

NON-HERMITIAN ORTHOGONAL POLYNOMIALS ON A TREFOIL

AHMAD B. BARHOUMI AND MAXIM L. YATTSELEV

ABSTRACT. We investigate asymptotic behavior of polynomials $Q_n(z)$ satisfying non-Hermitian orthogonality relations

$$\int_{\Delta} s^k Q_n(s) \rho(s) ds = 0, \quad k \in \{0, \dots, n-1\},$$

where Δ is a Chebotarëv (minimal capacity) contour connecting three non-collinear points and $\rho(s)$ is a Jacobi-type weight including a possible power-type singularity at the Chebotarëv center of Δ .

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1. INTRODUCTION

Let $f(z) = \sum_{k=1}^{\infty} f_k z^{-k}$ be a convergent power series. The n -th diagonal Padé approximant of $f(z)$ is a rational function $[n/n]_f(z) = P_n(z)/Q_n(z)$, where $\deg(P_n), \deg(Q_n) \leq n$ and $Q_n(z)$ is not identically zero, such that the linearized error function $R_n(z)$ satisfies

$$(1.1) \quad R_n(z) := (Q_n f - P_n)(z) = O\left(z^{-(n+1)}\right)$$

as $z \rightarrow \infty$. One can readily check that the above condition is nothing but a system of linear equations on the coefficients of the polynomials of $P_n(z), Q_n(z)$ that always has a non-trivial solution. This system is not necessarily uniquely solvable, but it is known that the rational function $[n/n]_f(z)$ is indeed unique. Hereafter, we shall understand that $Q_n(z)$ in (1.1) is monic and of minimal possible degree, which does make it unique.

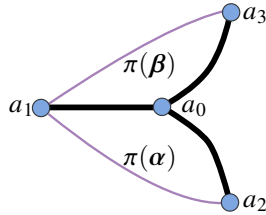
Our goal is to understand convergence properties of $[n/n]_f(z)$. It was shown by Herbert Stahl [23, 24, 25] that if $f(z)$ can be meromorphically continued along any path $\overline{\mathbb{C}} \setminus A$ for a polar set A and there exists a point in $\mathbb{C} \setminus A$ with at least two distinct continuations, then there exists a compact set Δ_f such that $f(z)$ has a single-valued meromorphic continuation into the complement of Δ_f (which is a branch cut for $f(z)$) and the diagonal Padé approximants converge to this continuation in logarithmic capacity. The set Δ_f is uniquely characterized as the branch cut of smallest logarithmic capacity that is set-theoretically minimal (if K is a branch cut with the same logarithmic capacity, then $\Delta_f \subseteq K$).

The minimal capacity contour Δ_f can also be characterized from the point of view of quadratic differentials. Assume for simplicity that $A = \{a_1, \dots, a_m\}$, $m \geq 2$, is a finite set. Suppose that every element of A is a branch point of $f(z)$. Then there exist auxiliary points $b_i, i \in \{1, \dots, m-2\}$, sometimes called Chebotarëv centers¹, which are not necessarily distinct nor disjoint from the elements of A , such that the set Δ_f consists of the critical trajectories of a rational quadratic differential

$$\frac{\prod_{i=1}^{m-2} (z - b_i)}{\prod_{i=1}^m (z - a_i)} dz^2.$$

Suppose that all the points b_i are disjoint from A and that each b_i appears either once or an even number of times and in the latter case does not belong to Δ_f (a generic situation). Assume further that $f(z)$ has either logarithmic or power branching at each a_i (i.e., behaves like $\log(z - a_i)$ or $(z - a_i)^{\alpha_i}$, $\alpha_i > -1$). Then the strong asymptotics of the corresponding diagonal Padé approximants was investigated by Aptekarev and the second author in [2]. Our overarching goal is to remove all the assumptions on contours Δ_f corresponding to finite sets A . As will become clear later, the main difficulty lies in the local analysis of the polynomials $Q_n(z)$ around the auxiliary points b_i . The first step in this direction was taken by the authors in [4] where we considered the case $m = 4$ and $b_1 = b_2 \in \Delta_f \setminus A$. Here, we consider that case $m = 4$ and $b_1 = b_2 \in A$.

¹In a somewhat different language, Chebotarëv posed a problem of finding a connected set of minimal logarithmic capacity containing a given finite set of points; descriptions of this set were independently given by Grötzsch [14] and Lavrentiev [19, 20].

FIGURE 1. Contour Δ .

Let us now change the notation slightly. Given three distinct non-collinear points a_1, a_2, a_3 , denote by a_0 their Chebotarëv center. That is, there are three disjoint, except for a_0 , analytic arcs, say $\Delta_1, \Delta_2, \Delta_3$ (Δ_i has endpoints a_i and a_0) such that

$$(1.2) \quad \frac{z(t) - a_0}{(z(t) - a_1)(z(t) - a_2)(z(t) - a_3)} (z'(t))^2 < 0$$

for any smooth parametrization $z(t)$ of any of the arcs Δ_i . Let $\Delta := \Delta_1 \cup \Delta_2 \cup \Delta_3$ (in our preceding notation $A = \{a_0, a_1, a_2, a_3\}$, $b_1 = b_2 = a_0$, and $\Delta_f = \Delta$), see Figure 1. Examples of functions $f(z)$ that lead to such minimal capacity contours include

$$(1.3) \quad \prod (z - a_i)^{\alpha_i} \quad \text{and} \quad \sum c_i \log(z - a_i),$$

where $\alpha_i \notin \mathbb{Z}$ while $\sum_{i=0}^4 \alpha_i \in \mathbb{Z}$ and $c_i \neq 0$ while $\sum_{i=0}^4 c_i = 0$. Let us point out that if $f(z)$ was given by either of the expressions above, but with $\alpha_0 = c_0 = 0$, then the asymptotics of the diagonal Padé approximants to $f(z)$ was obtained in [2], see also [21]. Moreover, asymptotics of the approximants for the case $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = -1/2$ is contained [27], see also [3, 26]. The reader might want to consult the Appendix D to see how these functions relate to the definition below.

In this work, we shall consider the following class of functions. Orient each arc Δ_i towards a_0 . Assume that the points a_1, a_2, a_3 are labeled counter-clockwise around a_0 .

Definition. We are interested in functions $f(z)$ of the form

$$(1.4) \quad f(z) = \frac{1}{2\pi i} \int_{\Delta} \frac{\rho(s) ds}{s - z},$$

where there exist exponents $\alpha_i > -1$, $i \in \{0, 1, 2, 3\}$, and branches of $(z - a_i)^{\alpha_i}$ holomorphic across $\Delta \setminus \{a_i\}$ for which the restriction $\rho_i(s)$ of $\rho(s)$ to $\Delta_i^\circ := \Delta_i \setminus \{a_0, a_i\}$ is such that $\rho_i(s)(s - a_0)^{-\alpha_0}(s - a_i)^{-\alpha_i}$ extends to a holomorphic and non-vanishing function in some neighborhood of Δ_i .

Notice that $\rho(s)$ is not defined at a_0 even if $\alpha_0 = 0$. However, in the latter case the values $\rho_i(a_0)$ are well-defined. It was assumed in [2] that $\rho_1(z) + \rho_2(z) + \rho_3(z) \equiv 0$ in some neighborhood of a_0 (no branching assumption). No such supposition is made here, neither in some neighborhood of a_0 nor at a_0 itself. The main advantage of the class of functions introduced in (1.4) is that the denominator polynomials $Q_n(z)$ can be equivalently characterized as non-Hermitian orthogonal polynomials satisfying

$$(1.5) \quad \int_{\Delta} s^k Q_n(s) \rho(s) ds = 0, \quad k \in \{0, \dots, n-1\}.$$

This paper is organized as follows. The next section is devoted to the description of the main term of the asymptotics of the polynomials $Q_n(z)$. The functions constructed in that section are known in integrable systems literature as Baker-Akhiezer functions and in the literature on non-Hermitian orthogonal polynomials are sometimes called Nuttall-Szegő functions. The propositions stated in the next section are proven in Section 4. Our main results on the asymptotics of the polynomials $Q_n(z)$ are stated in Section 3. Their proofs are given in Sections 5 and 6. As it happens, local behavior of the polynomials $Q_n(z)$ around a_0 is described by certain special functions that come from a 2×2 matrix function solving the so-called Painlevé XXXIV Riemann-Hilbert problem with Stokes parameters that depend on the weight $\rho(s)$ in a transcendental way. This connection is described

in Appendices A and B. Appendix C illustrates by an example that on the rotationally symmetric contour Δ asymptotics of $Q_n(z)$ can be very different for certain classes of weights from the rest of the cases.

2. NUTTALL-SZEGŐ FUNCTIONS

Similarly to orthogonal polynomials on an interval, the asymptotics of the polynomials $Q_n(z)$ is described by the term that captures the geometric rate of their growth, see (2.4), and a Szegő function of the weight $\rho(s)$, see (2.11). Both of these functions naturally live on a genus one Riemann surface associated with the contour Δ , see (2.1). However, the nature of the theory of functions on a Riemann surface leads to the introduction of the third term, in fact, a normal family of functions that depends on n , which are essentially ratios of the Riemann theta functions, see (2.17) (the necessity of these terms was already demonstrated by Akhiezer [1]).

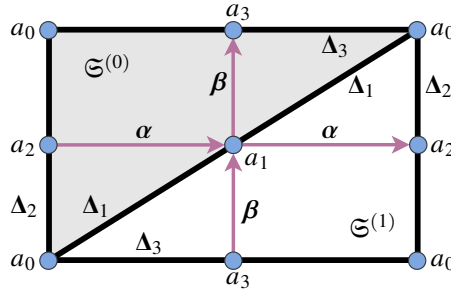


FIGURE 2. Surface \mathfrak{S} .

2.1. Riemann surface. Let \mathfrak{S} be a genus one two-sheeted Riemann surface defined by

$$(2.1) \quad \mathfrak{S} := \{z = (z, w) : w^2 = (z - a_0)(z - a_1)(z - a_2)(z - a_3)\}.$$

We realize \mathfrak{S} as a ramified cover of $\overline{\mathbb{C}}$ constructed in the following manner. Two copies of the extended complex plane are cut along each arc Δ_i° . These copies are glued together along the cuts in such a manner that the right (resp. left) side of the arc Δ_i° belonging to one copy is joined with the left (resp. right) side of the same arc Δ_i° only belonging to the other copy. Let

$$\pi, w : \mathfrak{S} \rightarrow \overline{\mathbb{C}}, \quad \pi(z) \mapsto z \quad \text{and} \quad w(z) \mapsto w.$$

We call π the canonical projection and label the sheets $\mathfrak{S}^{(0)}$ and $\mathfrak{S}^{(1)}$ (cut copies of the extended complex plane) so that $w(z) = w$ behaves like z^2 as $z \rightarrow \infty^{(0)}$. In our definition, the sheets are topologically closed and their intersection is equal to Δ , where

$$\Delta := \pi^{-1}(\Delta), \quad \Delta_i := \pi^{-1}(\Delta_i), \quad \text{and} \quad \mathbf{A} := \{a_0, a_1, a_2, a_3\}, \quad \pi(a_l) = a_l,$$

$i \in \{1, 2, 3\}$ and $l \in \{0, 1, 2, 3\}$. Thus, \mathbf{A} is the set of the ramification points of \mathfrak{S} and Δ is a union of three simple cycles that all intersect each other at a_0 . We orient each Δ_i so that $\mathfrak{S}^{(0)}$ remains on the left when Δ_i is traversed in the positive direction. We write

$$z^{(k)} := \pi^{-1}(z) \cap \mathfrak{S}^{(k)}, \quad z \in \overline{\mathbb{C}} \setminus \Delta,$$

where $k \in \{0, 1\}$, and use bold lower case letters such as $\mathbf{z}, \mathbf{t}, \mathbf{s}$ to indicate generic points on \mathfrak{S} with canonical projections z, t, s . We designate the symbol \cdot^* to stand for the conformal involution that sends $z^{(k)}$ into $z^{(1-k)}$, $k \in \{0, 1\}$ (we then extend this notion to Δ by continuity). Since \mathfrak{S} has genus 1, we need to select a homology basis. We choose cycles α, β to be involution-symmetric, i.e., $s \in \alpha$ if and only if $s^* \in \alpha$, and such that $\pi(\alpha), \pi(\beta)$ are smooth Jordan arcs joining a_1, a_2 and a_1, a_3 respectively, while α is oriented towards a_1 and β away from a_1 in $\mathfrak{S}^{(0)}$, see Figures 1 and 2. In what follows it will be sometimes convenient to set

$$\mathfrak{S}_\alpha := \mathfrak{S} \setminus \alpha \quad \text{and} \quad \mathfrak{S}_{\alpha, \beta} := \mathfrak{S} \setminus (\alpha \cup \beta).$$

Finally, given a function $F(z)$ defined on $\mathfrak{S} \setminus \Delta$, or on its subset, we shall set $F(z) := F(z^{(0)})$ and $F^*(z) := F(z^{(1)})$. That is, with a slight abuse of notation we use the same letter for a function on the top sheet as well as its pull-back to the complex plane while we use the involution symbol $*$ to mark the pull-back to the complex plane from the bottom sheet. In particular,

$$(2.2) \quad w(z) = \sqrt{(z - a_0)(z - a_1)(z - a_2)(z - a_3)}, \quad z \in \mathbb{C} \setminus \Delta,$$

is the branch such that $w(z) = z^2 + \mathcal{O}(z)$ as $z \rightarrow \infty$ and $w^*(z) = -w(z)$.

As it happens, the asymptotics of the polynomials $Q_n(z)$ as well as of the linearized error functions $R_n(z)$, see (1.1), is described by the solution of the following boundary value problem on Δ .

