ON AN IDENTITY BY ERCOLANI, LEGA, AND TIPPINGS

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ABSTRACT. In this note we prove that

$$j!2^N\binom{N+j-1}{j}{}_2F_1\left(\begin{matrix}-j,-2j\\-N-j+1\end{matrix};-1\right)=\sum_{l=0}^N\binom{N}{l}\prod_{i=0}^{j-1}2(2i+1+l),$$

where N and j are positive integers, which resolves a question posed by Ercolani, Lega, and Tippings.

In [2, Theorem 2.1], Ercolani, Lega, and Tippings have shown that the number of 2ν-valent maps with j vertices and two legs that can be embedded in a surface of genus $g\geq 1$ is given by

$$j!\left[2\nu(\nu-1)\binom{2\nu-1}{\nu-1}\right]^j\sum_{l=0}^{3g-1}a_l(g,\nu)\binom{2g-2+l+j}{j}{}_2F_1\left(\begin{matrix}-j,-\nu j\\2-2g-l-j\end{matrix};\frac{1}{1-\nu}\right)$$

for some coefficients $a_l(g,\nu)$. Based on numerical computations, they subsequently conjectured, see [1, Conjecture 5.1], that when $\nu=2$ the above expression can be stated without hypergeometric functions with the help of identity (1) further below. In this note we provide a proof of this fact.

Theorem 1. *Let N,j be positive integers. Then*

$$j!2^N\binom{N+j-1}{j}{}_2F_1\left(\begin{matrix}-j,-2j\\-N-j+1\end{matrix};-1\right)=\sum_{l=0}^N\binom{N}{l}\prod_{i=0}^{j-1}2(2i+1+l). \tag{1}$$

We use the following notation for the falling and rising factorials:

$$(a)^{\underline{0}}=(a)^{\overline{0}}:=1,\quad (a)^{\underline{n}}:=a(a-1)\cdots(a-n+1),\quad\text{and}\quad (a)^{\overline{n}}:=a(a+1)\cdots(a+n-1)$$

for $n\geq 1$. We prove (1) in three steps formulated as separate lemmas.

Lemma 2. *The right-hand side of (1) is equal to $2^NR_j(N)$, where $R_j(x)=\sum_{i=0}^jR(i,j)(x)^{\underline{i}}$ and $R(i,j)$ are numbers uniquely determined by the relations*

$$R(0,j)=2^j(2j-1)!!,\quad R(j,j)=1,\quad R(i,j+1)=2(2j+i+1)R(i,j)+R(i-1,j),$$

with the recurrence relation holding for $i\in\{1,2,\dots,j\}$.

Proof. Let $C_{i,j}$ be the following coefficients:

$$\prod_{i=0}^{j-1}(2i+1+x)=\sum_{k=0}^jC_{k,j}x^k.$$

Since

$$\prod_{i=0}^{j}(2i+1+x)=(2j+1+x)\sum_{k=0}^jC_{k,j}x^k=\sum_{k=0}^j(2j+1)C_{k,j}x^k+\sum_{k=1}^{j+1}C_{k-1,j}x^k,$$

it holds that

$$C_{0,j}=(2j-1)!!,\quad C_{j,j}=1,\quad C_{k,j+1}=(2j+1)C_{k,j}+C_{k-1,j},\ \ k\in\{1,2,\dots,j\}.$$

Exactly as in the case of the coefficients $R(i,j)$, the coefficients $C_{k,j}$ are uniquely defined by the above relations since the knowledge of all the coefficients on the level j allows one to compute all the coefficients on the level $j+1$ with the base case $C_{0,1}=2$ and $C_{1,1}=1$. Recall [3, Equation (26.8.10)] that

$$x^k=\sum_{i=1}^k\left\{\begin{matrix}k\\i\end{matrix}\right\}(x)^{\underline{i}},$$

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where $\left\{ {k \atop i} \right\}$ are Stirling numbers of the second kind. Therefore,

$$\prod_{i=0}^{j-1}(2i+1+x) = C_{0,j} + \sum_{i=1}^{j}\left(\sum_{k=i}^{j} C_{k,j}\left\{ {k \atop i} \right\}\right)(x)^{\underline{i}}.$$

Observe that

$$\begin{aligned}\sum_{l=0}^{N}\binom{N}{l}(l)^{\underline{i}} &= \sum_{l=0}^{N}\frac{L!}{l!(N-l)!}(l)(l-1)\cdots(l-i+1) \\ &= \sum_{l=i}^{N}\frac{N!}{(l-i)!(N-l)!} = \sum_{l=0}^{N-i}\frac{N!}{l!(N-i-l)!} = 2^{N-i}(N)^{\underline{i}}.\end{aligned}$$

