega$ be disjoint finely open sets such that $\Omega\backslash(E \cup F)$ is a base. Then $\Omega\backslash E$ and $\Omega\backslash F$ are bases as well.*

Proof. If $x \in (\Omega\backslash E) \cap (\Omega\backslash F) = \Omega\backslash(E \cup F)$, the latter set is non-thin at x by assumption and therefore so is $\Omega\backslash E$. If now $x \in (\Omega\backslash E) \cap F = F$, then the fine openness of F implies that $\Omega\backslash F$ is thin at x and so is $E \subset \Omega\backslash F$. However, if $\Omega\backslash E$ were also thin at x in this case, then $\Omega = E \cup (\Omega\backslash E)$ would be thin at x which is impossible. Hence, $\Omega\backslash E$ is non-thin at any of its points, therefore it is a base. □

The notion of a base generates a strong form of the domination principle, see [8, Theorem VIII.4].

Strong Domination Principle. *Let v be a non-negative superharmonic function in Ω such that $v \geqslant g(\sigma, \Omega; \cdot)$ quasi everywhere on a set E such that $\sigma(\Omega\backslash b(E)) = 0$. Then $v \geqslant g(\sigma, \Omega; \cdot)$ everywhere in Ω.*

The fine closure of E is equal to $\mathrm{clos}_{\mathfrak f}(E) = b(E) \cup i(E)$, where $i(E)$ is the set of finely isolated points of E, see [8, Proposition V.10]. The finely closed sets are precisely those for which $b(E) \subseteq E$. Note that $b(E) \cap i(E) = \varnothing$ and therefore, if V is finely open, we have that

$$b(\Omega\backslash V) = \Omega\backslash V', \quad V' := V \cup i(\Omega\backslash V), \tag{A.9}$$

where we observe that V' is in turn finely open. For any set E, the fine boundary $\partial_{\mathfrak f} E$ is finely closed and, as shown in [20, Lemma 12.3], it holds that

$$i(\partial_{\mathfrak f} E) = i(E) \cup i(\Omega\backslash E) \quad \text{and} \quad b(\partial_{\mathfrak f} E) = b(E) \cap b(\Omega\backslash E). \tag{A.10}$$

The next lemma connects fine topologies in D and $\mathfrak{R}$ (defined at the beginning of Section 2.1).

Lemma A.2. *The map $p : \mathfrak{R} \to D$ is finely open and finely continuous, that is, $p(V)$ and $p^{-1}(U)$ are finely open when $V \subset \mathfrak{R}$ and $U \subset D$ are finely open. Moreover, $i(\mathfrak{R}\backslash p^{-1}(U)) = p^{-1}(i(D\backslash U))$.*

Proof. Let $V \subset \mathfrak{R}$ be finely open and $\zeta \in V$. Denote by $\tilde O$ a Euclidean disk centered at $p(\zeta)$ of small enough radius so that O, the connected component of $p^{-1}(\tilde O)$ containing ζ, contains no ramification points of $\mathfrak{R}$ except possibly ζ itself (if $m(\zeta) > 1$) and $O \cap p^{-1}(p(\zeta)) = \{\zeta\}$. Since $E := O\backslash V$ is thin at ζ, there exists a superharmonic function v in O for which (A.8) takes place. *We claim* that

$$\tilde v(z) := \begin{cases} \sum_{w\in O\cap p^{-1}(z)} v(w), & z \in \tilde O\backslash\{p(\zeta)\}, \\ m(\zeta)v(\zeta), & z = p(\zeta), \end{cases} \tag{A.11}$$

is superharmonic on $\tilde O$. Indeed, by shrinking $\tilde O$ if needed, we may assume that v is the increasing limit of a sequence of continuous superharmonic functions v_n on O, see [52, Corollary 2.7.3] for a proof of this fact in the planar case that carries over to $\mathfrak{R}$ using local charts. Define $\tilde v_n$ similarly to (A.11), only replacing v with v_n. Clearly, $\tilde v_n$ is superharmonic on $\{z \in \tilde O : z \neq p(\zeta)\}$. Since it is bounded around $p(\zeta)$ by the continuity of v_n, the restriction $\tilde v_{n\lfloor \tilde O\backslash p(\zeta)}$ uniquely extends to a superharmonic function on $\tilde O$ by the Removability Theorem. Of necessity, the value at $p(\zeta)$ of this extension is given by

$$\liminf_{z\to p(\zeta), z\neq p(\zeta)} \tilde v_n(z) = m(\zeta)v_n(\zeta) = \tilde v_n(p(\zeta)),$$

where the first equality comes from the continuity of v_n at ζ. Hence, $\tilde v_n(z)$ is superharmonic on $\tilde O$ and so is its increasing limit $\tilde v$. *This proves the claim.* In another connection, the lower semicontinuity of v shows that the analog of (A.8) holds for $\tilde v$ when the limit inferior is taken along $p(E)$. As $\tilde O\backslash p(V) \subseteq p(E)$, we get that $D\backslash p(V)$ is thin at $p(\zeta)$ so that $p(V)$ is finely open, as claimed.

To show the second claim, observe that the lift of a function from D to $\mathfrak{R}$ preserves superharmonicity and that $\mathfrak{R}\backslash p^{-1}(U) = p^{-1}(D\backslash U)$. The identity $i(\mathfrak{R}\backslash p^{-1}(U)) = p^{-1}(i(D\backslash U))$ is now straightforward. □

In Lemma A.3 below, we single out for easy reference a basic geometric fact, used at several places in the paper. We say that a continuous injective map $\gamma : \mathbb{T} \to \mathfrak{R}$ is a *parametrized Jordan curve* in $\mathfrak{R}$ and we simply call the image $\gamma(\mathbb{T})$ a (non-parametrized) Jordan curve. On a hyperbolic Riemann surface $\mathfrak{R}$, any Jordan curve Γ homotopic to a point is uniquely the boundary of a (topological) disk $O \subset \mathfrak{R}$ [63, Theorem 2.4]; we say that O is the *interior* of Γ, and we write $O = \operatorname{int}\Gamma$. That Γ is homotopic to a point in particular holds if it is included in a simply connected open set U which is the domain of a chart. Recall also that $E \subset \mathfrak{R}$ is called *schlicht* over $U \subset \mathbb{C}$ if $U \supset p(E)$ and the restriction $p_{\lfloor E} : E \to U$ is injective.

Lemma A.3. *Let $E \subset \mathfrak{R}$ be schlicht over D and $\xi \in \mathbf{rp}(\mathfrak{R})$. There exists a neighborhood U_ξ of ξ such that no Jordan curve in U_ξ contains ξ in its interior and simultaneously is contained in E.*

Proof. Let U be a simply connected domain of a chart containing ξ, so that every Jordan curve $\mathbb{T} \to U$ is homotopic to a point. Let $O \subset D$ be a disk centered at $p(\xi)$ and O_ξ the connected component of $p^{-1}(O)$ containing ξ, with O small enough that $O_\xi \subset U$ and $O_\xi \cap \mathbf{rp}(\mathcal{R}) = \{\xi\}$; this is possible since $\mathcal{R}_*$ is compact. Then O_ξ is a neighborhood of ξ which is (isomorphic via a biholomorphic map fixing ξ to) an $m(\xi)$-sheeted cyclic covering of O. So, if we identify the homotopy groups $\pi_1(O_\xi\backslash\{\xi\})$ and $\pi_1(O\backslash\{p(\xi)\})$ with the infinite cyclic group generated by the symbol a, the induced morphism $p_* : \pi_1(O_\xi\backslash\{\xi\}) \to \pi_1(O\backslash\{p(\xi)\})$ is the map $a \mapsto a^{m(\xi)}$. Now, a parametrized Jordan curve $\gamma : \mathbb{T} \to O_\xi$ containing ξ in its interior is a generator of the fundamental group of $O_\xi\backslash\{\xi\}$. Hence, $p \circ \gamma$ is the $m(\xi)$-power of a generator of the fundamental group of $O\backslash\{p(\xi)\}$, in particular it has winding number $\pm m(\xi)$ with respect to $p(\xi)$. However, if γ is valued in E, then $p \circ \gamma : \mathbb{T} \to O$ is a parametrized Jordan curve because p is injective on E, and therefore it has winding number ± 1 with respect to $p(\xi)$. This contradicts the assumption that $m(\xi) > 1$, showing that $U_\xi := O_\xi$ satisfies our requirements. □

The next lemma, used in the proof of Lemma 3.11, depends on Lemma A.3.