Boundary Value Problem: BVP- Ψ . *Given $n \in \mathbb{N}$ and a function $\rho(s)$ on Δ as described after (1.4), find a function $\Psi(z)$ that is holomorphic in $\mathfrak{S} \setminus (\Delta \cup \{\infty^{(0)}\})$, has a pole of order n at $\infty^{(0)}$ and a zero of order at least $n - 1$ at $\infty^{(1)}$, and whose traces on Δ are continuous away from \mathbf{A} and satisfy*

$$(2.3) \quad \begin{cases} \Psi_+(s) = \Psi_-(s)/(\rho w_+)(s), & s \in \Delta, \\ |\Psi(z)| = \mathcal{O}(|z - a_l|^{-(2\alpha_l+1)/4}) & \text{as } z \rightarrow a_l, \quad l \in \{0, 1, 2, 3\}. \end{cases}$$

The solution of this boundary value problem is given in Theorem 1 further below. It is rather explicit and constructed as a product of three terms that are introduced in the following three subsections.

2.2. Geometric Term. Let

$$(2.4) \quad \Phi(z) := \exp \left\{ \int_{a_0}^z G \right\}, \quad G(z) := \frac{(z - a_0)dz}{w(z)},$$

for $z \in \mathfrak{S}_{\alpha, \beta}$, where the path of integration lies entirely in $\mathfrak{S}_{\alpha, \beta}$ and $G(z)$ is a meromorphic differential on \mathfrak{S} having two simple poles at $\infty^{(1)}$ and $\infty^{(0)}$ with respective residues 1 and -1 , whose period over any cycle on \mathfrak{S} is purely imaginary (the latter claim follows from (1.2)). The function $\Phi(z)$ is holomorphic and non-vanishing on $\mathfrak{S}_{\alpha, \beta}$ except for a simple pole at $\infty^{(0)}$ and a simple zero at $\infty^{(1)}$. Define (real) constants

$$(2.5) \quad \omega := -\frac{1}{2\pi i} \oint_{\beta} G \quad \text{and} \quad \tau := \frac{1}{2\pi i} \oint_{\alpha} G.$$

One can readily check that $\Phi(z)$ possesses continuous traces on both sides of each cycle of the canonical basis (in fact, it extends to a multiplicatively multi-valued function on the entire surface) that satisfy

$$(2.6) \quad \Phi_+(s) = \Phi_-(s) \begin{cases} e^{2\pi i \omega}, & s \in \alpha \setminus \{a_1\}, \\ e^{2\pi i \tau}, & s \in \beta \setminus \{a_1\}. \end{cases}$$

Since $w(z) = -w^*(z)$, it follows from (2.4) that $\Phi(z)\Phi^*(z) \equiv 1$. Moreover, since the constants in (2.5) are real, it also holds that $\log |\Phi(z)|$ is harmonic on $\mathfrak{S} \setminus \{\infty^{(0)}, \infty^{(1)}\}$. Furthermore, since Δ is the set of critical trajectories of the quadratic differential $-(z - a_0)^2 w^{-2}(z) dz^2$ and the domain $\overline{\mathbb{C}} \setminus \Delta$ contains a single critical point of this differential, namely, double pole at infinity, it follows from Basic Structure Theorem [16, Theorem 3.5] that $\overline{\mathbb{C}} \setminus \Delta$ is a circle domain for this differential, see [16, Definition 3.9]. In particular, it must be true that

$$(2.7) \quad -\log |\Phi^*(z)| = \log |\Phi(z)| > 0, \quad z \in \overline{\mathbb{C}} \setminus \Delta.$$

In particular, this means that $\log |\Phi(s)| \equiv 0$, $s \in \Delta$, and that $\log |\Phi(z)|$ is the Green function for $\overline{\mathbb{C}} \setminus \Delta$ with pole at infinity.

2.3. **Szegő Function.** Since \mathfrak{S} has genus one, it has one (up to a constant factor) holomorphic differential. Since \mathfrak{S} is a two-sheeted surface, it is a standard fact of the theory of Riemann surfaces that

$$(2.8) \quad \Omega(z) := \left(\oint_{\alpha} \frac{ds}{w(s)} \right)^{-1} \frac{dz}{w(z)}, \quad \mathbf{B} := \oint_{\beta} \Omega,$$

is the holomorphic differential on \mathfrak{S} normalized to have unit period on the α cycle of the homology basis (it was shown by Riemann that the constant \mathbf{B} has positive imaginary part). In particular, the function

$$(2.9) \quad a(z) := \int_{a_0}^z \Omega, \quad z \in \mathfrak{S}_{\alpha, \beta},$$

where the path of integration lies entirely in $\mathfrak{S}_{\alpha, \beta}$, extends to additively multi-valued analytic function on the entire surface \mathfrak{S} . Denote further by $\Omega_{z, z^*}(s)$ the meromorphic differential with two simple poles at z and z^* with respective residues 1 and -1 normalized to have a zero period on the α -cycle. When z does not lie on top of the point at infinity, it can be readily verified that

$$(2.10) \quad \Omega_{z, z^*}(s) = \frac{w(z)}{s-z} \frac{ds}{w(s)} - \left(\oint_{\alpha} \frac{w(z)}{t-z} \frac{dt}{w(t)} \right) \Omega(s).$$

The third kind differential $\Omega_{z, z^*}(s)$ can be thought of as a discontinuous Cauchy kernel on \mathfrak{S} , see [28]. The following proposition elaborates this point.

Proposition 1. *For a function $\rho(s)$ as described after (1.4) fix a branch of $\log(\rho w_+)(s)$ that is continuous on Δ_i° , $i \in \{1, 2, 3\}$. Put*

$$(2.11) \quad S_\rho(z) := \exp \left\{ -\frac{1}{4\pi i} \oint_{\Delta} \log(\rho w_+)(s) \Omega_{z, z^*}(s) \right\}.$$

Then $S_\rho(z)$ is a holomorphic and non-vanishing function in $\mathfrak{S}_\alpha \setminus \Delta$ with continuous traces on $(\Delta \cup \alpha) \setminus \mathbf{A}$ that satisfy

$$(2.12) \quad S_{\rho+}(s) = S_{\rho-}(s) \begin{cases} \exp\{2\pi i c_\rho\}, & s \in \alpha \setminus \mathbf{A}, \\ 1/(\rho w_+)(s), & s \in \Delta \setminus \mathbf{A}, \end{cases}$$

where the constant c_ρ is given by

$$(2.13) \quad c_\rho := \frac{1}{2\pi i} \oint_{\Delta} \log(\rho w_+) \Omega.$$

It also holds that $S_\rho(z)S_\rho(z^*) \equiv 1$, $z \in \mathfrak{S}_\alpha \setminus \Delta$, and

$$(2.14) \quad |S_\rho(z)| \sim |z - a_l|^{-(2\alpha_l+1)/4} \quad \text{as } z \rightarrow a_l, \quad l \in \{0, 1, 2, 3\}.$$

We call $S_\rho(z)$ the Szegő function of the weight $\rho(s)$. The Cauchy kernel $\Omega_{z, z^*}(s)$ is called discontinuous due to the presence of the jump of $S_\rho(z)$ across α . We prove Proposition 1 in Section 4.1.

2.4. **Adjustment Terms.** Due to the presence of the jumps of both $\Phi(z)$ and $S_\rho(z)$ across the cycles of the homology basis, we need to introduce an adjustment term that will cancel out these jumps. To this end, let

$$\theta(u) := \sum_{n \in \mathbb{Z}} \exp\{\pi i B n^2 + 2\pi i n u\}, \quad u \in \mathbb{C},$$

be the Riemann theta function associated with \mathbf{B} . Further, let $\text{Jac}(\mathfrak{S}) := \mathbb{C}/\{\mathbb{Z} + \mathbf{B}\mathbb{Z}\}$ be the Jacobi variety of \mathfrak{S} , where \mathbf{B} is given by (2.8). We shall represent elements of $\text{Jac}(\mathfrak{S})$ as equivalence classes $[s] = \{s + l + \mathbf{B}m : l, m \in \mathbb{Z}\}$, where $s \in \mathbb{C}$. It is known that Abel's map

$$(2.15) \quad z \in \mathfrak{S} \mapsto \mathbf{a}(z) \in \text{Jac}(\mathfrak{S}), \quad \mathbf{a}(z) := \left[\int_{a_0}^z \Omega \right],$$

is a bijection (clearly, it holds that $[a(z)] = \mathbf{a}(z)$ for $z \in \mathfrak{S}_{\alpha, \beta}$, see (2.9)). Inverting Abel's map (2.15) is known as the Jacobi inversion problem, see [7, Section III.6]. Since \mathfrak{S} has genus one, every Jacobi inversion problem has a unique solution.

Proposition 2. *Let ω, τ and c_ρ be as in (2.5) and (2.13), respectively. Set*

$$(2.16) \quad x_n := n\omega - [n\omega], \quad y_n := n\tau - [n\tau], \quad \text{and} \quad \mathbf{A} := (1 + \omega)/2 + \mathbf{B}(1 + \tau)/2,$$

where $[\cdot]$ is equal to the largest integer smaller or equal to its argument. Define

$$(2.17) \quad T_n(z) := \exp\{-2\pi i y_n \mathbf{a}(z)\} \frac{\theta(\mathbf{a}(z) - c_\rho - x_n - \mathbf{B}y_n + \mathbf{A})}{\theta(\mathbf{a}(z) + \mathbf{A})}, \quad z \in \mathfrak{S}_{\alpha, \beta}.$$

Each function $T_n(z)$ is meromorphic in the domain of its definition with continuous traces on $(\alpha \cup \beta) \setminus \{\mathbf{a}_1\}$ that satisfy

$$(2.18) \quad T_{n+}(s) = T_{n-}(s) \begin{cases} \exp\{-2\pi i(c_\rho + n\omega)\}, & s \in \alpha \setminus \{\mathbf{a}_1\}, \\ \exp\{-2\pi i n \tau\}, & s \in \beta \setminus \{\mathbf{a}_1\}. \end{cases}$$

In fact, $T_n(z)$ can be holomorphically continued across each of these cycles. It has precisely one pole, namely $\infty^{(1)}$, and exactly one zero, namely z_n , both simple (the function will become analytic and non-vanishing if $z_n = \infty^{(1)}$), where $z_n = z_n(\rho) \in \mathfrak{S}$ is the unique solution of the Jacobi inversion problem

$$(2.19) \quad \mathbf{a}(z_n) = [c_\rho + (n - 1/2)(\omega + \mathbf{B}\tau)].$$

We prove Proposition 2 in Section 4.2.

2.5. Nuttall-Szegő Functions. Now we are ready to describe the solutions of **BVP- Ψ** .

Theorem 1. *Fix $n \in \mathbb{N}$. Let c_ρ be given by (2.13) and z_n be the solution of (2.19). If $z_n = \infty^{(0)}$, then the boundary value problem **BVP- Ψ** does not have a solution. Otherwise, any solution of **BVP- Ψ** is of the form $c\Psi_n(z)$ for a non-zero constant c , where*

$$(2.20) \quad \Psi_n(z) := (\Phi^n S_\rho T_n)(z), \quad z \in \mathfrak{S} \setminus \Delta,$$

and the functions $\Phi(z)$, $S_\rho(z)$, and $T_n(z)$ are given by (2.4), (2.11), and (2.17), respectively.

We prove Theorem 1 in Section 4.3. Notice that in addition to the zero of order $n - 1$ at $\infty^{(1)}$, $\Psi_n(z)$ also has a simple zero at z_n (if z_n belongs to $\Delta \setminus \mathbf{A}$ the simplicity of the zero is understood as the simplicity of the zero of an analytic continuation of $\Psi_n(z)$ to a neighborhood of z_n ; of course, if $z_n = \infty^{(1)}$, then the zero at $\infty^{(1)}$ has order n) and the asymptotics of the behavior of $\Psi_n(z)$ around the points in \mathbf{A} can be improved to

$$(2.21) \quad |\Psi_n(z)| \sim |z - a_l|^{m_n(a_l)/2 - \alpha_l/2 - 1/4} \quad \text{as } z \rightarrow a_l,$$

$l \in \{0, 1, 2, 3\}$, where $m_n(\mathbf{a}) = 1$ if $z_n = \mathbf{a}$ and $m_n(\mathbf{a}) = 0$ otherwise.

Of course, the function $\Psi_n(z)$ is defined even when $z_n = \infty^{(0)}$. In this case it simply has a pole of order $n - 1$ at $\infty^{(0)}$ and therefore does not solve **BVP- Ψ** . It readily follows from (2.19) that if $z_n = z_m$ for some distinct indices n and m , then $[0] = [(n - m)(\omega + \mathbf{B}\tau)]$, which means that ω and τ are rational numbers with the denominators that divide $n - m$. Thus, either ω and τ are rational numbers, in which case $\{z_n\}$ is a periodic sequence, or $z_n = \infty^{(0)}$ for at most one index n . Unfortunately, for our asymptotic analysis we need to exclude not only indices for which $z_n = \infty^{(0)}$, but also those for which z_n is close $\infty^{(0)}$. To this end, given $\varepsilon > 0$ such that $\max_{s \in \Delta} |s| < 1/\varepsilon$, we let

$$(2.22) \quad \mathbb{N}_\varepsilon^{\text{reg}} := \left\{ n \in \mathbb{N} : z_n \notin U_\varepsilon^{(0)} \right\},$$

where $U_\varepsilon^{(k)}$ is a neighborhood of $\infty^{(k)}$ with natural projection equal to $\{|z| > 1/\varepsilon\}$.

Proposition 3. *If ω and τ are rational numbers with the smallest common denominator d , then there exists $\varepsilon_0 > 0$ such that either $\mathbb{N}_\varepsilon^{\text{reg}} = \mathbb{N}$ or $\mathbb{N}_\varepsilon^{\text{reg}} = \mathbb{N} \setminus (n_0 + d\mathbb{N})$ for some $n_0 \in \{0, \dots, d-1\}$ and all $\varepsilon < \varepsilon_0$. In the latter case it necessarily holds that $c_\rho = (p + \mathbf{B}q)/d$ for some integers p and q . Moreover, it holds that $d \geq 3$ and $d = 3$ only if Δ is the image under a linear map of*

$$(2.23) \quad \Delta_{\text{sym}} = \{s : s^3 \in [-1, 0]\}.$$

On the other hand, if at least one of the numbers ω and τ is irrational, then for any $N \in \mathbb{N}$ there exists $\varepsilon_N > 0$ such that

$$(2.24) \quad n \notin \mathbb{N}_\varepsilon^{\text{reg}} \quad \Rightarrow \quad n+1, \dots, n+N \in \mathbb{N}_\varepsilon^{\text{reg}}$$

for any $0 < \varepsilon \leq \varepsilon_N$. Moreover, there exists $\delta = \delta(\varepsilon)$ such that

$$(2.25) \quad n \in \mathbb{N}_\varepsilon^{\text{reg}} \quad \Rightarrow \quad z_{n-1} \notin U_\delta^{(1)}.$$

Furthermore, it is always true that $z_n = \infty^{(0)}$ if and only if $z_{n-1} = \infty^{(1)}$.