Therefore, the right-hand side of (1) is equal to

$$2^j\sum_{l=0}^{N}\binom{N}{l}\left(C_{0,j} + \sum_{i=1}^{j}\left(\sum_{k=i}^{j} C_{k,j}\left\{ {k \atop i} \right\}\right)(l)^{\underline{i}}\right) = 2^N R_j(N),$$

where $R_j(x) := \sum_{i=0}^{j} R(i,j)(x)^{\underline{i}}$ with

$$R(0,j) := 2^j(2j-1)!! \quad \text{and} \quad R(i,j) := 2^{j-i}\sum_{k=i}^{j} C_{k,j}\left\{ {k \atop i} \right\}, \; i \in \{1,2,\ldots,j\}.$$

Since $\left\{ {j \atop j} \right\} = 1$, see [3, Equation (26.8.4)], it indeed holds that $R(j,j) = 1$. Thus, we only need to establish the recurrence relation. The recurrence relation for $C_{k,j}$ yields that

$$\begin{aligned}R(i,j+1) &= 2^{j+1-i}\sum_{k=i}^{j+1} C_{k,j+1}\left\{ {k \atop i} \right\} = 2^{j+1-i}\sum_{k=i}^{j} C_{k,j+1}\left\{ {k \atop i} \right\} + 2^{j+1-i}\left\{ {j+1 \atop i} \right\} \\ &= 2(2j+1)R(i,j) + 2^{j+1-i}\sum_{k=i}^{j+1} C_{k-1,j}\left\{ {k \atop i} \right\}\end{aligned}$$

for any $i \in \{1,2,\ldots,j\}$, where we also used the fact that $C_{j,j} = 1$. Furthermore, we get from [3, Equation (26.8.22)] that

$$\begin{aligned}2^{j+1-i}\sum_{k=i}^{j+1} C_{k-1,j}\left\{ {k \atop i} \right\} &= 2^{j+1-i}\sum_{k=i-1}^{j} C_{k,j}\left\{ {k+1 \atop i} \right\} = 2^{j+1-i}\sum_{k=i-1}^{j} C_{k,j}\left(i\left\{ {k \atop i} \right\} + \left\{ {k \atop i-1} \right\}\right) \\ &= 2iR(i,j) + R(i-1,j),\end{aligned}$$

where, by convention, $\left\{ {i-1 \atop i} \right\} = 0$. This finishes the proof of the lemma. □

Lemma 3. *The left-hand side of* (1) *is equal to* $2^N L_j(N)$, *where* $L_j(x) = \sum_{i=0}^{j} L(i,j)(x)^{\underline{i}}$ *with*

$$L(0,j) = \frac{(2j)!}{j!} \quad \text{and} \quad L(i,j) = \frac{j!}{i!}\sum_{k=i}^{j}\binom{2j}{j+k}\binom{k-1}{i-1}, \quad i \in \{1,2,\ldots,j\}.$$

Proof. It follows from [3, Equation (15.2.1)] that the left-hand side of (1) is equal to $2^N L_j(N)$, where

$$L_j(x) = (x)^{\overline{j}}\sum_{k=0}^{j}\frac{(-j)^{\overline{k}}(-2j)^{\overline{k}}}{(-x-j+1)^{\overline{k}}}\frac{(-1)^k}{k!}.$$

Since

$$(-a)^{\overline{k}} = (-1)^k(a)^{\underline{k}} \quad \text{and} \quad \frac{(a)^{\overline{j}}}{(a+j-1)^{\underline{k}}} = (a)^{\overline{j-k}},$$

we have that

$$\begin{aligned}L_j(x) &= \sum_{k=0}^{j}(j)^{\underline{k}}(2j)^{\underline{k}}\frac{(x)^{\overline{j-k}}}{k!} = \sum_{k=0}^{j}\frac{j!}{(j-k)!}\frac{(2j)!}{(2j-k)!}\frac{(x)^{\overline{j-k}}}{k!} \\ &= \sum_{k=0}^{j}\binom{j}{k}\frac{(2j)!}{(2j-k)!}(x)^{\overline{j-k}} = \sum_{k=0}^{j}\binom{j}{k}\frac{(2j)!}{(j+k)!}(x)^{\overline{k}}.\end{aligned}$$

It is known [4] that

$$(x)^{\overline{k}} = \sum_{i=1}^{k} \binom{k-1}{i-1} \frac{k!}{i!} (x)^{\underline{i}}, \quad k \geq 1.$$

Hence,

$$\begin{aligned} L_j(x) &= \frac{(2j)!}{j!} + \sum_{k=1}^{j} \binom{j}{k} \frac{(2j)!}{(j+k)!} \sum_{i=1}^{k} \binom{k-1}{i-1} \frac{k!}{i!} (x)^{\underline{i}} \\ &= \frac{(2j)!}{j!} + \sum_{i=1}^{j} \left(\sum_{k=i}^{j} \binom{j}{k} \frac{(2j)!}{(j+k)!} \binom{k-1}{i-1} \frac{k!}{i!} \right) (x)^{\underline{i}}, \end{aligned}$$

which finishes the proof of the lemma. □

Lemma 4. *It holds that $L(i,j) = R(i,j)$ for all $i \in \{0,1,\dots,j\}$ and $j \geq 1$. In particular, (1) is true.*