Lemma A.4. *If $E \subset \mathcal{R}$ is schlicht over D and $\xi \in \mathbf{rp}(\mathcal{R})$, then $\mathcal{R}\backslash E$ is non-thin at ξ.*

Proof. Let (U_ξ, φ) be a chart around ξ, with U_ξ as in Lemma A.3 and $\varphi(\xi) = 0$. If $\mathcal{R}\backslash E$ is thin at ξ, then there is a circle $\mathbb{T}_r := \{|z| = r\} \subset \varphi(U_\xi)$ such that $\varphi^{-1}(\mathbb{T}_r) \subset E$. As $\varphi^{-1} : \mathbb{T}_r \to U_\xi$ is a Jordan curve in E that contains ξ in its interior, this contradicts Lemma A.3. □

A.7. **Regularity.** Thinness is intimately connected to the notion of a regular boundary point with respect to the Dirichlet problem. Given a Euclidean open set U with $\overline{U} \subset \Omega$, a point $\zeta \in \partial U$ is called *regular* if for any continuous function ψ on ∂U it holds that

$$\lim_{U \ni z \to \zeta} H_\psi(z) = \psi(\zeta),$$

where $H_\psi(z)$ is the Perron-Wiener-Brelot solution of the Dirichlet problem on U with boundary data ψ, see [8, Section VI.6, item γ)] for a description of the Perron-Wiener-Brelot process; other boundary points are called *irregular*. When all its boundary points are regular, we say that U itself is regular. It is known that $\zeta \in \partial U$ is regular if and only if

$$\lim_{U \ni z \to \zeta} g_U(z, w) = 0 \tag{A.12}$$

for some (and then any) w in each connected component of U. Moreover, ζ is irregular if and only if the complement of U is thin at ζ [8, Theorem VII.13]. This entails that regularity is a local notion, in particular each point of ∂U is regular as soon as the latter is locally connected, as follows from the analogous property in a Euclidean space [52, Theorem 4.2.2].

Let σ be a finite measure, compactly supported in U. As $g_U(z, w)$ is bounded for $w \in \operatorname{supp}\sigma$ and z outside of a neighborhood of the latter, see Section A.2, we get from (A.12) and the dominated convergence theorem that $g(\sigma, U; \cdot)$ extends continuously by zero to the set of regular points of ∂U. When σ is not compactly supported, a weaker result is stated in Section A.8 (Lemma A.5).

Regular points of *finely open* sets are defined analogously: when U is finely open, $\zeta \in \partial_{\mathfrak{f}} U$ is said to be regular if $\Omega\backslash U$ is non-thin at ζ. By (A.10), the set of regular points is then $b(\partial_{\mathfrak{f}} U)$, and if $\partial_{\mathfrak{f}} U$ is its own base one says that U is regular, see [20, Section IV.12].

The following results are the natural analogs, for Green potentials on regular open sets of hyperbolic surfaces, of their logarithmic counterparts in the plane, see [54, Theorems I.6.8 & I.6.9].

Principle of Descent and Lower Envelope Theorem. *Let U be a regular open set with compact closure $\overline{U} \subset \Omega$. If σ_n are positive measures on U with uniformly bounded masses that converge weak* to some measure σ as $n \to \infty$, then*

(1) [Principle of Descent]

$$\liminf_{n\to\infty} g(\sigma_n, U; z_n) \geqslant g(\sigma, U; z), \quad z_n \to z \in U. \tag{A.13}$$

(2) [Lower Envelope Theorem]

$$\liminf_{n\to\infty} g(\sigma_n, U; z) = g(\sigma, U; z) \quad \textit{for quasi every} \quad z \in U. \tag{A.14}$$

Proof. The arguments are a minor variation of those used in [54]. For $M > 0$ and $z \in U$, observe from (A.12) that $\varphi_{M,z}(w) := \min\{M, g_U(z,w)\}$ is continuous on $\overline{U}$ and zero on ∂U, locally uniformly with respect to z. Thus, $\varphi_{M,z}$ lies in the closure of the space $C_c(U)$ of continuous functions on U with compact support endowed with the sup norm. Moreover, $|\varphi_{M,z_n} - \varphi_{M,z}|$ is arbitrary small on U for n large enough, by the minimum principle and the continuity of Green functions off the diagonal. Hence, as $\sigma_n \overset{w*}{\to} \sigma$, we get that $\lim_n \int \varphi_{M,z_n} d\sigma_n = \int \varphi_{M,z} d\sigma$ and consequently, by monotone convergence, we deduce (A.13) from the relations

$$g(\sigma, U; z) = \lim_{M\to\infty} \int \varphi_{M,z} d\sigma = \lim_{M\to\infty} \lim_n \int \varphi_{M,z_n} d\sigma_n \leqslant \liminf_n g(\sigma_n, U; z_n).$$

Next, assume to the contrary that $g(\sigma, U; z) < \liminf_n g(\sigma_n, U, z)$ for $z \in K$, where $K \subset U$ is such that $\mathrm{cap}_U(K) > 0$. Clearly, we may suppose that K is compact and so we can find a nonzero measure σ_*, supported on K, such that $g(\sigma_*, U; \cdot)$ is continuous on U, see Section A.4. Then by Fatou's lemma it holds that

$$\int g(\sigma, U; z) d\sigma_*(z) < \int \liminf_n g(\sigma_n, U; z) d\sigma_*(z) \leqslant \liminf_n \int g(\sigma_n, U; z) d\sigma_*(z). \tag{A.15}$$

Moreover, as $g(\sigma_*, U; \cdot)$ extends continuously by zero on ∂U, see discussion after (A.12), it lies in the closure of $C_c(U)$. Therefore, by Fubini's theorem,

$$\lim_n \int g(\sigma_n, U; z) d\sigma_*(z) = \lim_n \int g(\sigma_*, U; z) d\sigma_n(z) = \int g(\sigma_*, U; z) d\sigma(z) = \int g(\sigma, U; z) d\sigma_*(z),$$

thereby contradicting (A.15). □

A.8. **Superlevel Sets of Green Potentials.** In this section we restrict attention to a planar simply connected domain D, which is the interior of a Jordan curve T. In this case, any conformal map $\phi : \mathbb{D} \to D$ extends to a homeomorphism from $\overline{\mathbb{D}}$ onto $\overline{D}$ [49, Theorem 2.6], that we continue to denote with ϕ. Clearly, such a domain D is regular. As mentioned in the previous subsection, if σ is a finite Borel measure compactly supported in D, then $g(\sigma, D; \cdot)$ continuously extends by zero to T. If σ is not compactly supported this may not hold, but when F is a relatively closed subset of $\mathbb{D}$ with a limit point $\xi \in \mathbb{T}$, it was shown in [39] that