We prove Proposition 3 in Section 4.4.

The reason we need to go from excluding indices for which z_n is equal to $\infty^{(0)}$ to sequences $\mathbb{N}_\varepsilon^{\text{reg}}$ is that we need to control asymptotic behavior of $\Psi_n(z)$. More precisely, for our analysis it is important to know that functions $T_n(z)$ and their reciprocals form normal families in certain subdomains of \mathfrak{S} .

Proposition 4. *Let $R > \max_{s \in \Delta} |s|$. There exists a constant $M_R > 0$ such that*

$$(2.26) \quad \begin{cases} |T_n(z)| \leq M_R, & z \notin U_{1/R}^{(1)}, \\ |T_n(z)| \leq M_R \frac{|z|}{R}, & z \in U_{1/R}^{(1)}, \end{cases}$$

for all $n \in \mathbb{N}$. Furthermore, let $\mathbb{N}_\varepsilon^{\text{reg}}$ and $\delta(\varepsilon)$ be as in Proposition 3. Then there exists a constant $M_\varepsilon > 0$ such that

$$(2.27) \quad \begin{cases} |T_n^{-1}(z)| \leq M_\varepsilon, & z \in U_{\varepsilon/2}^{(0)}, \\ |T_{n-1}^{-1}(z)| \leq M_\varepsilon |z|^{-1}, & z \in U_{\delta(\varepsilon)/2}^{(1)}, \end{cases}$$

for $n \in \mathbb{N}_\varepsilon^{\text{reg}}$.

We prove Proposition 4 in Section 4.5.

2.6. Local Obstruction. To prove our main results on asymptotic behavior of orthogonal polynomials we utilize matrix Riemann-Hilbert analysis. As typical for such an approach, we need to construct a local parametrix around a_0 that models the behavior of the polynomials and their functions of the second kind there. In our case, this is done with the help of the matrix that solves Riemann-Hilbert problems that characterizes solutions of Painlevé XXXIV equation with the Painlevé variable equal to zero. It is known that such a matrix does not exist when the corresponding Painlevé function has a pole at zero. In this case we need to modify our construction and this modification requires placing an additional requirement on the sequence of allowable indices. More precisely, let us write

$$(2.28) \quad T_n(z) := t_0^{(n)} + t_1^{(n)}(z - a_0)^{1/2} + t_2^{(n)}(z - a_0) + \dots$$

for z close to a_0 lying in the sector between Δ_1 and Δ_3 for some fixed determination of $(z - a_0)^{1/2}$ that is analytic in this sector. Let h_1 be the constant defined further below in (6.2) and (6.4) (it is the second coefficient of the Puiseux series at a_0 of a certain function that depends on the geometry via $w(z)$ and the weight $\rho(s)$ via $S_\rho(z)$ and the function $r(z)$ defined in (5.12)). Define

$$(2.29) \quad T_n := \begin{pmatrix} 2t_2^{(n)} - h_1 t_1^{(n)} & 2t_2^{(n-1)} - h_1 t_1^{(n-1)} \\ t_0^{(n)} & t_0^{(n-1)} \end{pmatrix}.$$

We need to require that the determinants of these matrices are separated away from zero.

Proposition 5. For each $\varepsilon > 0$ small enough there exists V_ε , a neighborhood of \mathbf{a}_0 that can possibly be empty, such that if $\mathbf{z}_n \notin V_\varepsilon$, then either n or $n + 1$ belongs to $\mathbb{N}_\varepsilon^{\text{sing}}$, where

$$\mathbb{N}_\varepsilon^{\text{sing}} := \left\{ n \in \mathbb{N}_\varepsilon^{\text{reg}} : |\det(\mathbf{T}_n)| \geq \varepsilon \right\}.$$

We prove Proposition 5 in Section 4.6. Let us note that there are no particular reasons to couple the constants in the definition of $\mathbb{N}_\varepsilon^{\text{sing}}$ and $\mathbb{N}_\varepsilon^{\text{reg}}$ except esthetic ones. That is, we could have defined $\mathbb{N}_{\varepsilon_1, \varepsilon_2}^{\text{sing}} \subseteq \mathbb{N}_{\varepsilon_1}^{\text{reg}}$.

Proposition 5 has the following implications. When at least one of the constants ω and τ is irrational, given a natural number N , ε can be taken small enough so that $\mathbb{N}_\varepsilon^{\text{reg}}$ contains a consecutive block of N integers after each missing index by Proposition 3. The same arguments as in the proof of that proposition can be employed to argue that $\mathbf{z}_n \in V_\varepsilon$ for at most one index in this block. Thus, according to Proposition 5, essentially at least half of the indices in this block will belong to $\mathbb{N}_\varepsilon^{\text{sing}}$. If the constants ω and τ are rational with the smallest common denominator $d \geq 3$, then the sequence $\{\mathbf{z}_n\}$ is d -periodic and at most one index in this period can be such that the corresponding solution of the Jacobi inversion problem (2.19) is equal to \mathbf{a}_0 . Hence, it is always true that

$$\mathbb{N} \setminus (n_0 + d\mathbb{N}) \cup (n_1 + d\mathbb{N}) \cup (n_1 + 1 + d\mathbb{N}) \subseteq \mathbb{N}_\varepsilon^{\text{sing}},$$

where $n_0, n_1 \in \{0, \dots, d-1\}$ are such that $\mathbf{z}_{n_0} = \infty^{(0)}$ and $\mathbf{z}_{n_1} = \mathbf{a}_0$. Thus, $\mathbb{N}_\varepsilon^{\text{sing}}$ is again necessarily infinite, unless of course $d = 3$. As we prove in Proposition 3, in the latter case Δ is a linear image of Δ_{sym} . Further below in Appendix C we construct a class of examples of weights of orthogonality on Δ_{sym} that do require the restriction to $\mathbb{N}_\varepsilon^{\text{sing}}$ and for which it is indeed the case that $\mathbf{z}_0 = \infty^{(0)}$, $\mathbf{z}_1 = \mathbf{a}_0$, and $\mathbf{z}_2 = \infty^{(1)}$, while $\mathbb{N}_\varepsilon^{\text{sing}} = \emptyset$. Therefore neither of the forthcoming asymptotic results applies. Of course, one might surmise that this is simply due to the technical limitations of our methods. However, we do analyze the orthogonal polynomials for this class of weights and show that the main term of their asymptotics is actually *different* from $\Psi_n(z)$ constructed in Theorem 1. That is, the polynomials considered in Appendix C do have different asymptotic behavior as compared to the polynomials in the rest of the cases, which does indicate that the obstruction that occurs during the construction of local parametrices is not a mere technicality.

It is also curious to observe that the condition defining $\mathbb{N}_\varepsilon^{\text{reg}}$ can be equivalently restated, see (5.5), (5.6), and (6.7) further below, as

$$\varepsilon \leq \left| \det \begin{pmatrix} t_1^{(n)} & t_1^{(n-1)} \\ t_0^{(n)} & t_0^{(n-1)} \end{pmatrix} \right| \leq \varepsilon^{-1}.$$

3. MAIN RESULTS

To state our main results on the asymptotics of the diagonal Padé approximants to functions of the form (1.4), we need to separate the weights $\rho(s)$ in (1.4) into two classes. To this end, let $\rho_i(s)$ be the restriction to Δ_i° of $\rho(s)$ and $(z - a_0)^{\alpha_0}$ be a fixed branch whose branch cut splits the sector between Δ_2 and Δ_3 . Define

$$(3.1) \quad \varrho_i(s) := \rho_i(s)/(s - a_0)^{\alpha_0}, \quad s \in \Delta_i^\circ, \quad i \in \{1, 2, 3\},$$

which, according to our assumptions stated right after (1.4), extend to analytic non-vanishing functions in some neighborhood of a_0 . Define constants

$$(3.2) \quad \begin{cases} b_1(\rho) := -\frac{e^{\pi i \alpha_0} \varrho_2(a_0) + e^{-\pi i \alpha_0} \varrho_3(a_0)}{\varrho_1(a_0)}, \\ b_i(\rho) := -\frac{\varrho_{i-1}(a_0) + \varrho_{i+1}(a_0)}{e^{(-1)^i \pi i \alpha_0} \varrho_i(a_0)}, \quad i \in \{2, 3\}, \end{cases}$$

where subindices are understood cyclically in $\{1, 2, 3\}$. The parameters α_0 and $b_i(\rho)$, $i \in \{1, 2, 3\}$, uniquely determine a 2×2 sectionally analytic matrix function $\Phi_{\alpha_0}(\zeta; x)$ as a solution of the Riemann-Hilbert problem $\text{RHP-}\Phi_\alpha$, see Appendix B, where $x \in \mathbb{C}$ is a parameter that appears in

the description of the asymptotic behavior at infinity. As we explain in Appendix B, $\Phi_{\alpha_0}(\zeta; x)$ is meromorphic in x .

Definition. In what follows we shall say that a weight $\rho(s)$ on Δ belongs to the class W^{reg} if the matrix function $\Phi_{\alpha_0}(\zeta; 0)$ solving Riemann-Hilbert problem $\text{RHP-}\Phi_{\alpha}$ with $\alpha = \alpha_0$, $b_i = b_i(\rho)$, $i \in \{1, 2, 3\}$, and $x = 0$ exists. That is, if $\Phi_{\alpha_0}(\zeta; x)$ does not have a pole at $x = 0$. Otherwise, we shall say that the weight $\rho(s)$ belongs to the class W^{sing} .

It can be readily verified that parameters (3.2) satisfy the relation

$$(3.3) \quad b_1 + b_2 + b_3 - b_1 b_2 b_3 = 2 \cos(\pi \alpha_0).$$

We shall call any solution of (3.3) a set of Stokes parameters. Any set of Stokes parameters satisfying (3.3) leads to its own matrix $\Phi_{\alpha_0}(\zeta; x)$. Some of these matrices will have a pole at $x = 0$ and some will not. Essentially any such set can be obtained via (3.2) (in particular, this means that the class W^{sing} is non-empty). Indeed, given a set of Stokes parameters (3.3) for which none is equal to $e^{\alpha_0 \pi i}$ (in this case it is also true that none is equal to $e^{-\alpha_0 \pi i}$ and a product of no two parameters is equal to 1, hence, the third one is uniquely determined by the other two), they are realized by a weight $\rho(s)$ such that

$$\frac{\varrho_1(a_0)}{\varrho_2(a_0)} = -\frac{1 - b_2 b_3}{1 - e^{-\alpha_0 \pi i} b_3} \quad \text{and} \quad \frac{\varrho_3(a_0)}{\varrho_2(a_0)} = \frac{1 - e^{\alpha_0 \pi i} b_2}{1 - e^{-\alpha_0 \pi i} b_3}.$$

On the other hand, if one of the Stokes parameters in (3.3) is equal to $e^{\alpha_0 \pi i}$ or $e^{-\alpha_0 \pi i}$, then one (whichever) of the other two must be equal to $e^{-\alpha_0 \pi i}$ or $e^{\alpha_0 \pi i}$, respectively, and the third one could be absolutely arbitrary. However, this is not the case for the parameters $b_i(\rho)$. If one of them happens to be $e^{\alpha_0 \pi i}$ or $e^{-\alpha_0 \pi i}$, then the other two are uniquely determined.

Unfortunately, currently it is unknown how to determine solely from the Stokes parameters $b_i(\rho)$ whether $\rho \in W^{\text{reg}}$ or $\rho \in W^{\text{sing}}$. Such a determination amounts to distinguishing Stokes parameters that lead to finite initial conditions at $x = 0$ for the corresponding solution of Painlevé XXXIV equation, see (B.3) and (B.8), further below, and those leading to solutions with a pole at $x = 0$.

Theorem 2. Let $f(z)$ be a function given by (1.4) with $\alpha_0 \in (-1, 1)$ and $\rho \in W^{\text{reg}}$. Let $Q_n(z)$ be the minimal degree monic denominator of the diagonal Padé approximants $[n/n]_f(z)$ and $R_n(z)$ be the corresponding linearized error function (1.1). Further, let $\Psi_n(z)$ be given by (2.20) and $\mathbb{N}_{\varepsilon}^{\text{reg}}$ be as in Proposition 3. Then it holds for all $n \in \mathbb{N}_{\varepsilon}^{\text{reg}}$ large enough that

$$(3.4) \quad \begin{cases} Q_n(z) &= \gamma_n(1 + \nu_{n1}(z))\Psi_n(z) + \gamma_n \nu_{n2}(z)\Psi_{n-1}(z), \\ (wR_n)(z) &= \gamma_n(1 + \nu_{n1}(z))\Psi_n^*(z) + \gamma_n \nu_{n2}(z)\Psi_{n-1}^*(z), \end{cases}$$

for $z \in \overline{\mathbb{C}} \setminus \Delta$ (in particular, $\deg(Q_n) = n$ for such indices n), where $\gamma_n := \lim_{z \rightarrow \infty} z^n \Psi_n^{-1}(z)$ is the normalizing constant, $w(z)$ is given by (2.2), and the functions $\nu_{ni}(s)$ vanish at infinity and satisfy²

$$(3.5) \quad \nu_{ni}(z) = O_{\varepsilon}(n^{-1/3})$$

locally uniformly on $\overline{\mathbb{C}} \setminus \Delta$. Moreover, it holds for all $n \in \mathbb{N}_{\varepsilon}^{\text{reg}}$ large enough that

$$(3.6) \quad \begin{cases} Q_n(s) &= \gamma_n(1 + \nu_{n1}(s))(\Psi_{n+}(s) + \Psi_{n-}(s)) + \gamma_n \nu_{n2}(s)(\Psi_{n-1+}(s) + \Psi_{n-1-}(s)), \\ (wR_n)_{\pm}(s) &= \gamma_n(1 + \nu_{n1}(s))\Psi_{n\pm}^*(s) + \gamma_n \nu_{n2}(s)\Psi_{n-1\pm}^*(s), \end{cases}$$

for $s \in \Delta^{\circ}$, where the functions $\nu_{ni}(s)$ can be analytically continued into a neighborhood of Δ° and satisfy (3.5) locally uniformly on Δ° .