Proof. Clearly, $L(0,j) = R(0,j)$ and $L(j,j) = R(j,j) = 1$. Thus, we only need to show that

$$L(i,j+1) = 2(2j+i+1)L(i,j) + L(i-1,j), \quad i \in \{1,2,\dots,j\},$$

since this recurrence relation and the marginals $L(0,j)$, $L(j,j)$ uniquely determine the whole table $L(i,j)$. In what follows, we agree that binomial coefficients with out-of-range indices are set to zero. In what follows we repeatedly use the elementary identity

$$\binom{n+1}{l} = \binom{n}{l} + \binom{n}{l-1}.$$

Using this identity twice and our convention concerning binomial coefficients with out-of-range indices, we get that

$$\begin{aligned} L(i,j+1) &= 2(j+1)L(i,j) + \frac{(j+1)!}{i!} \sum_{k=i}^{j+1} \left[\binom{2j}{j+k+1} + \binom{2j}{j+k-1} \right] \binom{k-1}{i-1} \\ &= 2(j+1)L(i,j) + \frac{(j+1)!}{i!} \left[\sum_{k=i+1}^{j} \binom{2j}{j+k} \binom{k-2}{i-1} + \sum_{k=i-1}^{j} \binom{2j}{j+k} \binom{k}{i-1} \right], \end{aligned}$$

where the second row is obtained simply by a change of summation indices. Therefore, when $i = 1$, we get that

$$\begin{aligned} L(1,j+1) &= 4(j+1)L(1,j) + (j+1)! \left[\binom{2j}{j} - \binom{2j}{j+1} \right] \\ &= 4(j+1)L(1,j) + L(0,j)[j+1-j], \end{aligned}$$

which establishes the desired recurrence relation for $i = 1$. On the other hand, when $i > 1$, the above sum in square brackets can further be rewritten with the help of the elementary identity as

$$\begin{aligned} \sum_{k=i+1}^{j} \binom{2j}{j+k} \left[\binom{k-1}{i-1} - \binom{k-2}{i-2} \right] + \sum_{k=i-1}^{j} \binom{2j}{j+l} \left[\binom{k-1}{i-1} + \binom{k-1}{i-2} \right] = \\ 2\sum_{k=i}^{j} \binom{2j}{j+k} \binom{k-1}{i-1} + \sum_{k=i-1}^{j} \binom{2j}{j+k} \left[\binom{k-1}{i-2} - \binom{k-2}{i-2} \right]. \end{aligned}$$

Thus, using the elementary identity once more, we get that

$$L(i,j+1) = 4(j+1)L(i,j) + \frac{(j+1)!}{i!} \sum_{k=i-1}^{j} \binom{2j}{j+k} \binom{k-2}{i-3},$$

where we understand that $\binom{i-3}{i-3} = 1$ even if $i = 2$. That is, to prove the lemma we need to show that

$$2(i-1)L(i,j) + L(i-1,j) = \frac{(j+1)!}{i!} \sum_{k=i-1}^{j} \binom{2j}{j+k} \binom{k-2}{i-3},$$

or equivalently that

$$\sum_{k=i-1}^{j} \binom{2j}{j+k} \left[2(i-1)\binom{k-1}{i-1} + i\binom{k-1}{i-2} - (j+1)\binom{k-2}{i-3} \right] = 0.$$

When $i=2$, the left-hand side above becomes

$$\sum_{k=2}^{j} 2k\binom{2j}{j+k} - (j-1)\binom{2j}{j+1} = \sum_{k=2}^{j}(j+k)\binom{2j}{j+k} - \sum_{k=1}^{j-1}(j-k)\binom{2j}{j+k}.$$

Similarly, when $i>2$, the left-hand side in question is equal to

$$\sum_{k=i-1}^{j}\binom{2j}{j+k}\frac{(k-2)!}{(k-i+1)!(i-2)!}\big[2(k-1)(k-i)+1+i(k-1)-(j+1)(i-2)\big] =$$
$$\sum_{k=i-1}^{j}\binom{2j}{j+k}\frac{(k-2)!}{(k-i+1)!(i-2)!}\big[(k-i+1)(j+k)-(j-k)(k-1)\big],$$

which is the same as

$$\sum_{k=i}^{j}(j+k)\binom{2j}{j+k}\binom{k-2}{i-2} - \sum_{k=i-1}^{j-1}(j-k)\binom{2j}{j+k}\binom{k-1}{i-2}$$

(as shown just before, this formula is valid for $i=2$ as well). This difference is indeed equal to zero as the second sum above can be rewritten as

$$\sum_{k=i-1}^{j-1}(j+k+1)\binom{2j}{j+k+1}\binom{k-1}{i-2} = \sum_{l=i}^{j}(j+l)\binom{2j}{j+l}\binom{l-2}{i-2}. \qquad \square$$

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