$$\lim_{\epsilon\to 0} \mathrm{cap}_{\mathbb{D}}(F \cap \{|z-\xi| < \epsilon\}) > 0 \quad \Rightarrow \quad \liminf_{F \ni z \to \xi} g(\sigma, \mathbb{D}; z) = 0, \tag{A.16}$$

and if the rightmost limit holds for every finite measure σ, then the implication can be reversed. This result is in fact stated in [39] with $g(\sigma, \mathbb{D}; z)$ replaced by $(1-|z|)g(\nu, \mathbb{D}; z)$ where ν is any measure whose Green potential is not identically $+\infty$, but the latter condition is equivalent to saying that the measure $d\sigma(z) := (1-|z|)d\nu(z)$ is finite and then convergence to zero along F of the limit inferior of $(1-|z|)g(\nu, \mathbb{D}; z)$ and of $g(\sigma, \mathbb{D}; z)$ are equivalent, see [39, Section 3, Lemma]. It is also pointed out in [39, Equation (2.5)] that the leftmost limit in (A.16) can be equivalently replaced by $\mathrm{cap}_{\mathbb{D}}(F \cap \{|z-\xi| < \epsilon\}) = \infty$ for every $\epsilon > 0$ (note that instead of the Greenian capacity $\mathrm{cap}_{\mathbb{D}}(E)$ that we use, [39] employs the hyperbolic capacity $\exp\{-1/\mathrm{cap}_{\mathbb{D}}(E)\}$). In Lemma A.5 below, we derive a useful consequence of (A.16). With the notation of this lemma, we stress that a stronger conclusion in fact holds quasi everywhere, namely U_ϵ is thin at quasi every point of T (this can be deduced from general properties of balayage covered in Section A.9). The interest of Lemma A.5 lies with the fact that its conclusion holds at *every* point of T.

Lemma A.5. *Let σ be a finite measure in D, and for $\epsilon > 0$ set $U_\epsilon := \{z \in D : g(\sigma, D; z) > \epsilon\}$. Let $\xi \in \overline{U}_\epsilon \cap T$ and $\phi : \mathbb{D} \to D$ be a conformal map such that $\phi(1) = \xi$. Then there exists a closed set $R_\epsilon \subset [0, 1]$ such that $R_\epsilon \cap (1 - \delta, 1)$ is non-polar for any $\delta \in (0, 1)$, and for each $r \in R_\epsilon$ one has $U_\epsilon \cap \phi(\{z \in \mathbb{D} : |1 - z| = 1 - r\}) = \varnothing$.*

Proof. Suppose initially that $D = \mathbb{D}$. Without loss of generality, we can assume that $\xi = 1$. First, we shall show that

$$\lim_{\delta\to 0} \operatorname{cap}_{\mathbb{D}}(U_\epsilon \cap D_\delta) = 0, \tag{A.17}$$

where $D_\delta := \{z \in \mathbb{D} : |z - 1| < \delta\}$. For $\zeta \in \mathbb{D}$, let $S(\zeta) := \{z \in \mathbb{D} : g_{\mathbb{D}}(z, \zeta) \geqslant \log 2\}$. It was shown in [39, Section 3, Lemma] that there exists $\delta_0 = \delta(\epsilon) > 0$ for which

$$\int_{\mathbb{D}\setminus S(\zeta)} g_{\mathbb{D}}(z, \zeta) d\sigma(z) \leqslant \epsilon/2, \quad |\zeta| > 1 - \delta_0.$$

In particular, this inequality holds for $\zeta \in D_{\delta_0}$. Hence, for any compact subset $F \subset U_\epsilon \cap D_{\delta_0}$, it holds when $\zeta \in F$ that

$$h(\zeta) := \int_{S(\zeta)} g_{\mathbb{D}}(z, \zeta) d\sigma(z) = g(\sigma, \mathbb{D}; \zeta) - \int_{\mathbb{D}\setminus S(\zeta)} g_{\mathbb{D}}(z, \zeta) d\sigma(z) > \epsilon/2. \tag{A.18}$$

Assume to the contrary that the limit in (A.17) is larger that $2\eta > 0$ (the limit must exist as $\operatorname{cap}_{\mathbb{D}}(U_\epsilon \cap D_\delta)$ decreases with δ). Since U_ϵ is an open set, we get from (A.5) that for any $\delta > 0$ there exists a compact set $F_\delta \subset U_\epsilon \cap D_\delta$ for which $\operatorname{cap}_{\mathbb{D}}(F_\delta) \geqslant \eta$. This entails that there exist a sequence $\delta_n \to 0$ and disjoint compact sets $F_n \subset U_\epsilon \cap D_{\delta_n}$ with $\operatorname{cap}_{\mathbb{D}}(F_n) \geqslant \eta$. Let ν_n be the Green equilibrium distribution on F_n and $F_n^* := \{z \in S(\zeta) : \text{for some } \zeta \in F_n\}$. In view of (A.18),

$$\epsilon/2 \leqslant \int_{F_n} h(\zeta) d\nu_n(\zeta) \leqslant \frac{1}{\eta} \sigma(F_n^*) \to 0,$$

where the second inequality and the fact that $\lim_n \sigma(F_n^*) \to 0$ can be established as in the proof of [39, Theorem 1] (compare to p. 486 of that reference). This contradiction proves (A.17).

Let $\mathsf{T} : D_1 \to (0, 1)$ be defined by $z \mapsto \mathsf{T}z := 1 - |1 - z|$ and put $V_\epsilon := U_\epsilon \cap D_1$. Denoting by $\mathsf{T}V_\epsilon$ the set $\{\mathsf{T}\zeta : \zeta \in V_\epsilon\}$, we claim that

$$\lim_{\delta\to 0} \operatorname{cap}_{\mathbb{D}}(\mathsf{T}V_\epsilon \cap D_\delta) = 0. \tag{A.19}$$

Before proving (A.19), let us show why it implies the lemma. For this, consider $R_\epsilon := [0, 1]\setminus\mathsf{T}V_\epsilon$, which is a closed set. If the conclusion of the lemma were not true, there would exist $\delta_0 > 0$ such that $R_\epsilon \cap (1 - \delta_0, 1)$ is polar. By definition of T this would imply that $\operatorname{cap}_{\mathbb{D}}(\mathsf{T}V_\epsilon \cap D_\delta) = \operatorname{cap}_{\mathbb{D}}((1 - \delta, 1)) = \infty$ for any $\delta < \delta_0$ (the last equality follows at once from the definition of the Greenian capacity), which contradicts (A.19).