Theorem 2 is proved in Section 5. To prove it we use by now classical approach of Fokas, Its, and Kitaev [10, 11] connecting orthogonal polynomials to matrix Riemann-Hilbert problems. The RH problem is then analyzed via the nonlinear steepest descent method of Deift and Zhou [6].

The error estimates in (3.5) can be improved in some cases.

²The notation $f(n) = O_{\varepsilon}(g(n))$ means that there exists a constant C_{ε} depending on ε such that $|f(n)| \leq C_{\varepsilon}|g(n)|$.

- It is known, see [2], that if $\alpha_0 = 0$ and $\rho_1(z) + \rho_2(z) + \rho_3(z) \equiv 0$ in some neighborhood of a_0 , then the error estimates improve to $\mathcal{O}_\varepsilon(n^{-1})$ (in this case $b_i(\rho) = 1$ and $\Phi_0(\zeta; 0)$ admits an explicit construction that uses Airy functions, so such weights do belong to W^{reg}).
- If $\rho(s) = \rho^*(s)/w_+(s)$ for some function $\rho^*(s)$ that has an analytic extension to some neighborhood of Δ , then the error estimates are geometric, see [27] (in this case $b_i(\rho) = 0$ and $\Phi_{-1/2}(\zeta; x)$ is equal to the right-hand side of **RHP- Φ_α** (d) with $U_n = V_n \equiv 0$).

In the case of singular weights, the following theorem takes place.

Theorem 3. *Let $f(z)$ be a function given by (1.4) with $\alpha_0 \in (-1, 1)$ and $\rho \in W^{\text{sing}}$. Assume further that*

$$(3.7) \quad \varrho'_1(a_0)/\varrho_1(a_0) = \varrho'_2(a_0)/\varrho_2(a_0) = \varrho'_3(a_0)/\varrho_3(a_0).$$

Let $Q_n(z)$, $R_n(z)$, and $\Psi_n(z)$ be as in Theorem 2. Let the matrices \mathbf{T}_n be as in (2.29) and $\mathbb{N}_\varepsilon^{\text{sing}}$ be as in Proposition 5. Then it holds for all $n \in \mathbb{N}_\varepsilon^{\text{sing}}$ large enough that asymptotic formulae (3.4) and (3.6) remain valid but with $v_{n1}(z)$ and $v_{n2}(z)$ replaced by

$$v_{n1}(z) - \frac{2}{z - a_0} \frac{t_0^{(n-1)} t_0^{(n)}}{\det(\mathbf{T}_n)} \quad \text{and} \quad v_{n2}(z) + \frac{2}{z - a_0} \frac{(t_0^{(n)})^2}{\det(\mathbf{T}_n)},$$

respectively, where the error functions $v_{ni}(z)$ possess the same properties as in Theorem 2.

Let us point out that the new term added to the error functions $v_{ni}(z)$ is not asymptotically small. The presence of the condition (3.7) is connected to the local analysis around a_0 and seems to be inescapable for our techniques. In particular, we needed this condition for our analysis of non-Hermitian orthogonal polynomials on a cross in [4], see the definitions of classes \mathcal{W}_ℓ on page 2. Its nature is not entirely transparent to us at this stage. However, as we explain in Appendix D, functions of the form (1.3) do satisfy (3.7). Theorem 3 is proved in Section 6.

4. PROOF OF PROPOSITIONS 1–5 AND THEOREM 1

4.1. **Proof of Proposition 1.** Orient $\pi(\alpha)$ towards a_1 . Let

$$H(z) := \frac{w(z)}{2\pi i} \int_{\pi(\alpha)} \frac{1}{t - z} \frac{dt}{w(t)}, \quad z \in \mathbb{C} \setminus (\Delta \cup \pi(\alpha)).$$

Then $H(z)$ is an analytic function in its domain of definition (the integral itself is analytic in $\overline{\mathbb{C}} \setminus \pi(\alpha)$) that behaves like

$$(4.1) \quad H(z) = -\frac{z}{2\pi i} \int_{\pi(\alpha)} \frac{ds}{w(s)} + \mathcal{O}(1) = -\frac{z}{4\pi i} \oint_{\alpha} \frac{ds}{w(s)} + \mathcal{O}(1)$$

as $z \rightarrow \infty$. We get from Plemelj-Sokhotski formulae [12, Section I.4.2] that

$$(4.2) \quad H_+(s) - H_-(s) = 1, \quad s \in \pi(\alpha) \setminus \{a_1, a_3\}.$$

It further follows from [12, Section 8.3] that $H(z)$ is bounded around a_1 and a_3 . Set

$$(4.3) \quad \Lambda(z) := \frac{w(z)}{2\pi i} \int_{\Delta} \frac{\log(\rho w_+)(s)}{s - z} \frac{ds}{w_+(s)}, \quad z \in \mathbb{C} \setminus \Delta.$$

This is a holomorphic function in its domain of definition that satisfies

$$(4.4) \quad \Lambda(z) = -\frac{z}{2\pi i} \int_{\Delta} \frac{\log(\rho w_+)(s)}{w_+(s)} ds + \mathcal{O}(1) = -\frac{zc_\rho}{2} \oint_{\alpha} \frac{ds}{w(s)} + \mathcal{O}(1)$$

as $z \rightarrow \infty$, see (2.13) and (2.8). It follows from Plemelj-Sokhotski formulae that

$$(4.5) \quad \Lambda_+(s) + \Lambda_-(s) = \log(\rho_i w_+)(s), \quad s \in \Delta_i^\circ, \quad i \in \{1, 2, 3\}.$$

Write $\Lambda(z) = \Lambda_1(z) + \Lambda_2(z) + \Lambda_3(z)$, where $\Lambda_i(z)$ is given by (4.3) with Δ replaced by Δ_i . Fix $i \in \{1, 2, 3\}$. Let $\sqrt{z - a_i}$ and $(z - a_i)^{\alpha_i}$ be branches whose branch cuts include Δ_i and such that

$w(z) = v_i(z)\sqrt{z - a_i}$ and $\rho_i(s) = \tilde{\rho}_i(s)(s - a_i)_+^{\alpha_i}$ for some functions $v_i(z)$ and $\tilde{\rho}_i(z)$ analytic in a vicinity of a_i . Then we get from [12, Sections I.8.3-6] that

$$\begin{aligned} \Lambda_i(z) &= \frac{v_i(z)\sqrt{z - a_i}}{2\pi i} \int_{\Delta_i} \frac{(\alpha_i + 1/2) \log_+(s - a_i) + \log(\tilde{\rho}_i v_i)(s)}{s - z} \frac{ds}{v_i(s)\sqrt{s - a_i}} \\ &= \frac{v_i(z)\sqrt{z - a_i}}{2\pi i} \int_{\Delta_i} \frac{(\alpha_i + 1/2) \log_+(s - a_i)}{s - z} \frac{ds}{v_i(s)\sqrt{s - a_i}} + \mathcal{O}(1) \\ (4.6) \quad &= (\alpha_i/2 + 1/4) \log(z - a_i) + \mathcal{O}(1) \end{aligned}$$

as $z \rightarrow a_i$, where $\log(z - a_i)$ has a branch cut that includes Δ_i .

Similarly, it follows from the conditions placed on $\rho_i(s)$ that $\rho_i(s)/(s - a_0)_+^{\alpha_0}$ extends to an analytic function in a vicinity of a_0 , where $(z - a_0)^{\alpha_0}$ is a branch with the cut along Δ_i . Moreover, to get a function analytic in a vicinity of a_0 out of $w(z)$, we need to divide it by $\sqrt{z - a_0}$ with the jump along Δ_i and then multiply by -1 in the sector opposite to Δ_i . Hence, it holds that

$$(4.7) \quad \Lambda_i(z) = v_i(\alpha_0/2 + 1/4)L_i(z) + \mathcal{O}(1)$$

as $z \rightarrow a_0$, where $L_i(z) := \log(z - a_0)$ has a branch cut that includes Δ_i and v_i is equal to 1 in the sectors adjacent to Δ_i and is equal to -1 in the sector opposite to Δ_i . Since

$$S_\rho(z^{(k)}) = \exp\{(-1)^{k+1}(\Lambda(z) - 2\pi i c_\rho H(z))\},$$

see (2.10) and (2.11), the claims of the proposition follow ((4.1) and (4.4) give analyticity at infinity; (4.2) and (4.5) yield (2.12); (4.6) and (4.7) give (2.14)).

4.2. Proof of Proposition 2. Before we start proving this proposition in earnest, let us make an observation. Recall the differentials $\Omega_{z,z^*}(s)$ introduced in (2.11) and the differential $G(s)$ defined in (2.4). One can easily check that

$$G(s) = \Omega_{\infty^{(1)}, \infty^{(0)}}(s) + 2\pi i \tau \Omega(s),$$

where $\Omega(s)$ is given by (2.8). It further follows from Riemann's relations [7, Section III.3.6] that

$$\oint_{\beta} \Omega_{\infty^{(1)}, \infty^{(0)}} = 2\pi i \int_{\infty^{(0)}}^{\infty^{(1)}} \Omega,$$

where the path of integration lies entirely in $\mathfrak{S}_{\alpha, \beta}$. Therefore, we can immediately deduce from the last two identities, (2.8), and (2.5) that

$$(4.8) \quad \int_{\infty^{(1)}}^{\infty^{(0)}} \Omega = \omega + B\tau \quad \text{and} \quad \int_{a_0}^{\infty^{(k)}} \Omega = (-1)^k(\omega + B\tau)/2, \quad k \in \{0, 1\},$$

where we used symmetry $\Omega(z^*) = -\Omega(z)$ for the last conclusion.

It is well known and can be checked directly that $\theta(u)$ enjoys the following periodicity properties:

$$(4.9) \quad \theta(u + l + Bm) = \exp\{-\pi i Bm^2 - 2\pi i mu\}\theta(u), \quad l, m \in \mathbb{Z}.$$

It is further known that $\theta(u) = 0$ if and only if $[u] = [(B+1)/2]$. As already mentioned, the function $a(z)$ from (2.9) extends to a multi-valued analytic function on \mathfrak{S} . Therefore, it has continuous traces on $(\alpha \cup \beta) \setminus \{a_1\}$ and these traces satisfy

$$(4.10) \quad a_+(s) - a_-(s) = \begin{cases} -B, & s \in \alpha \setminus \{a_1\}, \\ 1, & s \in \beta \setminus \{a_1\}, \end{cases}$$

due to normalization of $\Omega(z)$ and the definition of B , see (2.8). Relations (4.9) and (4.10) imply that $T_n(z)$ does indeed extend to a multiplicatively multi-valued function on \mathfrak{S} . Furthermore, they also yield that

$$\begin{aligned} T_{n+}(s) &= \exp\{-2\pi i y_n(a_-(s) - B)\} \frac{\theta(a_-(s) - c_\rho - x_n - B y_n + A - B)}{\theta(a_-(s) + A - B)} \\ &= T_{n-}(s) \exp\{2\pi i(-c_\rho - x_n)\}, \quad s \in \alpha \setminus \{a_1\}, \end{aligned}$$

from which the first relation in (2.18) follows (recall that x_n and $n\omega$ differ by an integer). The proof of the second one is analogous. Moreover, since

$$[(1 + \mathbf{B})/2] = [a(\mathbf{z}) - c_\rho - x_n - \mathbf{B}y_n + \mathbf{A}] = [a(\mathbf{z}) - a(\mathbf{z}_n) + (1 + \mathbf{B})/2]$$

holds only when $\mathbf{z} = \mathbf{z}_n$ by the unique solvability of the Jacobi inversion problem (we used (2.19) and (2.16) for the second equality), $T_n(\mathbf{z})$ does indeed vanish solely at \mathbf{z}_n . The location of the pole of $T_n(\mathbf{z})$ can be identified similarly using (4.8).

4.3. Proof of Theorem 1. Fix $n \in \mathbb{N}$ and let $\Psi_n(\mathbf{z})$ be given by (2.20). It follows from the properties of $\Phi(\mathbf{z})$, $S_\rho(\mathbf{z})$, and $T_n(\mathbf{z})$, including jump relations (2.6), (2.12), and (2.18), and local behavior (2.14) that $\Psi_n(\mathbf{z})$ solves **BVP- Ψ** when $\mathbf{z}_n \neq \infty^{(0)}$ (when $\mathbf{z}_n = \infty^{(0)}$, the pole at $\infty^{(0)}$ is of order $n - 1$). Let $\Psi(\mathbf{z})$ be any solution of **BVP- Ψ** , if it exists. Consider the function $F(\mathbf{z}) := \Psi(\mathbf{z})/\Psi_n(\mathbf{z})$. Then $F(\mathbf{z})$ is meromorphic in $\mathfrak{S} \setminus \Delta$ with at most one pole, namely \mathbf{z}_n , which is simple if present. It also holds that $F_+(s) = F_-(s)$ for $s \in \Delta \setminus \mathbf{A}$. Since these traces are continuous on $\Delta \setminus \mathbf{A}$ (apart from a possible simple pole at \mathbf{z}_n when the latter lies on $\Delta \setminus \mathbf{A}$), the analytic continuation principle yields that $F(\mathbf{z})$ is analytic in $\mathfrak{S} \setminus (\{\mathbf{z}_n\} \cup \mathbf{A})$ with at most a simple pole at \mathbf{z}_n when $\mathbf{z}_n \notin \mathbf{A}$. Finally, it follows from (2.3) and (2.21) that every \mathbf{a}_i is a removable singularity of $F(\mathbf{z})$ unless it coincides with \mathbf{z}_n , in which case it can be a simple pole. Altogether, $F(\mathbf{z})$ is a rational function on \mathfrak{S} with at most one pole, which is simple. Hence, $F(\mathbf{z})$ is a constant. That is, $\Psi(\mathbf{z}) = c\Psi_n(\mathbf{z})$, which finishes the proof of the theorem.