We are now left to demonstrate (A.19). Assume to the contrary that it does not hold, i.e., there exists $\eta > 0$ such that for any $\delta > 0$ there is a compact set $F_\delta \subset \mathsf{T}V_\epsilon \cap D_\delta$ for which $\operatorname{cap}_{\mathbb{D}}(F_\delta) \geqslant \eta$. The previous inequality means that there exists a probability measure μ_δ supported on F_δ such that

$$\int\int g_{\mathbb{D}}(x, y) d\mu_\delta(x) d\mu_\delta(y) \leqslant \frac{1}{\eta}.$$

Since U_ϵ is open, so is V_ϵ and one easily sees that each $z \in V_\epsilon \cap D_\delta$ has a neighborhood, say O_z, whose closure is contained in $V_\epsilon \cap D_\delta$ and whose circular projection $\mathsf{T}O_z$ is an open subinterval of $(0, 1)$. These subintervals form an open cover of F_δ, which necessarily has a finite subcover, say $\mathsf{T}O_{z_1}, \ldots, \mathsf{T}O_{z_N}$. The closure K_δ of $O_{z_1} \cup \cdots \cup O_{z_N}$ is a compact subset of $V_\epsilon \cap D_\delta$, and clearly

$F_\delta \subset \mathsf{T}K_\delta$. In particular, there exists a probability measure ν_δ on K_δ such that $\nu_\delta\mathsf{T}^{-1} = \mu_\delta$, see for example [52, Theorem A.4.4]. Then, by Fubini's theorem, it holds that

$$\frac{1}{\eta} \geqslant \int\int g_{\mathbb{D}}(x,y)d\mu_\delta(x)\mu_\delta(y) = \int\int g_{\mathbb{D}}(\mathsf{T}z,\mathsf{T}w)d\nu_\delta(z)d\nu_\delta(w),$$

and if we can show that $g_{\mathbb{D}}(\mathsf{T}z,\mathsf{T}w) \geqslant g_{\mathbb{D}}(z,w)$ then we will deduce from the above estimate that

$$\mathrm{cap}_{\mathbb{D}}(U_\epsilon \cap D_\delta) = \mathrm{cap}_{\mathbb{D}}(V_\epsilon \cap D_\delta) \geqslant \mathrm{cap}_{\mathbb{D}}(K_\delta) \geqslant \eta > 0 \quad \text{for any } \delta > 0,$$

which of course contradicts (A.17). Hence, the proof has been reduced to the verification of $g_{\mathbb{D}}(\mathsf{T}z,\mathsf{T}w) \geqslant g_{\mathbb{D}}(z,w)$ for $z,w \in D_1$, that we now carry out. Since $g_{\mathbb{D}}(z,w) = \log|(1-z\bar{w})/(z-w)|$ and T is real-valued, we need to show that

$$E := |1-\mathsf{T}z\mathsf{T}w|^2|z-w|^2 - |1-z\bar{w}|^2|\mathsf{T}z-\mathsf{T}w|^2 \geqslant 0. \tag{A.20}$$

Set $a\xi := 1-z$ and $b\eta := 1-w$, where $a,b \in (0,1)$ and $|\xi| = |\eta| = 1$ with $\mathrm{Re}\xi, \mathrm{Re}\eta \in (0,1)$. Then

$$\begin{aligned} E &= |a+b-ab|^2|a\xi-b\eta|^2 - |a\xi+b\bar{\eta}-ab\xi\bar{\eta}|^2|a-b|^2 \\ &= (S+2abU)(T-2abV)-(S+2abW)(T-2ab) \end{aligned}$$

where $S := a^2+b^2+(ab)^2$, $T := a^2+b^2$, $U := 1-a-b$, $V := \mathrm{Re}(\xi\bar{\eta})$, and $W := \mathrm{Re}(\xi\eta-a\eta-b\xi)$. Therefore,

$$\begin{aligned} E &= 2ab\big(S(1-V)+T(U-W)+2ab(W-UV)\big) \\ &= 2ab\big((S+2abU)(1-V)+(T-2ab)(U-W)\big) \\ &= 2ab\big((a+b-ab)^2(1-V)+(a-b)^2(U-W)\big). \end{aligned}$$

Because $V \leqslant 1$, the above expression can be estimated from below as

$$\begin{aligned} E &\geqslant 2ab(a-b)^2(1-V+U-W) \\ &= 2ab(a-b)^2\big(2-\mathrm{Re}(\xi\bar{\eta}+\xi\eta)-a(1-\mathrm{Re}\eta)-b(1-\mathrm{Re}\xi)\big), \end{aligned}$$

and since $1-\mathrm{Re}\eta$, $1-\mathrm{Re}\xi$, $1-a$ as well as $1-b$ are all positive, it therefore holds that

$$\begin{aligned} E &\geqslant 2ab(a-b)^2\big(\mathrm{Re}\eta+\mathrm{Re}\xi-\mathrm{Re}(\xi\bar{\eta}+\xi\eta)\big) \\ &= 2ab(a-b)^2\big(\mathrm{Re}\eta+\mathrm{Re}\xi-2\mathrm{Re}\eta\mathrm{Re}\xi\big) \\ &= 2ab(a-b)^2\big(\mathrm{Re}\eta(1-\mathrm{Re}\xi)+\mathrm{Re}\xi(1-\mathrm{Re}\eta)\big). \end{aligned}$$

As $\mathrm{Re}\xi, \mathrm{Re}\eta \in [0,1]$, this establishes (A.20) and completes the proof of the lemma when $D = \mathbb{D}$.

Finally, it remains to reduce the case of a general domain D to the one of the unit disk. Using [52, Theorem A.4.4] once more, let ν be a finite measure in $\overline{\mathbb{D}}$ such that $\nu\phi^{-1} = \sigma$. Then

$$\begin{aligned} g(\sigma,D;\phi(z)) = \int g_D(\phi(z),\zeta)d\sigma(\zeta) &= \int g_D(\phi(z),\phi(w))d\nu(w) \\ &= \int g_{\mathbb{D}}(z,w)d\nu(w) = g(\nu,\mathbb{D};z), \quad z \in \mathbb{D}, \end{aligned}$$

by conformal equivalence of Green functions. Since $\phi : \overline{\mathbb{D}} \to \overline{D}$ is a bijection, the superlevel set $\{z \in \mathbb{D} : g(\nu,\mathbb{D};z) > \epsilon\}$ is equal to $\phi^{-1}(U_\epsilon)$, from which the desired result follows. □

A.9. **Balayage.** Let v be a non-negative superharmonic function on Ω and E be a subset of Ω. The *balayage function* (or *regularized reduction*) of v relative to E, denoted by $\mathcal{B}_v^E$, is the lower semi-continuous regularization of

$$\inf\left\{u \mid u \text{ is superharmonic and positive in } \Omega,\ u \geqslant v \text{ on } E\right\}, \tag{A.21}$$

see [29, Section 5.3] for an account on $\mathbb{R}^n$ that carries over to Ω without change; in fact, $\mathcal{B}_v^E$ coincides with the infimum in (A.21) except perhaps on a polar set where lower semi-continuous regularization may modify the value. The balayage function $\mathcal{B}_v^E$ is superharmonic in Ω, harmonic in $\Omega\backslash\overline{E}$, and equal to v on $b(E)$ [8, Section VIII.1]. Clearly, $\mathcal{B}_v^E \leqslant v$ everywhere, since v qualifies as one of the functions u in (A.21). The balayage function $\mathcal{B}_v^E$ does not change if E gets replaced by $b(E)$ or by $\mathrm{clos}_\mathrm{f}(E)$; in fact, it remains invariant if E is altered by a polar set.

When E is compact, the Strong Domination Principle and properties of the Green equilibrium potential $g(\mu_{\Omega,E},\Omega;\cdot)$ imply that $\mathcal{B}_1^E = \mathrm{cap}_\Omega(E)g(\mu_{\Omega,E},\Omega;\cdot)$. Thus, it follows from the left continuity of $\mathcal{B}_1^E$ with respect to E, see [8, Section VI.10 e)], and the monotone convergence theorem that the outer Greenian capacity of an arbitrary set $E\subset\Omega$ is the mass of $\Delta\mathcal{B}_1^E$ (in fact, this is the way the outer capacity is defined in [8, Section VIII.4]). From this and [8, Theorem VIII.12], we deduce in particular that E is polar if and only if $\mathrm{cap}_\Omega(E)=0$, justifying a claim made in Section A.5.