4.4. Proof of Proposition 3. Below we always assume that all ε_N satisfy $\max_{s \in \Delta} |s| < \varepsilon_N^{-1}$.

Because Jacobi inversion problems are uniquely solvable in the considered case, it follows from (2.15) and (4.8) that if $\mathbf{z}_n = \infty^{(k)}$, then

$$[c_\rho + (n - 1/2)(\omega + \mathbf{B}\tau)] = [(-1)^k(\omega + \mathbf{B}\tau)/2]$$

and therefore $\mathbf{z}_{n-(-1)^k} = \infty^{(1-k)}$, again by (2.15) and (4.8), since

$$[c_\rho + (n - 1/2 - (-1)^k)(\omega + \mathbf{B}\tau)] = [-(-1)^k(\omega + \mathbf{B}\tau)/2].$$

This proves the very last claim of the proposition.

If ω and τ are rational numbers with the smallest common denominator d , then

$$[c_\rho + (n + d - 1/2)(\omega + \mathbf{B}\tau)] = [c_\rho + (n - 1/2)(\omega + \mathbf{B}\tau)]$$

and therefore the sequence $\{\mathbf{z}_n\}$ is d -periodic. Thus, if none of $\mathbf{z}_0, \dots, \mathbf{z}_{d-1}$ coincides with $\infty^{(0)}$, choose ε_0 small enough so that $\mathbf{z}_0, \dots, \mathbf{z}_{d-1} \notin \mathbf{U}_{\varepsilon_0}^{(0)}$, in which case $\mathbb{N}_\varepsilon^{\text{reg}} = \mathbb{N}$ for all $\varepsilon < \varepsilon_0$. If $\mathbf{z}_{n_0} = \infty^{(0)}$ for some $n_0 \in \{0, \dots, d - 1\}$, then of course $\mathbf{z}_{n_0+kd} = \infty^{(0)}$ for all $k \geq 0$ and $\mathbf{z}_{l+kd} \neq \infty^{(0)}$ for $l \in \{0, \dots, d - 1\} \setminus \{n_0\}$, since otherwise $l - n_0$ would be divisible by d , which is impossible. Hence, by periodicity, $\mathbb{N}_\varepsilon^{\text{reg}} = \mathbb{N} \setminus (n_0 + d\mathbb{N})$ for all $\varepsilon < \varepsilon_0$, where ε_0 is such that $\mathbf{z}_l \notin \mathbf{U}_{\varepsilon_0}^{(0)}$ for $l \in \{0, \dots, d - 1\} \setminus \{n_0\}$. Moreover, in this case we get that there exist integers l and m such that

$$c_\rho = ((1 - n_0)\omega + l) + \mathbf{B}((1 - n_0)\tau + m) = (p/d) + \mathbf{B}(q/d)$$

by (2.15) and (4.8), where p and q are again integers.

Now, recall the definition of the constants ω and τ in (2.5). We can equivalently write

$$(4.11) \quad \omega = I_1 + I_3 \quad \text{and} \quad \tau = -(I_1 + I_2), \quad I_i := -\frac{1}{\pi i} \int_{\Delta_i} \frac{(s - a_0)ds}{w_+(s)}$$

(recall that each Δ_i is oriented towards a_0). It follows from the Cauchy integral formula that

$$(4.12) \quad 1 = \frac{1}{2\pi i} \oint_L \frac{(s - a_0)ds}{w(s)} = -\frac{1}{\pi i} \int_\Delta \frac{(s - a_0)ds}{w_+(s)} = I_1 + I_2 + I_3,$$

where L is any smooth Jordan curve that encircles Δ in the counter-clockwise direction. We also get from (1.2) that the integrand in (4.12), i.e.,

$$(4.13) \quad \frac{(s - a_0) ds}{w_+(s) \pi i}$$

is real and non-vanishing on each Δ_i° . In particular, $I_i \neq 0$, $i \in \{1, 2, 3\}$. Moreover, it is known, see [22, Theorem 8.1], that there exists a conformal map $\phi(z)$ in a neighborhood of a_0 such that the differential (1.2) can be locally written as $\phi d\phi^2$. If we normalize $\phi(z)$ so that it maps Δ_1 into non-positive reals (locally around a_0), then $\sqrt{\phi} d\phi$ is equal to (4.13) in the sector delimited by Δ_2 and Δ_3 that contains Δ_1 and minus the differential in (4.13) in the other sector. This yields that $I_i > 0$, $i \in \{1, 2, 3\}$. In particular, the latter claim implies that $\omega \in (0, 1)$, $\tau \in (-1, 0)$, and for no Δ it can hold that $\omega = -\tau = 1/2$ as $\omega - \tau = 1 + I_1 > 1$, that is, $d \geq 3$.

Assume that $d = 3$. Then $I_i = 1/3$, $i \in \{1, 2, 3\}$, as these are positive rational numbers smaller than 1 with denominator 3 by (4.11) and (4.12). It readily follows from its symmetries that this is indeed the case for Δ_{sym} . Let as before $\Phi(z)$ stand for the pull-back of $\Phi(z)$ from $\mathfrak{S}^{(0)}$ onto $\mathbb{C} \setminus \Delta$. Denote by $\Phi_\Delta(z)$ the analytic continuation of $\Phi(z)$ from infinity across $\pi(\alpha)$ and $\pi(\beta)$ to a holomorphic function in $\mathbb{C} \setminus \Delta$. In fact, $\Phi_\Delta(z)$ is again given by (2.4), where $z \in \mathfrak{S}^{(0)} \setminus \Delta$ and the path of integration also belongs to $\mathfrak{S}^{(0)} \setminus \Delta$, except for a_0 , and proceeds from a_0 into the sector delimited by Δ_2 and Δ_3 , recall Figure 1. As was explained after (2.6), $\Phi_\Delta(z)$ is a conformal map of $\mathbb{C} \setminus \Delta$ onto $\{|z| > 1\}$ that maps infinity into infinity. This map also sends $\partial(\mathbb{C} \setminus \Delta)$, the boundary of $\mathbb{C} \setminus \Delta$ understood as the collection of limit points of sequences in $\mathbb{C} \setminus \Delta$, bijectively onto the unit circle. Clearly, $\partial(\mathbb{C} \setminus \Delta)$ consists of two copies of each $s \in \Delta_i^\circ$, say s_+ and s_- , as accessed from the $+$ and $-$ sides of Δ_i , one copy of each a_1, a_2, a_3 , and three copies of a_0 , say $a_{0,12}, a_{0,23}$, and $a_{0,31}$, where $a_{0,i,j}$ is accessed from the sector delimited by Δ_i and Δ_j . Then

$$\begin{cases} \Phi_\Delta(a_{0,23}) = 1, \\ \Phi_\Delta(a_{0,12}) = e^{\pi i(-2I_2)} = e^{-2\pi i/3}, \\ \Phi_\Delta(a_{0,31}) = e^{\pi i(2I_3)} = e^{2\pi i/3}, \end{cases} \quad \text{and} \quad \begin{cases} \Phi_\Delta(a_1) = e^{\pi i(2I_3+I_1)} = -1, \\ \Phi_\Delta(a_2) = e^{\pi i(-I_2)} = e^{-\pi i/3}, \\ \Phi_\Delta(a_3) = e^{\pi iI_3} = e^{\pi i/3}. \end{cases}$$

Moreover, as $w_+(s) = -w_-(s)$, it also holds that $\Phi_\Delta(s_+)/\Phi_\Delta(a_i) = \overline{\Phi_\Delta(s_-)/\Phi_\Delta(a_i)}$ for any $s \in \Delta_i^\circ$. So, if Δ^1 and Δ^2 are two such contours, then $F(z) := (\Phi_{\Delta^2}^{-1} \circ \Phi_{\Delta^1})(z)$ is a conformal map of $\mathbb{C} \setminus \Delta^1$ onto $\mathbb{C} \setminus \Delta^2$ that maps infinity into infinity and $\partial(\mathbb{C} \setminus \Delta^1)$ continuously and bijectively onto $\partial(\mathbb{C} \setminus \Delta^2)$. The just described properties also yield that

$$F(s_+) = F(s_-), \quad s \in (\Delta_i^1)^\circ, \quad \text{and} \quad F(a_l(\Delta^1)) = a_l(\Delta^2)$$

for $i \in \{1, 2, 3\}$ and $l \in \{0, 1, 2, 3\}$. Thus, $F(z)$ is continuous and therefore holomorphic in the entire complex plane. Since $F(z)$ is conformal around infinity, it has a simple pole there, which means that it is a linear function that maps Δ^1 onto Δ^2 , as claimed.

Assume now that at least one the numbers ω and τ is irrational. As explained before the statement of the proposition, z_n can be equal to $\infty^{(0)}$ for at most one index n . Fix $N \geq 1$. Let $x_{-1}, x_0, x_1, \dots, x_N$ be the unique points on \mathfrak{S} such that

$$\mathbf{a}(x_l) = \left[\frac{\omega + B\tau}{2} + (\omega + B\tau)l \right], \quad l \in \{-1, 0, \dots, N\}.$$

It follows from (4.8) that $x_{-1} = \infty^{(1)}$ and $x_0 = \infty^{(0)}$. It also holds that none of x_1, \dots, x_N is equal to $\infty^{(0)}$ and they are all distinct (if $x_i = x_j$ for $i \neq j$, then ω and τ are necessarily rational). Fix $\epsilon_* > 0$ small enough so that the sets

$$O_l := \left\{ [s] \mid \exists x : [x] = \mathbf{a}(x_l), |s - x| < \epsilon_* \right\} \subset \text{Jac}(\mathfrak{S})$$

are disjoint. Recall that the Abel's map $\mathbf{a}(z)$ is a holomorphic bijection between \mathfrak{S} and $\text{Jac}(\mathfrak{S})$. Set

$$U_l := \mathbf{a}^{-1}(O_l), \quad l \in \{-1, 0, \dots, N\}.$$

Notice that if $[s] \in O_l$, then $[s + \omega + B\tau] \in O_{l+1}$. Hence, the domains U_l are disjoint and

$$z_{n+k} \in U_k, \quad k \in \{-1, \dots, N-1\} \quad \Rightarrow \quad z_{n+l} \in U_l, \quad l \in \{k+1, \dots, N\}.$$

Thus, to prove (2.24) it remains to choose ε_N small enough so that $U_{\varepsilon_N}^{(0)} \subseteq U_0$, and to prove (2.25), given $0 < \varepsilon < \varepsilon_N$, to choose ε_* and $\delta(\varepsilon)$ so that $U_0 \subseteq U_{\varepsilon}^{(0)}$ and $U_{\delta(\varepsilon)}^{(1)} \subseteq U_{-1}$.

4.5. Proof of Proposition 4. Since $a(z)$ is a branch of an additively multi-valued function on \mathfrak{S} , see (4.10), it holds that K_R , the closure of $a(\mathfrak{S}_{\alpha,\beta} \setminus U_{1/R}^{(1)})$ is a compact set. Moreover, K_R is necessarily disjoint from any point on the lattice $[-(\omega + B\tau)/2]$ by (4.8). Hence, there exists a constant $M_{1,R}$ such that

$$\left| e^{-2\pi i y_n u} \theta^{-1}(u + \mathbf{A}) \right| \leq M_{1,R}, \quad u \in K_R$$

(recall also that $y_n \in [0, 1)$). There also exists a constant $M_{2,R}$ such that

$$|\theta(u + \mathbf{A})| \leq M_{2,R}, \quad u \in \left\{ v - x - B y - c_\rho : v \in K_R, x, y \in [0, 1] \right\},$$

by compactness of the latter set. The first inequality in (2.26) now holds with $M_R = M_{1,R} M_{2,R}$. The second inequality follows from the maximum modulus principle applied in $U_{1/R}^{(1)}$. The proof of (2.27) is identical.

4.6. Proof of Proposition 5. Given $\mathbf{x} \in \mathfrak{S}_{\alpha,\beta}$, let $x(\mathbf{x}), y(\mathbf{x})$ be real numbers such that

$$\mathbf{a}(\mathbf{x}) = [c_\rho + x(\mathbf{x}) + B y(\mathbf{x}) - (\omega + B)/2], \quad x(z_0) = y(z_0) = 0,$$

recall (2.19), and that continuously depend on \mathbf{x} . Observe that these functions continuously extend to the boundary of $\mathfrak{S}_{\alpha,\beta}$, $x(\mathbf{x})$ is in fact continuous across α and $x_+(\mathbf{x}) - x_-(\mathbf{x}) = 1$ for $\mathbf{x} \in \beta$ while $y(\mathbf{x})$ is continuous across β and satisfies $y_+(\mathbf{x}) - y_-(\mathbf{x}) = -1$ for $\mathbf{x} \in \alpha$, see (4.10). In particular, it follows from the chosen normalization, (2.19), and (2.16) that either $x(z_n) = x_n$ or $x(z_n) = x_n - 1$ and the same is true about $y(z_n)$ and y_n . Define

$$T(z; \mathbf{x}) := \exp \left\{ -2\pi i y(\mathbf{x}) a(z) \right\} \frac{\theta(a(z) - c_\rho - x(\mathbf{x}) - B y(\mathbf{x}) + \mathbf{A})}{\theta(a(z) + \mathbf{A})}, \quad z \in \mathfrak{S}_{\alpha,\beta}.$$

In particular, it follows from (4.9) that

$$(4.14) \quad \begin{cases} T(z; \mathbf{z}_n) = T_n(z), & y(\mathbf{z}_n) = y_n, \\ T(z; \mathbf{z}_n) = e^{2\pi i (c_\rho + x_n + B y_n - \mathbf{A} - B/2)} T_n(z), & y(\mathbf{z}_n) = y_n - 1. \end{cases}$$

Furthermore, similarly to (2.18), it holds that

$$(4.15) \quad T_+(s; \mathbf{x}) = T_-(s; \mathbf{x}) \begin{cases} \exp \left\{ -2\pi i (c_\rho + x(\mathbf{x})) \right\}, & s \in \alpha \setminus \{\mathbf{a}_1\}, \\ \exp \left\{ -2\pi i y(\mathbf{x}) \right\}, & s \in \beta \setminus \{\mathbf{a}_1\}. \end{cases}$$

It is also true that $T(s; \mathbf{x})$ is a holomorphic non-vanishing function except for a simple zero at \mathbf{x} and a simple pole at $\infty^{(1)}$. Finally, with the same determination of $(z - a_0)^{1/2}$ as in (2.28) we can write

$$T(z; \mathbf{x}) := t_0(\mathbf{x}) + t_1(\mathbf{x})(z - a_0)^{1/2} + t_2(\mathbf{x})(z - a_0) + \dots$$

for z close to a_0 and lying in the sector between Δ_1 and Δ_3 . Notice that the coefficients $t_l(\mathbf{x})$ are continuous functions of $\mathbf{x} \in \mathfrak{S}_\alpha$ (the change of $x(\mathbf{x})$ by an integer does not affect $T(z; \mathbf{x})$) that continuously extend to both sides of α .