If v is the Green potential of a positive Borel measure σ, then $\mathcal{B}_v^E$ is a Green potential as well [8, Section VI.11] and the measure σ^E such that $\mathcal{B}_v^E = g(\sigma^E,\Omega;\cdot)$ is called the *balayage* of σ relative to E. The measure σ^E is characterized as the unique measure satisfying

$$\sigma^E\big(\Omega\backslash b(E)\big)=0 \quad\text{and}\quad g(\sigma,\Omega;z)=g\big(\sigma^E,\Omega;z\big),\quad z\in b(E), \tag{A.22}$$

see [8, Theorem VIII.3]. From (A.22), one deduces at once that

$$(\sigma^E)^F=\sigma^F,\qquad F\subset E\subset\Omega. \tag{A.23}$$

Moreover, it holds by [8, Section VI.12, Equation (13)] that

$$\sigma^E(B)=\int\delta_x^E(B)d\sigma(x),\qquad B\quad\text{Borel}, \tag{A.24}$$

while it follows from [8, Section VI.12, Equation (9)] and Fubini's theorem that

$$g\big(\sigma^E,\Omega;z\big)=\int g(\sigma,\Omega;x)d\delta_z^E(x). \tag{A.25}$$

Since $g(\sigma^E,\Omega;\cdot)\leqslant g(\sigma,\Omega;\cdot)$ for any Radon measure σ by the just discussed properties of balayage, it follows from Fubini's theorem that $\int v d\sigma^E \leqslant \int v d\sigma$ for any Green potential v. Since any non-negative superharmonic function v is an increasing limit of potentials, see [10, Lemma 1.1], monotone convergence yields that $\int v d\sigma^E \leqslant \int v d\sigma$ remains valid for such functions as well, see also [20, Section I.3]. In particular, the mass of σ^E cannot exceed the mass of σ.

The *fine support* of a Radon measure σ, denoted by $\mathrm{supp}_\mathrm{f}\,\sigma$ when it exists, is the smallest finely closed carrier of σ. A sufficient condition for its existence is that σ does not charge polar sets, in which case $\mathrm{supp}_\mathrm{f}\,\sigma$ is its own base, see [8, Theorem VII.12]. If σ is admissible, meaning that $g(\sigma,\Omega;\cdot)\not\equiv+\infty$, see Section A.3, and if $\sigma(F)=0$ for some polar set F, then $\sigma^E(F)=0$ for any $E\subset\Omega$ [19, Theorem 1] (as usual, $\sigma(F)$ means the outer σ-measure of F when the latter is not Borel). In particular, if σ is admissible and $\sigma(b(E))=0$, then σ^E does not charge polar sets since it is carried by $b(E)$. Thus, $\mathrm{supp}_\mathrm{f}\,\sigma^E$ exists in this case. An important special case is handled by the following lemma, the first item of which follows from the preceding discussion.

Lemma A.6. [19, Corollary 1 to Theorem 4] [20, Corollaries 2 & 3 to Theorem 12.7]

(i) *If W is finely open and either $z\in W$ or $z\in i(\Omega\backslash W)$, then $\delta_z^{\Omega\backslash W}$ does not charge polar sets.*

(ii) *Let V be a regular finely open set, $z \in V$, and V_z the fine component of V containing z. Then V_z is regular, and $\Omega\backslash V_z$ is largest among all the bases B such that $\delta_z^B = \delta_z^{\Omega\backslash V}$. Moreover, the fine support of $\delta_z^{\Omega\backslash V}$ exists and*

$$\operatorname{supp}_{\mathfrak{f}} \delta_z^{\Omega\backslash V} = \partial_{\mathfrak{f}} V_z \subset \partial_{\mathfrak{f}} V. \tag{A.26}$$

(iii) *Let U be a fine domain and $z \in U$ or $z \in i(\Omega\backslash U)$. Then the fine support of $\delta_z^{\Omega\backslash U}$ exists and*

$$\operatorname{supp}_{\mathfrak{f}} \delta_z^{\Omega\backslash U} = b(\partial_{\mathfrak{f}} U) = b(\Omega\backslash U) \cap \partial_{\mathfrak{f}} U. \tag{A.27}$$

Let now $O \subset \Omega$ be (Euclidean) open, $z \in O$, and O_z the connected component of O containing z. If we let V be the regular finely open set obtained by adjoining to O the polar set $i(\Omega\backslash O)$ and V_z the fine component containing z, then we get from [19, Theorem 6] (see discussion there) that $O_z = V_z\backslash i(\Omega\backslash O)$. Thus, since the balayage function remains the same if the set relative to which it is defined is altered by a polar set, Lemma A.6 (ii) implies that

$$\delta_z^{\Omega\backslash O} = \delta_z^{\Omega\backslash V} = \delta_z^{\Omega\backslash V_z} = \delta_z^{\Omega\backslash O_z}$$

and the latter is carried by the regular points of ∂O_z. Moreover, if O has compact closure in Ω, then $\delta_z^{\Omega\backslash O}$ is a probability measure, and for h a harmonic function in O with continuous extension to $\overline{O}$:

$$h(z) = \int h\, d\delta_z^{\Omega\backslash O}, \qquad z \in O. \tag{A.28}$$

Indeed, (A.28) follows from [8, Section VI.12, application 1] since h is the Perron-Wiener-Brelot solution of the Dirichlet problem[4] in O with boundary data $h_{\lfloor\partial O}$; see [8, Section VI.6, item γ)]. Equality in (A.28) shows that the measure $\delta_z^{\Omega\backslash O}$ does not depend on Ω, provided that the latter is hyperbolic and compactly contains $\overline{O}$. It is called the *harmonic measure* for O (at z). More general versions of (A.28), involving the fine Dirichlet problem and cases where $\overline{O}$ is non-compact, are stated in Theorem A.8 and Lemma A.9 further below.

When $\Omega \subset \mathbb{C}$ and $\overline{O}$ is compact in Ω, it follows from [54, Chapter II, Theorem 5.1] that

$$g(\sigma, O, z) = V^{\sigma}(z) - V^{\sigma^{\Omega\backslash O}}(z), \qquad z \in \Omega, \tag{A.29}$$

where V^{σ} is the logarithmic potential of σ and the left-hand side is interpreted as 0 for $z \in b(\Omega\backslash O)$. More general versions when $\infty \in \Omega$ may be found in [54]. If moreover O is a domain with $K \subset O$ a non-polar compact set such that $\Omega\backslash K$ regular and $O\backslash K$ is non-thin at every point of K, then

$$\operatorname{supp}_{\mathfrak{f}} \mu_{O,K} = K. \tag{A.30}$$

Indeed, K is its own base and $K = \partial_{\mathfrak{f}}(O\backslash K)$ by assumption, while $\mu_{O,K}$ is the balayage onto K of the equilibrium measure on the plate ∂O of the condenser $(\partial O, K)$ [54, Chapter VIII, Theorem 2.6]. Thus, (A.30) follows from (A.24) and (A.27).

A.10. **Green Potentials in D and on $\mathcal{R}$.** In this subsection, we connect Green functions and potentials on the domain D and surface $\mathcal{R}$ defined in Section 2.1. First, let us show that

$$g_D(x, y) = \sum_{z \in p^{-1}(x)} m(z) g_{\mathcal{R}}(z, w), \tag{A.31}$$

where $m(z)$ is the ramification order of $\mathcal{R}$ at z and w is an arbitrary element of the fiber $p^{-1}(y)$. To check (A.31), note that for fixed $y \notin p(\mathbf{rp}(\mathcal{R}))$ and $w \in \mathcal{R}$ with $p(w) = y$, the right-hand side is well defined and harmonic as a function of $x \in D\backslash(p(\mathbf{rp}(\mathcal{R})) \cup \{y\})$. Thus, it is harmonic for $x \in D\backslash\{y\}$ by the Removability Theorem and the continuity of Green functions off the diagonal. Moreover, the right-hand side clearly has a logarithmic singularity at y, and since $\lim_{z\to\partial\mathcal{R}} g_{\mathcal{R}}(z, w) = 0$ by the

[4]In fact, the right-hand side of (A.28) is the Perron-Wiener-Brelot solution of the Dirichlet problem on O with boundary data h as soon as the latter is summable against $\delta_z^{\Omega\backslash O}$ for one (and then any) z in each component of O.

regularity of $\mathcal{R}$ in $\mathcal{R}_*$, its largest harmonic minorant is zero. This proves (A.31) when $y \notin p(\mathbf{rp}(\mathcal{R}))$, and the general case follows by continuity of Green functions off the diagonal. Consequently, if ν is a Radon measure on $\mathcal{R}$ and $p_*(\nu)$ denotes its pushforward under p (the measure on D such that $p_*(\nu)(B) = \nu(p^{-1}(B))$ for a Borel set B), integrating (A.31) against $p_*(\nu)$ with respect to y and changing variables yields

$$g(p_*(\nu), D; x) = \sum_{z \in p^{-1}(x)} m(z) g(\nu, \mathcal{R}; z). \tag{A.32}$$

In the other direction, for a Radon measure σ on D, let us define $\widehat{\sigma}$ by

$$\widehat{\sigma}(B) := \int_{p(B)} \sum_{z \in p^{-1}(x) \cap B} m(z) d\sigma(x), \quad B \subset \mathcal{R}, \quad B \text{ Borel}. \tag{A.33}$$

As $p(B)$ is Borel when B is Borel, one easily checks that (A.33) defines a Radon measure on $\mathcal{R}$. In fact, one can verify that $\widehat{\sigma} = \sigma^* + \sum_{z \in \mathbf{rp}(\mathcal{R})} m(z) \sigma(\{p(z)\}) \delta_z$, where σ^* is the pullback measure resulting from Carathéodory's construction as applied to the map $B \mapsto \sigma(p(B) \backslash p(\mathbf{rp}(\mathcal{R})))$ defined on Borel subsets of $\mathcal{R}$, see [16, Theorem 2.10.10].

Partitioning $D \backslash p(\mathbf{rp}(\mathcal{R}))$ into countably many Borel sets B_k such that $p : p^{-1}(B_k) \to B_k$ induces a homeomorphism on each connected component of $p^{-1}(B_k)$, and invoking the Removability Theorem to proceed by superharmonicity from the case where $\zeta \notin \mathbf{rp}(\mathcal{R})$, one deduces from (A.33) and (A.31) that

$$g(\widehat{\sigma}, \mathcal{R}; \zeta) = g(\sigma, D; p(\zeta)), \quad \zeta \in \mathcal{R}. \tag{A.34}$$

As a consequence of definition (A.33), we claim that if a sequence $\{\mu_n\}$ of finite positive measures supported on a fixed compact set $K \subset D$ converges weak* to μ on D, then the sequence $\{\widehat{\mu}_n\}$ converges weak* to $\widehat{\mu}$ on $\mathcal{R}$. Indeed, the total mass of μ_n is necessarily bounded independently of n by some $C > 0$ (this follows from the Banach-Steinhaus principle) and therefore, in view of (A.33), the total mass of $\widehat{\mu}_n$ is bounded by MC, where M is the number of sheets of $\mathcal{R}$. Hence, an arbitrary subsequence of $\{\widehat{\mu}_n\}$ has a subsequence, say $\{\widehat{\mu}_{n_k}\}$, that converges weak* on $\mathcal{R}$ to some finite measure, say s. It follows from the Lower Envelope Theorem that $\liminf_k g(\widehat{\mu}_{n_k}, \mathcal{R}; z) = g(s, \mathcal{R}; z)$ for quasi every $z \in \mathcal{R}$. Similarly, (A.34) and the Lower Envelope Theorem, applied this time to $\{\mu_{n_k}\}$, yield that $\liminf_k g(\widehat{\mu}_{n_k}, \mathcal{R}; z) = g(\mu, D; p(z)) = g(\widehat{\mu}, \mathcal{R}; z)$ for quasi every $z \in \mathcal{R}$. Thus, $g(s, \mathcal{R}; \cdot) = g(\widehat{\mu}, \mathcal{R}; \cdot)$ quasi everywhere on $\mathcal{R}$, and the claim follows by taking Laplacians on both sides of this equality.

In the previous construction, $\mathcal{R}$ may of course be replaced by another saturated connected bordered surface $\mathcal{S} \subset \mathcal{R}_*$ with bounded projection such that $\overline{\mathcal{R}} \subset \mathcal{S}$. Therefore,

$$\left(\mu_n \overset{w*}{\to} \mu \quad \text{in} \quad \overline{D}\right) \quad \Rightarrow \quad \left(\widehat{\mu}_n \overset{w*}{\to} \widehat{\mu} \quad \text{in} \quad \overline{\mathcal{R}}\right),$$

because $\{\mu_n\}$ also converges weak* in $p(\mathcal{S}) \supset \overline{D}$ whence the measures $\widehat{\mu}_n$ converge weak* on $\mathcal{S}$, while having their supports contained in $\overline{\mathcal{R}}$. The "hat measure" constructed in (A.33) is instrumental both in the proof of Lemma 3.9 and of the following technical result, needed in the paper.

Lemma A.7. *Let σ be a measure in D. Given $E \subset D$, set $\widehat{E} := p^{-1}(E)$. Then it holds that*

$$g\left(\sigma^E, D; p(z)\right) = \int g(\sigma, D; p(x)) d\delta_z^{\widehat{E}}(x).$$

Proof. If u is superharmonic on D, it is obvious from (A.34) that

$$\left(u(z) \geqslant g(\sigma, D; z), \quad z \in E\right) \quad \Rightarrow \quad \left(u(p(\xi)) \geqslant g(\widehat{\sigma}, \mathcal{R}; \xi), \quad \xi \in \widehat{E}\right).$$

As $u \circ p$ is superharmonic on $\mathcal{R}$, this and the definition of balayage imply that

$$g\left(\sigma^E, D; p(\zeta)\right) \geqslant g\left(\widehat{\sigma}^{\widehat{E}}, \mathcal{R}; \zeta\right), \quad \zeta \in \mathcal{R}. \tag{A.35}$$

Conversely, averaging over $\zeta \in p^{-1}(\{z\})$, equation (A.34) yields that

$$g(\sigma, D; z) = \frac{1}{M} \sum_{\zeta \in p^{-1}(\{z\})} m(\zeta) g(\widehat{\sigma}, \mathcal{R}; \zeta), \quad z \in D, \tag{A.36}$$

with M being the total number of sheets of $\mathcal{R}$, entailing when v is superharmonic on $\mathcal{R}$ that

$$\left(v(\zeta) \geqslant g(\widehat{\sigma}, \mathcal{R}; \zeta), \quad \zeta \in \widehat{E}\right) \quad \Rightarrow \quad \left(\frac{1}{M} \sum_{\zeta \in p^{-1}(z)} m(\zeta) v(\zeta) \geqslant g(\sigma, D; z), \quad z \in D\right).$$