Recall (2.9). Given $\mathbf{x} \in \mathfrak{S}$, let $\mathbf{x}_{\pm 1} \in \mathfrak{S}$ be the unique points such that

$$(4.16) \quad \mathbf{a}(\mathbf{x}_{\pm 1}) = [a(\mathbf{x}) \pm (\omega + B\tau)].$$

Clearly, \mathbf{x}_1 and \mathbf{x}_{-1} continuously depend on \mathbf{x} . Notice that $(\mathbf{x}_1)_{-1} = \mathbf{x}$ and that $(\mathbf{z}_n)_{\pm 1} = \mathbf{z}_{n \pm 1}$. Define

$$T(\mathbf{x}) := \begin{pmatrix} 2t_2(\mathbf{x}) - h_1 t_1(\mathbf{x}) & 2t_2(\mathbf{x}_{-1}) - h_1 t_1(\mathbf{x}_{-1}) \\ t_0(\mathbf{x}) & t_0(\mathbf{x}_{-1}) \end{pmatrix},$$

where the constant h_1 is the same as in (2.29), whatever it might be. Consider the function

$$d(\mathbf{x}) := |\det(\mathbf{T}(\mathbf{x}))| + |\det(\mathbf{T}(\mathbf{x}_1))|.$$

As we have explained, this is a continuous function of $\mathbf{x} \in \mathfrak{S}_\alpha$ that continuously extends to both sides of α . Our next goal is to prove that $\mathbf{x} = \mathbf{a}_0$ is the only possible zero of $d(\mathbf{x})$ (it might happen that this function does not vanish at all).

Assume that $d(\mathbf{x}) = 0$, in which case $\det(\mathbf{T}(\mathbf{x})) = \det(\mathbf{T}(\mathbf{x}_1)) = 0$. This means that rows of $\mathbf{T}(\mathbf{x})$ and $\mathbf{T}(\mathbf{x}_1)$ are linearly dependent and since these matrices share a column, there exists a constant c such that

$$2t_2(\mathbf{y}) - h_1 t_1(\mathbf{y}) + c t_0(\mathbf{y}) = 0, \quad \mathbf{y} \in \{\mathbf{x}_{-1}, \mathbf{x}, \mathbf{x}_1\}.$$

Hence, the matrix $[t_l(\mathbf{y})]$ has zero determinant, where l is the column index and $\mathbf{y} \in \{\mathbf{x}_{-1}, \mathbf{x}, \mathbf{x}_1\}$ is the row one. By taking linear combinations of rows rather than columns, we get that there exist constants c_{-1}, c_0, c_1 , not all equal to zero, such that

$$(4.17) \quad c_{-1} t_l(\mathbf{x}_{-1}) + c_0 t_l(\mathbf{x}) + c_1 t_l(\mathbf{x}_1) = 0, \quad l \in \{0, 1, 2\}.$$

In another connection, since the integrand in (2.4) behaves like $\sqrt{z - a_0}$ around \mathbf{a}_0 , it holds that

$$(4.18) \quad \Phi(z) = 1 + \mathcal{O}\left((z - a_0)^{3/2}\right)$$

as $z \rightarrow a_0$. Consider the function

$$\begin{aligned} F(z) &:= c_{-1} \frac{T(z; \mathbf{x}_{-1})}{\Phi(z)T(z; \mathbf{x})} + c_0 + c_1 \frac{\Phi(z)T(z; \mathbf{x}_1)}{T(z; \mathbf{x})} \\ &= \frac{c_{-1}T(z; \mathbf{x}_{-1}) + c_0\Phi(z)T(z; \mathbf{x}) + c_1\Phi^2(z)T(z; \mathbf{x}_1)}{\Phi(z)T(z; \mathbf{x})}. \end{aligned}$$

It follows from (2.6), (4.15), and (4.16) that this is a rational function on the entire surface \mathfrak{S} . Moreover, the first representation readily yields that $F(z)$ has exactly three poles, all simple, at \mathbf{x} , $\infty^{(0)}$, and $\infty^{(1)}$, while the second formula together with (4.17) and (4.18) clearly shows that it has at least a triple zero at \mathbf{a}_0 (unless $\mathbf{x} = \mathbf{a}_0$, in which case it is at least a double zero). Since $F(z)$ must have equal number of poles and zeros, its zero/pole divisor is equal to

$$3\mathbf{a}_0 - \mathbf{x} - \infty^{(0)} - \infty^{(1)}.$$

However, this means that the function $F(z)/(z - a_0)$ has a zero/pole divisor $\mathbf{a}_0 - \mathbf{x}$. Since there are no rational functions with at most one pole besides constants, $F(z) = C(z - a_0)$ for some constant C and it must hold that $\mathbf{a}_0 = \mathbf{x}$ as claimed.

Now, it follows from (2.29) and (4.14) that

$$|\det(\mathbf{T}_n)| = Y_{n-1} Y_n |\det(\mathbf{T}(\mathbf{z}_n))|, \quad Y_n = \begin{cases} 1, & y(\mathbf{z}_n) = y_n, \\ e^{\pi \operatorname{Im}((2-2y_n+\tau)\mathbf{B}-2c_\rho)}, & y(\mathbf{z}_n) = y_n - 1. \end{cases}$$

Since $\operatorname{Im}(\mathbf{B}) > 0$, $\tau \in (-1, 0)$, and $y_n \in [0, 1)$, we have that $Y_n \geq Y := e^{-\pi \operatorname{Im}(2c_\rho + \mathbf{B})}$ for all $n \in \mathbb{N}$. Let $\mathbf{V}_\varepsilon := d^{-1}([0, 2Y^{-2}\varepsilon])$. Then for all ε small enough this set is either empty (when $d(\mathbf{x})$ does not vanish) or it is a neighborhood of \mathbf{a}_0 by continuity of $d(\mathbf{x})$. Then if $\mathbf{z}_n \notin \mathbf{V}_\varepsilon$, it holds that

$$|\det(\mathbf{T}_n)| + |\det(\mathbf{T}_{n+1})| \geq Y^2 d(\mathbf{z}_n) \geq 2\varepsilon,$$

which yields that $|\det(\mathbf{T}_m)| \geq |\det(\mathbf{T}(\mathbf{z}_m))| \geq \varepsilon$ for either $m = n$ or $m = n + 1$.

5. PROOF OF THEOREM 2

5.1. Initial Riemann-Hilbert Problem. Recall the definition of $R_n(z)$ in (1.1) as well as the definition of $f(z)$ in (1.4). It follows from the known behavior of Cauchy integrals around the endpoints

of arcs of integration [12, Sections 8.2-4] (we can consider $f(z)$ as a sum of three Cauchy integrals over the arcs $\Delta_1, \Delta_2, \Delta_3$) that

$$|f(z)|, |R_n(z)| = \mathcal{O}(\psi_{\alpha_l}(z - a_l)), \quad \psi_{\alpha}(z) := \begin{cases} 1, & \alpha > 0, \\ \log |z|, & \alpha = 0, \\ |z|^{\alpha}, & \alpha < 0, \end{cases}$$

for $l \in \{0, 1, 2, 3\}$ (it was pointed out in [2] that these functions are bounded around a_0 when $\alpha_0 = 0$ and $\rho_1(0) + \rho_2(0) + \rho_3(0) = 0$, where, as before, $\rho_i(s)$ is the restriction of $\rho(s)$ to $\Delta_i^{\circ} = \Delta_i \setminus \{a_0, a_i\}$). Assume now that

$$(5.1) \quad \deg(Q_n) = n \quad \text{and} \quad R_{n-1}(z) = k_n^{-1} z^{-n} + \mathcal{O}(z^{-n-1}),$$

where the second relation must hold as $z \rightarrow \infty$ for some finite non-zero constant k_n . Let

$$(5.2) \quad Y(z) := \begin{pmatrix} Q_n(z) & R_n(z) \\ k_{n-1} Q_{n-1}(z) & k_{n-1} R_{n-1}(z) \end{pmatrix}.$$

Then $Y(z)$ solves the following Riemann-Hilbert problem (RHP- Y):

- (a) $Y(z)$ is analytic in $\mathbb{C} \setminus \Delta$ and $\lim_{z \rightarrow \infty} Y(z) z^{-n\sigma_3} = I$, where $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$;
- (b) $Y(z)$ has continuous traces on each Δ_i° , $i \in \{1, 2, 3\}$, that satisfy

$$Y_+(s) = Y_-(s) \begin{pmatrix} 1 & \rho_i(s) \\ 0 & 1 \end{pmatrix}, \quad s \in \Delta_i^{\circ};$$

- (c) $Y(z) = \mathcal{O} \begin{pmatrix} 1 & \psi_{\alpha_l}(z - a_l) \\ 1 & \psi_{\alpha_l}(z - a_l) \end{pmatrix}$ as $\Delta \ni z \rightarrow a_l$, $l \in \{0, 1, 2, 3\}$.

It is by now standard to show, see for example [2, 27], that if a solution of RHP- Y exists then it has the form (5.2) and (5.1) holds. Notice that $\det(Y(z))$ is holomorphic in $\mathbb{C} \setminus \Delta$, equal to 1 at infinity, is continuous across Δ° and can have at most removable singularity at each a_l . Hence, $\det(Y(z)) \equiv 1$.

5.2. Opening of Lenses. Now, we construct a system of arcs, the lens, around Δ as on Figure 3. Fix

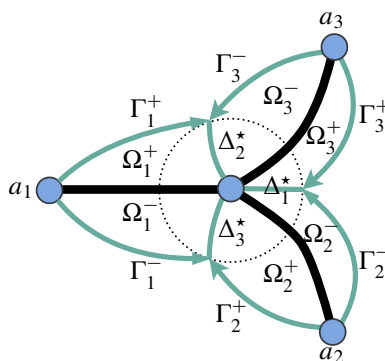


FIGURE 3. Contour Γ . The dotted circle represents ∂U_0 .

$\delta > 0$ such that $\Delta \cap U_0$ is a smaller trefoil, where $U_0 := \{z : |z - a_0| < \delta\}$, see the dotted circle on Figure 3. We are assuming that δ is small enough so that $\bar{U}_0 \cap (\pi(\alpha) \cup \pi(\beta)) = \emptyset$. Denote by S_i the sector of $U_0 \setminus \Delta$ for which $\partial S_i \cap \Delta_i = \{a_0\}$ (i.e., the sector opposite to Δ_i). Further, let U_0^{\pm} be the connected components of $U_0 \setminus (\Delta_1 \cup \Delta_1^*)$ that border Δ_1^+ (U_0^+ intersects Δ_3 and U_0^- intersects Δ_2), where Δ_i^* is an open analytic arc that splits the sector S_i into two halves (its endpoints are a_0 and some point on $\partial S_i \setminus \Delta$) and has the same tangent vector at a_0 as Δ_i (we shall completely fix Δ_i^* later in Section 5.4, see (5.9)), see Figure 3. We orient each Δ_i^* towards a_0 . Further, let Γ_i^+ and Γ_i^- be open smooth arcs that connect a_i to Δ_{i-1}^* and Δ_{i+1}^* , respectively, where the subindices are

understood cyclically within $\{1, 2, 3\}$, a convention we abide by throughout the rest of the paper. We assume that all these arcs do not intersect except for the common endpoints and let

$$\Gamma := \Delta \bigcup \left(\cup_i \Delta_i^* \right) \bigcup \left(\cup_i (\Gamma_i^+ \cup \Gamma_i^-) \right).$$

We let Ω_i^\pm to denote bounded components of the complement of Γ that contain Γ_i^\pm as a part of their boundary and Γ° be the subset of those points in Γ that possess a well defined tangent.

Recall our assumption made after (1.4) that $\rho_i(s)(s - a_0)^{-\alpha_0}(s - a_i)^{-\alpha_i}$ extends to an analytic function in some neighborhood of Δ_i . We shall suppose that this neighborhood contains the closure of $\Omega_i^+ \cup \Omega_i^-$ and that the branch of $(z - a_i)^{\alpha_i}$ is analytic throughout this closure, except, of course, at a_i . Recall also that we fixed a branch of $(z - a_0)^{\alpha_0}$ whose branch cut lies, in part, in S_1 and we shall further assume that this cut contains Δ_1^* . Finally, recall (3.1) and suppose that U_0 is small enough so that every $\varrho_i(z)$ is analytic in its closure.

Define

$$(5.3) \quad \mathbf{X}(z) := \mathbf{Y}(z) \begin{cases} \begin{pmatrix} 1 & 0 \\ \mp 1/\rho_i(z) & 1 \end{pmatrix}, & z \in \Omega_i^\pm, \\ \mathbf{I}, & \text{otherwise.} \end{cases}$$

Clearly, if $\mathbf{Y}(z)$ is a solution of **RHP-Y**, then $\mathbf{X}(z)$ is a solution of the following Riemann-Hilbert problem (**RHP-X**):

- (a) $\mathbf{X}(z)$ is analytic in $\mathbb{C} \setminus \Delta$ and $\lim_{z \rightarrow \infty} \mathbf{X}(z)z^{-n\sigma_3} = \mathbf{I}$;
- (b) $\mathbf{X}(z)$ has continuous traces on Γ° that satisfy

$$\mathbf{X}_+(s) = \mathbf{X}_-(s) \begin{cases} \begin{pmatrix} 0 & \rho_i(s) \\ -\rho_i^{-1}(s) & 0 \end{pmatrix}, & s \in \Delta_i^\circ, \\ \begin{pmatrix} 1 & 0 \\ \rho_i^{-1}(s) & 1 \end{pmatrix}, & s \in \Gamma_i^\pm, \\ \begin{pmatrix} 1 & 0 \\ \rho_{i+1+}^{-1}(s) + \rho_{i-1-}^{-1}(s) & 1 \end{pmatrix}, & s \in \Delta_i^*, \end{cases}$$

where taking traces is important only on Δ_1^* ;

- (c) for each $l \in \{0, 1, 2, 3\}$ it holds that

$$\mathbf{X}(z) = \mathcal{O} \begin{pmatrix} 1 & |z - a_l|^{\alpha_l} \\ 1 & |z - a_l|^{\alpha_l} \end{pmatrix} \quad \text{and} \quad \mathbf{X}(z) = \mathcal{O} \begin{pmatrix} \log |z - a_l| & \log |z - a_l| \\ \log |z - a_l| & \log |z - a_l| \end{pmatrix}$$

as $\Gamma \not\ni z \rightarrow a_l$ when $\alpha_l < 0$ and $\alpha_l = 0$, respectively, and

$$\mathbf{X}(z) = \mathcal{O} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{X}(z) = \mathcal{O} \begin{pmatrix} |z - a_l|^{-\alpha_l} & 1 \\ |z - a_l|^{-\alpha_l} & 1 \end{pmatrix}$$

as $\Gamma \not\ni z \rightarrow a_l$, from within the unbounded and bounded components of the complement of Γ , respectively, when $\alpha_l > 0$.

Conversely, assume that $\mathbf{X}(z)$ is a solution of **RHP-X** and $\mathbf{Y}(z)$ is defined by inverting (5.3). It is quite easy to see that thus defined $\mathbf{Y}(z)$ satisfies **RHP-Y(a,b)**. Furthermore, since the second columns of $\mathbf{X}(z)$ and $\mathbf{Y}(z)$ coincide, the second column of $\mathbf{Y}(z)$ satisfies **RHP-Y(c)** as well. In fact, the same holds for the first column of $\mathbf{Y}(z)$ if $\alpha_l < 0$. In other cases, we see from **RHP-Y(b)** that the first column of $\mathbf{Y}(z)$ can have at most an isolated singularity at a_l . However, if $\alpha_l = 0$, then blowing up at a_l is at most logarithmic and therefore singularity must be removable. Similarly, when $\alpha_l > 0$ and $l \in \{1, 2, 3\}$, the first column of $\mathbf{Y}(z)$ remains bounded as $z \rightarrow a_l$ from outside the lens and therefore the singularity at a_l cannot be polar. Furthermore, in this case $|z - a_l|^{\alpha_l} \mathbf{Y}(z)$ is bounded around a_l and therefore the singularity cannot be essential. Thus, it is again removable. Finally, since a_0 cannot be approached from outside the lens, we rule out polar (and essential) singularity at a_0 due to the requirement $\alpha_0 < 1$.

5.3. Parametrices. Below we assume that the index n is such that $z_n \neq \infty^{(0)}$, see (2.19).

The global parametrix is obtained from **RHP-X** by ignoring local behavior around a_i and discarding the jumps on Δ_i^* and Γ_i^\pm . Namely, let $\Psi_n(z)$ be given by (2.20). Recall our convention that $\Psi_n(z)$ and $\Psi_n^*(z)$ stand for the pull-backs of $\Psi_n(z)$ from $\mathfrak{S}^{(0)}$ and $\mathfrak{S}^{(1)}$, respectively. Define

$$N(z) := \begin{pmatrix} \gamma_n \Psi_n(z) & \gamma_n \Psi_n^*(z)/w(z) \\ \gamma_{n-1}^* \Psi_{n-1}(z) & \gamma_{n-1}^* \Psi_{n-1}^*(z)/w(z) \end{pmatrix}$$

for $z \notin \Delta$, where the constants γ_n, γ_{n-1}^* are chosen so that $\lim_{z \rightarrow \infty} N(z)z^{-n\sigma_3} = \mathbf{I}$. By the restriction placed on the indices n , these constants are finite and non-zero, see the last claim of Proposition 3. Then $N(z)$ is a solution of the following Riemann-Hilbert problem (**RHP-N**):

- (a) $N(z)$ is analytic in $\mathbb{C} \setminus \Delta$ and $\lim_{z \rightarrow \infty} N(z)z^{-n\sigma_3} = \mathbf{I}$;
- (b) $N(z)$ has continuous traces on each $\Delta_i^\circ, i \in \{1, 2, 3\}$, that satisfy

$$N_+(s) = N_-(s) \begin{pmatrix} 0 & \rho_i(s) \\ -\rho_i^{-1}(s) & 0 \end{pmatrix}, \quad s \in \Delta_i^\circ;$$

- (c) $N(z) = \mathcal{O} \begin{pmatrix} |z - a_l|^{-(2\alpha_l+1)/4} & |z - a_l|^{(2\alpha_l-1)/4} \\ |z - a_l|^{-(2\alpha_l+1)/4} & |z - a_l|^{(2\alpha_l-1)/4} \end{pmatrix}$ as $z \rightarrow a_l, l \in \{0, 1, 2, 3\}$.

The properties of $N(z)$ described above follow easily from the requirements of **BVP-Ψ**.

In fact, it will be more convenient for us later to work with separate factors of $N(z)$ rather than $N(z)$ itself. To this end we write $N(z) := \mathbf{C} \mathbf{M}^{\text{reg}}(z) \Phi^{n\sigma_3}(z)$, where

$$(5.4) \quad \mathbf{C} := \begin{pmatrix} \gamma_n & 0 \\ 0 & \gamma_{n-1}^* \end{pmatrix} \quad \text{and} \quad \mathbf{M}^{\text{reg}}(z) := \begin{pmatrix} T_n(z) & T_n^*(z)/w(z) \\ (T_{n-1}\Phi^*)(z) & (T_{n-1}^*\Phi)(z)/w(z) \end{pmatrix} S_\rho^{\sigma_3}(z),$$

where we used the facts that $\Phi(z)\Phi^*(z) \equiv 1$ and $S_\rho(z)S_\rho^*(z) \equiv 1$, see Proposition 1. As in the case of $Y(z)$, it is straightforward to argue that $\det(N(z)) \equiv 1$ in \mathbb{C} . Hence, it holds that

$$\det(\mathbf{M}^{\text{reg}}(z)) \equiv 1/\det(\mathbf{C}) = 1/(\gamma_n \gamma_{n-1}^*).$$

Let $\mathbb{N}_\varepsilon^{\text{reg}}$ be as in (2.22). Then it follows from the above identity, (5.4), and (2.20) that

$$(5.5) \quad (\gamma_n \gamma_{n-1}^*)^{-1} = T_n(\infty) \lim_{z \rightarrow \infty} \frac{T_{n-1}^*(z)\Phi(z)}{w(z)}.$$

It is well known (and is not very important for us here) that $\lim_{z \rightarrow \infty} |\Phi(z)/z| = 1/\text{cap}(\Delta)$ is the reciprocal of the logarithmic capacity of Δ (in any case, it is a non-zero and finite constant). Hence, it follows from (2.26)–(2.27) that

$$(5.6) \quad \text{cap}(\Delta) R M_R^{-2} \leq |\gamma_n \gamma_{n-1}^*| \leq \text{cap}(\Delta) M_\varepsilon^2, \quad n \in \mathbb{N}_\varepsilon^{\text{reg}},$$

for any given $R > \max_{s \in \Delta} |s|$. We further get from (2.26)–(2.27) and the above inequalities that

$$(5.7) \quad \begin{cases} \mathbf{M}^{\text{reg}}(z) & = \mathbf{O}_{K,\varepsilon}(1) \text{diag}(1, 1/w(z)) S_\rho^{\sigma_3}(z) \\ \mathbf{M}^{\text{reg}}(z)^{-1} & = S_\rho^{-\sigma_3}(z) \text{diag}(1/w(z), 1) \mathbf{O}_{K,\varepsilon}(1) \end{cases}$$

on any compact set K as $\mathbb{N}_\varepsilon^{\text{reg}} \ni n \rightarrow \infty$.

Assume now that $\delta > 0$ is small enough so that $U_i := \{z : |z - a_i| < \delta\}, i \in \{1, 2, 3\}$, intersects only Δ_i , i.e., $U_i \cap \Delta_j = \emptyset$ for $j \neq i$, and that $\rho_i(s)(s - a_i)^{-\alpha_i}$ extends analytically to its closure, which is disjoint from the closure of U_0 . Local parametrix around a_i is obtained by solving **RHP-X** in U_i . That is, we need to solve the following Riemann-Hilbert problem (**RHP-P_i**):

- (a) $\mathbf{P}_i(z)$ is analytic in $U_i \setminus \Gamma$;
- (b,c) $\mathbf{P}_i(z)$ satisfies **RHP-X**(b,c) within U_i ;
- (d) it holds that $\mathbf{P}_i(s) = (\mathbf{I} + \mathcal{O}(1/n)) \mathbf{M}^{\text{reg}}(s) \Phi^{n\sigma_3}(s)$ uniformly on ∂U_i .

The solution of $\text{RHP-}\mathbf{P}_i$ is well known and was first derived in [18] in the context of orthogonal polynomials on a segment with the help of the modified Bessel and Hankel functions. Since then it appeared in countless papers and was adopted to the context of non-Hermitian orthogonal polynomials on symmetric contours in [2]. Therefore, we shall skip the explicit construction.

Around a_0 , we look only for an approximate parametrix. More precisely, we would like to solve the following Riemann-Hilbert problem ($\text{RHP-}\mathbf{P}_0$):

- (a) $\mathbf{P}_0(z)$ is analytic in $U_0 \setminus \Gamma$;
- (b) $\mathbf{P}_0(z)$ satisfies $\text{RHP-}\mathbf{X}$ (b) within U_0 except the 21-entries of the jump matrices on Δ_i^* , $i \in \{1, 2, 3\}$, are replaced by

$$\kappa_{i,i-1}(s)\rho_{i+1+}^{-1}(s) + \kappa_{i,i+1}(s)\rho_{i-1-}^{-1}(s),$$

where $\kappa_{i,j}(z) := (\varrho_i(z)\varrho_j(a_0))/(\varrho_i(a_0)\varrho_j(z))$ (recall (3.1));

- (c) $\mathbf{P}_0(z)$ satisfies $\text{RHP-}\mathbf{X}$ (c) within U_0 ;
- (d) it holds that $\mathbf{P}_0(s) = (\mathbf{I} + \mathcal{O}(n^{-1/3}))\mathbf{M}^{\text{reg}}(s)\Phi^{n\sigma_3}(s)$ uniformly on ∂U_0 .

We solve $\text{RHP-}\mathbf{P}_0$ in the next subsection. The presence of the functions $\kappa_{i,j}(z)$ is precisely what makes this parametrix approximate. Of course, it would be preferable to solve $\text{RHP-}\mathbf{X}$ locally around a_0 exactly, that is, with the jumps appearing in $\text{RHP-}\mathbf{X}$ (b). However, currently, we do not know how to achieve this. Notice also that if $\varrho_i(s) = c_i\varrho(s)$ for some function $\varrho(z)$ analytic around a_0 and possibly different constants c_i , which is exactly the case when the approximated function has the form (1.3), see Appendix D, then $\kappa_{i,j}(z) \equiv 1$ and the parametrix is exact.

5.4. Approximate Local Parametrix around a_0 . Solution of $\text{RHP-}\mathbf{P}_0$ is based on the solution of $\text{RHP-}\Phi_\alpha$ from Appendix B. The knowledge of the statement of that Riemann-Hilbert problem is sufficient for understanding the construction further below.