Now, the function $z \mapsto \sum_{\zeta \in p^{-1}(z)} m(\zeta) v(\zeta)$ is well-defined and superharmonic on $D \setminus p(\mathbf{rp}(\mathcal{R}))$ and therefore on the whole domain D by the Removability Theorem. Thus, by the definition of balayage, we obtain when v is superharmonic on $\mathcal{R}$ that

$$\left(v(\zeta) \geqslant g(\widehat{\sigma}, \mathcal{R}; \zeta), \quad \zeta \in \widehat{E}\right) \quad \Rightarrow \quad \left(\frac{1}{M} \sum_{\zeta \in p^{-1}(z)} m(\zeta) v(\zeta) \geqslant g(\sigma^E, D; z), \quad z \in D\right),$$

and taking the infimum over v before taking the lower semi-continuous regularization gives us, by virtue of the Strong Domination Principle, that

$$\frac{1}{M} \sum_{\zeta \in p^{-1}(z)} m(\zeta) g(\widehat{\sigma}^{\widehat{E}}, \mathcal{R}; \zeta) \geqslant g(\sigma^E, D; z), \quad z \in D.$$

Combining the above estimate with (A.35), we deduce that

$$g\big(\sigma^E, D; p(z)\big) = g\big(\widehat{\sigma}^{\widehat{E}}, \mathcal{R}; z\big), \quad z \in \mathcal{R},$$

and the conclusion now follows from (A.25) and (A.34). □

A.11. **Dirichlet problem.** The Dirichlet problem on a domain consists in finding a harmonic function in that domain with given boundary data. In the fine Dirichlet problem, one looks for a finely harmonic function on a fine domain to meet prescribed boundary data. A real-valued function h on a fine domain V is *finely harmonic* if it is finely continuous, and if the fine topology on V has a basis comprised of finely open sets E with $\mathrm{clos}_{\mathsf{f}}(E) \subset V$ such that

$$h(z) = \int h d\delta_z^{\Omega \setminus E} \quad \text{for every} \quad z \in E$$

(in particular h must be integrable with respect to $\delta_z^{\Omega \setminus E}$ for all $z \in E$ and each E); one may even assume that E is regular and has compact closure (with respect to the Euclidean topology) in V, see [20, Sections 8 & 14]. Note that a function harmonic in a domain is finely harmonic on any fine subdomain, see [20, Theorem 8.7].

If V is a regular finely open set (recall that it means $\Omega \setminus V$ is its own base), then $b(\partial_{\mathsf{f}} V) = \partial_{\mathsf{f}} V$ by (A.10) whence the result below is a special case of [20, Theorem 14.1] and its proof.

Theorem A.8. *Let $V \subset \Omega$ be a finely open set such that $\Omega \setminus V$ is non-thin at every point of itself, i.e., such that V is regular. If ψ is a finely continuous function on $\partial_{\mathsf{f}} V$, majorized in absolute value there by a finite semi-bounded potential on Ω, say g, then*

$$h_\psi(z) := \int \psi d\delta_z^{\Omega \setminus V} = \int \psi d\delta_z^{\partial_{\mathsf{f}} V}, \quad z \in V, \tag{A.37}$$

is the unique finely continuous extension of ψ to $\mathrm{clos}_{\mathsf{f}}(V)$ that is finely harmonic in V and is majorized in absolute value there by a semi-bounded potential. In fact, it holds that $|h_\psi| \leqslant g$ on $\mathrm{clos}_{\mathsf{f}}(V)$.

The lemma below addresses the question as to when constant functions solve the fine Dirichlet problem on $V \subset \Omega$ or, equivalently by (A.37), as to when the balayage of δ_z out of V, $z \in V$, has unit mass. We recall that an overline, as in $\overline{V}$, or a ∂ sign, as in $\partial\Omega$, refer respectively to the closure and boundary with respect to the Euclidean topology induced by the ambient Riemanian manifold ($\mathcal{R}_*$ or

$\mathbb{C}$). In contrast, fine closures and fine boundaries as in $\partial_{\mathfrak f}V$ and $\mathrm{clos}_{\mathfrak f}(V)$ refer to the fine topology on Ω; thus, $\partial\Omega$ is "invisible" from the point of view of fine topology in Ω, and if $V\subset\Omega$ then $\partial V\cap\partial\Omega$ is disjoint from $\partial_{\mathfrak f}V$ as the latter is included in Ω.

Lemma A.9. *Let V be a proper nonempty regular fine domain in Ω, which itself is regular within the ambient Riemann surface ($\mathcal{R}_*$ or $\mathbb{C}$). Then it holds for $z\in V$ that*

$$\int d\delta_z^{\Omega\setminus V}\begin{cases} =1 & \text{if } \overline{V}\cap\partial\Omega=\varnothing,\\ <1 & \text{if } \overline{\partial_{\mathfrak f}V}\cap\partial\Omega=\varnothing \text{ and } \overline{V}\cap\partial\Omega\neq\varnothing.\end{cases}\tag{A.38}$$

Moreover, if either condition on the right-hand side of (A.38) *holds, then for any harmonic function h on Ω one has*

$$h(z)=\int h d\delta_z^{\Omega\setminus V},\qquad z\in V,\tag{A.39}$$

provided that $|h|$ is majorized on V by a semi-bounded potential in Ω when $\overline{V}\cap\partial\Omega\neq\varnothing$.

Proof. Let $K\subset\Omega$ be non-polar and compact. Set, for brevity, $g_K:=\mathrm{cap}_\Omega(K)g(\mu_{\Omega,K},\Omega;\cdot)$, where $\mu_{\Omega,K}$ indicates, as in Section A.4, the Green equilibrium distribution on K. Since $\mu_{\Omega,K}$ has finite energy, g_K is semi-bounded. As $\partial\Omega$ is regular in the ambient Riemann surface, g_K extends continuously by zero to $\partial\Omega$, see Section A.7. Since $\mathrm{cap}_\Omega(K)>0$, it holds that $g_K\leqslant 1$ in Ω, see the paragraph after (A.5), and $g_K=1$ on $b(K)$ because $g_K=\mathcal{B}_1^K$, see Section A.9. Moreover, $g_K<1$ in each connected component U of $\Omega\setminus K$ such that $\partial U\cap\partial\Omega\neq\varnothing$ by the maximum principle for harmonic functions.

When $\overline{V}$ is compactly included in Ω, we may put $K:=\overline{V}$ in what precedes, and then $\mathrm{cap}_\Omega(K)>0$ as otherwise V itself would be polar and therefore empty, since it is finely open. The infimum of g_K on K is attained by lower semi-continuity, and it is strictly positive because nonzero Green potentials are never zero. Therefore, if h is harmonic on Ω, the potential cg_K majorizes $|h|$ on K for sufficiently large $c>0$. The uniqueness part of Theorem A.8 now implies that $h_{\lfloor V}$ is the solution of the fine Dirichlet problem with boundary data $h_{\lfloor\partial_{\mathfrak f}V}$. Hence, (A.39) is just (A.37) while the upper equality in (A.38) follows by taking $h\equiv 1$.

Assume next that $\overline{V}\cap\partial\Omega\neq\varnothing$ and $\overline{\partial_{\mathfrak f}V}\cap\partial\Omega=\varnothing$. Then $K:=\overline{\partial_{\mathfrak f}V}$ is a compact subset of Ω which is non-polar, for if $\partial_{\mathfrak f}V$ were polar, then either V or $\Omega\setminus V$ would be polar [19, Theorem 2] and V would be either empty or irregular, a contradiction. Note that a subdomain of $\Omega\setminus K$ is also a fine domain [19, Theorem 2] and thus, if it contains both a point in V and a point in $\Omega\setminus V$, then it must contain a point in $\partial_{\mathfrak f}V$ which is impossible by the definition of K. Hence, $V\setminus K$ is Euclidean open.

Let U be a connected component of $V\setminus K$ such that $\partial U\cap\partial\Omega\neq\varnothing$; it exists because $\overline{V}\cap\partial\Omega\neq\varnothing$. Since $\partial_{\mathfrak f}V=b(\partial_{\mathfrak f}V)\subseteq b(K)$ by assumption, $g_K=1$ on $\partial_{\mathfrak f}V$ (see the beginning of the proof). Hence, as g_K is a semi-bounded potential on Ω, it follows from Theorem A.8 that $h_{\mathbf 1}\leqslant g_K$ in V, where $h_{\mathbf 1}(z)$ is the solution of the fine Dirichlet problem on V with boundary data identically 1 on $\partial_{\mathfrak f}V$, see (A.37). In particular, $h_{\mathbf 1}\leqslant g_K<1$ in U by the maximum principle for harmonic functions.

Let $F\subset U$ be a closed disk. Observe that $V\setminus F$ is a fine domain as otherwise F would finely disconnect U whereas $U\setminus F$ is a domain and therefore also a fine domain. Put $\delta_z^*:=\delta_z^{(\Omega\setminus V)\cup F}$, $z\in V\setminus F$, and observe that $\delta^*_{z\lfloor\partial_{\mathfrak f}F}$ is a non-trivial measure by (A.27) and since $\partial_{\mathfrak f}((\Omega\setminus V)\cup F)=\partial_{\mathfrak f}V\cup\partial_{\mathfrak f}F$, where the union is disjoint and $b(\partial_{\mathfrak f}F)=\partial_{\mathfrak f}F=\partial F$. We now get from (A.23) and (A.27), applied to the fine domain $V\setminus F$, that

$$\delta_z^{\Omega\setminus V}=\left(\delta_z^*\right)^{\Omega\setminus V}=\left(\delta^*_{z\lfloor\partial_{\mathfrak f}V}\right)^{\Omega\setminus V}+\left(\delta^*_{z\lfloor\partial_{\mathfrak f}F}\right)^{\Omega\setminus V}=\delta^*_{z\lfloor\partial_{\mathfrak f}V}+\left(\delta^*_{z\lfloor\partial_{\mathfrak f}F}\right)^{\Omega\setminus V},\tag{A.40}$$

where we observe that the balayage out of V does not change measures supported on $b(\partial_{\mathfrak f}V)=\partial_{\mathfrak f}V$ by (A.22). Since $h_{\mathbf 1}<1$ on $\partial_{\mathfrak f}F$ as $\partial_{\mathfrak f}F\subset U$, we get from (A.24) and (A.26) that

$$0<\left(\delta^*_{z\lfloor\partial_{\mathfrak f}F}\right)^{\Omega\setminus V}(\partial_{\mathfrak f}V)=\int\delta_x^{\Omega\setminus V}(\partial_{\mathfrak f}V)d\delta^*_{z\lfloor\partial_{\mathfrak f}F}(x)=\int h_{\mathbf 1}(x)d\delta^*_{z\lfloor\partial_{\mathfrak f}F}(x)<\delta^*_{z\lfloor\partial_{\mathfrak f}F}(\partial_{\mathfrak f}F)$$

which implies, in view of (A.40), that $h_{\mathbf{1}}(z) = \delta_z^{\Omega\setminus V}(\partial_{\mathsf{f}} V) < \delta_z^*((\Omega\setminus V)\cup F) \leqslant 1$ as claimed.

Finally, let h be a harmonic function on Ω which is majorized on V in absolute value by a semi-bounded potential. Because h is also finely harmonic on V and finely continuous on $\mathrm{clos}_{\mathsf{f}}(V)$, we deduce from Theorem A.8 that it is the solution of the fine Dirichlet problem on V with boundary data $h_{\lfloor \partial_{\mathsf{f}} V}$ and that (A.39) holds. □

Lemma A.10. *Let $V\subset\mathcal{R}$ be a proper regular fine domain such that $\overline{\partial_{\mathsf{f}} V}\cap\partial\mathcal{R}=\varnothing$ and $\overline{V}\cap\partial\mathcal{R}\neq\varnothing$ (the lower assumption on the right-hand side of (A.38) when $\Omega=\mathcal{R}$). Let further h be a harmonic function in $\mathcal{R}$ such that $\lim_{z\to\xi} h(z)=0$ for every $\xi\in\overline{V}\cap\partial\mathcal{R}$. Then identity (A.39) holds.*

Proof. In view of Lemma A.9, it is enough to show that h is majorized on V by a semi-bounded potential. Note, as in the proof of Lemma A.9, that $\partial_{\mathsf{f}} V$ is non-polar. Let us show that $\partial\mathcal{R}\cap\overline{V}$ consists of a union of connected components of $\partial\mathcal{R}$. Indeed, any such component Γ is a 1-dimensional compact topological submanifold of $\mathcal{R}_*$, and as such it has a tubular neighborhood N that may be chosen so thin that $N\cap\overline{\partial_{\mathsf{f}} V}=\varnothing$. Then, if $\zeta_1,\zeta_2\in\Gamma$ and $\zeta_1\in\overline{V}$ while $\zeta_2\notin\overline{V}$, we can find $z_1\in N\cap V$ close to ζ_1 and $z_2\in N\cap\Omega\setminus V$ close to ζ_2. The points z_1,z_2 can be joined by a smooth arc contained in N. However, such an arc is finely connected [19, Theorem 7], but cannot meet $\overline{\partial_{\mathsf{f}} V}$ by construction, a contradiction that proves our claim.

Assume first that $D=\mathbb{D}$ is the unit disk. Any function u harmonic in an annular region $\{r<|z|<1\}$ that extends continuously to $\mathbb{T}$ by zero can be harmonically extended to $\{r<|z|<1/r\}$ by reflection, i.e., by setting $u(z):=-u(1/\bar z)$ for $z\in\{1<|z|<1/r\}$. Due to the smoothness of this extension it necessarily holds that $|u(z)|\leqslant C_\rho(1-|z|)$ for $r<\rho\leqslant|z|\leqslant1$. As h is harmonic on $\mathcal{R}$, this principle used around each of the finitely many connected components of $\overline{V}\cap\partial\mathcal{R}$ yields that $|h(z)|\leqslant C(1-|p(z)|)$ for $z\in V$ and some constant $C>0$. On the other hand, the function $g_r(z):=-\log\max\{r,|z|\}$ is a continuous (thus bounded and therefore semi-bounded) potential in $\mathbb{D}$ for any $r\in(0,1)$ (this is the Green equilibrium potential of $\{|z|\leqslant r\}$). It can be readily verified that $g_r(z)\geqslant(1-|z|)$ in $\mathbb{D}$ when $r\leqslant e^{-1}$. Thus, $|h(z)|\leqslant Cg_r(p(z))$, $z\in V$, for any such r. As $g_r(p(z))$ is a (bounded) potential on $\mathcal{R}$ by (A.34), the claim of the corollary follows.

In the general case, let $\phi:\mathbb{D}\to D$ be a conformal map. Recall that ϕ extends to a homeomorphism from $\overline{\mathbb{D}}$ onto $\overline{D}$. Clearly, ϕ is also a homeomorphism for the fine topology since v is superharmonic (resp. harmonic) on D if and only if so is $v\circ\phi$ on $\mathbb{D}$. Moreover, g is a bounded potential on D if and only if $g\circ\phi$ is such a potential on $\mathbb{D}$. Hence, the result just proven on the disk carries over to D by conformal mapping. □

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