Recall the definition of Σ_2, Σ_3 in Appendix B. Define

$$(5.8) \quad \zeta^3(z) := \left(\frac{3}{2} \int_{a_0}^z \frac{(s-a_0)}{w(s)} ds \right)^2, \quad z \in (S_1 \cup S_2 \cup S_3) \cap U_0.$$

Since $w_+(s) = -w_-(s)$ for $s \in \Delta$, $\zeta^3(z)$ extends to a holomorphic function in U_0 , which, upon easy verification, has a triple zero at a_0 (we already used this observation in (4.18)). Furthermore, it readily follows from (1.2) that $\zeta^3(s)$ is negative for $s \in \Delta \cap U_0$ (except for a_0 , of course). Hence, we can select a branch $\zeta(z)$ which is holomorphic in U_0 and such that $\zeta(s) < 0$ when $s \in \Delta_1 \cap U_0$. Since $\zeta(z)$ has a simple zero at a_0 , we can decrease the radius of U_0 if necessary so that $\zeta(z)$ is univalent in U_0 . Moreover, as we had some freedom in choosing the arcs Δ_i^* , we now fix them so that

$$(5.9) \quad \zeta(\Delta_1^*) \subset (0, \infty) \quad \text{and} \quad \zeta(\Delta_i^*) \subset \Sigma_i, \quad i \in \{2, 3\}.$$

In what follows, all the roots of $\zeta(z)$ are principal, that is, they remain positive on Δ_1^* and have a branch cut along Δ_1 . In particular,

$$(5.10) \quad \zeta_+^{1/4}(s) = i\zeta_-^{1/4}(s), \quad s \in \Delta_1 \cap U_0.$$

Recall the definition of $\Phi(z)$ in (2.4) and that $|\Phi(z)| > 1$ for $z \in U_0 \setminus \Delta$, see (2.7). Thus, it also holds that

$$(5.11) \quad e^{-(2/3)\zeta^{3/2}(z)} = \begin{cases} \Phi^*(z), & z \in S_1 \cap U_0, \\ \Phi(z), & z \in (S_2 \cup S_3) \cap U_0. \end{cases}$$

Let $\varrho_i(s)$ be given by (3.1), where for definiteness we fix a branch of $(z-a_0)^{\alpha_0}$ that is analytic in $U_0 \setminus \Delta_1^*$. Define

$$E_{a_0}(z) := \begin{cases} e^{\pm \pi i \alpha_0 / 2}, & z \in S_1 \cap U_0^\pm, \\ e^{\mp \pi i \alpha_0 / 2}, & z \in (S_2 \cup S_3) \cap U_0^\pm. \end{cases}$$

Further, let $r_i(z)$, $i \in \{1, 2, 3\}$, be a function holomorphic in S_i given by

$$(5.12) \quad r_i(z) := \varsigma_i i(z - a_0)^{\alpha_0/2} E_{\alpha_0}(z) \left(\frac{\varrho_{i-1}(z) \varrho_{i+1}(z)}{\varrho_i(z)} \right)^{1/2},$$

where $\varsigma_1 = \varsigma_2 = -\varsigma_3 = 1$ and we use the same square root branches of $\varrho_j(z)$ in all $r_i(z)$. Since $E_{\alpha_0+}(s)E_{\alpha_0-}(s) \equiv 1$ for $s \in \Delta_i^\circ \cap U_0$, it holds that

$$(5.13) \quad r_{i-1-}(s)r_{i+1+}(s) = \varsigma_i \rho_i(s), \quad s \in \Delta_i^\circ \cap U_0.$$

We further set $r(z) := r_i(z)$ for $z \in S_i$.

Recall that $\rho \in W^{\text{reg}}$. Let $\Phi_{\alpha_0}(\zeta; 0)$ be the solution of **RHP- Φ_α** with $\alpha = \alpha_0$ and the Stokes parameters given by (3.2), see Appendix B (it exists by the very definition of the class W^{reg}). Then a solution of **RHP- P_0** is given by

$$(5.14) \quad P_0(z) := E_0^{\text{reg}}(z) \Phi_{\alpha_0}(n^{2/3} \zeta(z); 0) J(z) r^{-\sigma_3}(z),$$

where $E_0^{\text{reg}}(z)$ is a holomorphic matrix function in U_0 that we shall specify further below in (5.16) and

$$(5.15) \quad J(z) := \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & z \in S_1 \cap U_0, \\ I, & z \in (S_2 \cup S_3) \cap U_0. \end{cases}$$

Indeed, it follows from (5.9) that $P_0(z)$ is a holomorphic matrix function in $U_0 \setminus \Gamma$, i.e., **RHP- P_0** (a) is satisfied. We further get from **RHP- Φ_α** (b) and (5.13) that

$$P_{0-}^{-1}(s) P_{0+}(s) = r_{3-}^{\sigma_3}(s) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} r_{2+}^{-\sigma_3}(s) = \begin{pmatrix} 0 & \rho_1(s) \\ -1/\rho_1(s) & 0 \end{pmatrix}$$

for $s \in \Delta_1^\circ \cap U_0$,

$$P_{0-}^{-1}(s) P_{0+}(s) = r_{1-}^{\sigma_3}(s) J_{-}^{-1}(s) r_{3+}^{-\sigma_3}(s) = \begin{pmatrix} 0 & \rho_2(s) \\ -1/\rho_2(s) & 0 \end{pmatrix}$$

for $s \in \Delta_2^\circ \cap U_0$, and

$$P_{0-}^{-1}(s) P_{0+}(s) = r_{2-}^{\sigma_3}(s) J_{+}(s) r_{1+}^{-\sigma_3}(s) = \begin{pmatrix} 0 & \rho_3(s) \\ -1/\rho_3(s) & 0 \end{pmatrix}$$

for $s \in \Delta_3^\circ \cap U_0$. Moreover, using (3.2) we get that

$$P_{0-}^{-1}(s) P_{0+}(s) = r_1^{\sigma_3}(s) J^{-1}(s) \begin{pmatrix} 1 & -b_1(\rho) \\ 0 & 1 \end{pmatrix} J(s) r_1^{-\sigma_3}(s) = \begin{pmatrix} 1 & 0 \\ \frac{\kappa_{1,2}(s)}{\rho_{2+}(s)} + \frac{\kappa_{1,3}(s)}{\rho_{3-}(s)} & 1 \end{pmatrix}$$

for $s \in \Delta_1^* \cap U_0$ (notice that the orientations of $\zeta(\Delta_1^*)$ and Σ_1 are opposite to each other), and

$$P_{0-}^{-1}(s) P_{0+}(s) = r_l^{\sigma_3}(s) \begin{pmatrix} 1 & 0 \\ b_l(\rho) & 1 \end{pmatrix} r_l^{-\sigma_3}(s) = \begin{pmatrix} 1 & 0 \\ \frac{\kappa_{l,l-1}(s)}{\rho_{l+1}(s)} + \frac{\kappa_{l,l+1}(s)}{\rho_{l-1}(s)} & 1 \end{pmatrix}$$

for $s \in \Delta_l^* \cap U_0$, $l \in \{2, 3\}$. Hence, $P_0(z)$, defined in (5.14), fulfills **RHP- P_0** (b). Next, it follows from **RHP- Φ_α** (c), (5.14), and (5.12) that

$$P_0(z) = E_0^{\text{reg}}(z) \begin{cases} \mathcal{O}(|\zeta(z)|^{-|\alpha_0|/2}), & \alpha_0 \neq 0 \\ \mathcal{O}(\log |\zeta(z)|), & \alpha_0 = 0 \end{cases} \mathcal{O}(|z - a_0|^{-\alpha_0 \sigma_3/2}).$$

Since multiplication by $E_0^{\text{reg}}(z)$ on the left does not mix the entries from different columns of the product of the other two factors above, $P_0(z)$ does indeed satisfy **RHP- P_0** (c).

Finally, let

$$(5.16) \quad E_0^{\text{reg}}(z) := M^{\text{reg}}(z) r^{\sigma_3}(z) J^{-1}(z) \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \frac{(n^{2/3} \zeta(z))^{\sigma_3/4}}{\sqrt{2}}.$$

We get from **RHP- Φ_α** (d), (5.7), (5.11), and (5.15) that

$$(5.17) \quad \begin{aligned} \mathbf{P}_0(s) &= \mathbf{M}^{\text{reg}}(s) r^{\sigma_3}(s) \left(\mathbf{I} + \mathcal{O}(n^{-1/3}) \right) r^{-\sigma_3}(s) \Phi^{n\sigma_3}(s) \\ &= \left(\mathbf{I} + \mathcal{O}(n^{-1/3}) \right) \mathbf{M}^{\text{reg}}(s) \Phi^{n\sigma_3}(s), \end{aligned}$$

which is exactly what was required in **RHP- \mathbf{P}_0** (d). Thus, it only remains to prove that $\mathbf{E}_0^{\text{reg}}(z)$ is analytic in U_0 . To this end, this matrix is clearly analytic in $U_0 \setminus \Delta$. It also holds that

$$\begin{aligned} (\mathbf{E}_{0-}^{\text{reg}})^{-1}(s) \mathbf{E}_{0+}^{\text{reg}}(s) &= n^{-\sigma_3/6} \frac{\zeta_-^{-\sigma_3/4}(s)}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \frac{\zeta_+^{\sigma_3/4}(s)}{\sqrt{2}} n^{\sigma_3/6} \\ &= n^{-\sigma_3/6} \zeta_-^{-\sigma_3/4}(s) \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \zeta_+^{\sigma_3/4}(s) n^{\sigma_3/6} = \mathbf{I} \end{aligned}$$

for $s \in \Delta_1^\circ \cap U_0$, where we used **RHP- \mathbf{N}** (b), (5.13), (5.15), and (5.10). Similarly, we have for $s \in \Delta_l^\circ \cap U_0$, $l \in \{2, 3\}$, that $(\mathbf{E}_{0-}^{\text{reg}})^{-1}(s) \mathbf{E}_{0+}^{\text{reg}}(s) = \mathbf{I}$, because it holds there that

$$\mathbf{J}_-(s) \begin{pmatrix} 0 & (-1)^l \\ (-1)^{l-1} & 0 \end{pmatrix} \mathbf{J}_+^{-1}(s) = \mathbf{I}.$$

Hence, $\mathbf{E}_0^{\text{reg}}(z)$ is analytic in $U_0 \setminus \{a_0\}$. Furthermore, we get from **RHP- \mathbf{N}** (c) and (5.12) that

$$\mathbf{E}_0^{\text{reg}}(z) = \mathcal{O}\left(|z - a_0|^{-1/4}\right) \mathbf{J}_+^{-1}(s) \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \frac{\zeta^{\sigma_3/4}(z)}{\sqrt{2}} n^{\sigma_3/6} = \mathcal{O}\left(|z - a_0|^{-1/2}\right).$$

Therefore, a_0 cannot be a polar singularity of $\mathbf{E}_0^{\text{reg}}(z)$ and must be a point of analyticity.

5.5. Small-Norm Riemann-Hilbert Problem. Let \mathbf{C} and $\mathbf{M}^{\text{reg}}(z)$ be matrices defined in (5.4) and $\mathbf{P}_l(z)$, $l \in \{0, 1, 2, 3\}$, be the matrix functions solving **RHP- \mathbf{P}_0** and **RHP- \mathbf{P}_i** , $i \in \{1, 2, 3\}$. Further, let $U := U_0 \cup U_1 \cup U_2 \cup U_3$, whose boundary we orient clockwise. We are looking for the solution

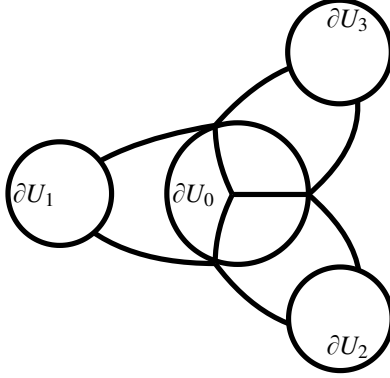


FIGURE 4. Contour Σ and the circles ∂U_l .

of **RHP- \mathbf{X}** in the form

$$(5.18) \quad \mathbf{X}(z) = \mathbf{CZ}(z) \begin{cases} \mathbf{M}^{\text{reg}}(z) \Phi^{n\sigma_3}, & z \notin (\Gamma \cup \bar{U}), \\ \mathbf{P}_l(z), & z \in U_l \setminus \Gamma, \quad l \in \{0, 1, 2, 3\}. \end{cases}$$

Let $\Sigma := \partial U \cup (\Gamma \setminus (\Delta^\circ \cup U_1 \cup U_2 \cup U_3))$, see Figure 4. Combining the above equation with **RHP- \mathbf{X}** , we see that $\mathbf{Z}(z)$ should solve the following Riemann-Hilbert problem (**RHP- \mathbf{Z}**):

- (a) $\mathbf{Z}(z)$ is analytic in $\mathbb{C} \setminus \Sigma$ and $\lim_{z \rightarrow \infty} \mathbf{Z}(z) = \mathbf{I}$;

(b) $\mathbf{Z}(z)$ has continuous traces on Σ° that satisfy

$$\mathbf{Z}_+(s) = \mathbf{Z}_-(s) \begin{cases} \mathbf{P}_l(s) \mathbf{N}_*^{-1}(s), & s \in \partial U, \\ \mathbf{N}_*(s) \begin{pmatrix} 1 & 0 \\ \rho_i^{-1}(s) & 1 \end{pmatrix} \mathbf{N}_*^{-1}(s), & s \in \Gamma_i^\pm \setminus \bar{U}, \end{cases}$$

where $\mathbf{N}_*(z) := \mathbf{C}^{-1} \mathbf{N}^{\text{reg}}(z) = \mathbf{M}^{\text{reg}}(z) \Phi^{n\sigma_3}(z)$ and we understand that $\mathbf{P}_l(s)$ is used when $s \in \partial U_l$, $l \in \{0, 1, 2, 3\}$, in the first jump relation, and

$$\mathbf{Z}_+(s) = \mathbf{Z}_-(s) \mathbf{P}_{0-}(s) \begin{pmatrix} 1 & 0 \\ \rho_{i+1+}^{-1}(s) + \rho_{i-1-}^{-1}(s) & 1 \end{pmatrix} \mathbf{P}_{0+}^{-1}(s), \quad s \in \Delta_i^*.$$

Note that if the jump relations **RHP- \mathbf{P}_0** (b) on Δ_i^* are exactly the same as the ones in **RHP- \mathbf{X}** (b), then the jump relations on $\Delta_i^* \cap U_0$ in **RHP- \mathbf{Z}** (b) above are absent.

As usual in this line of arguments, we shall show that all the jump matrices in **RHP- \mathbf{Z}** (b) are uniformly close to \mathbf{I} along any sequence $\mathbb{N}_\varepsilon^{\text{reg}}$. Indeed, it follows from **RHP- \mathbf{P}_0** (d) and **RHP- \mathbf{P}_i** (d), $i \in \{1, 2, 3\}$, that the jump of $\mathbf{Z}(z)$ on ∂U can be estimated as

$$(5.19) \quad \mathbf{I} + \mathcal{O}_\varepsilon(n^{-1/3}) \quad \text{on } \partial U,$$

where the constants in $\mathcal{O}_\varepsilon(\cdot)$ are independent of n but do depend on ε . Furthermore, we get from (5.7) as well as (2.7) that the jump of $\mathbf{Z}(z)$ on $\Gamma_i^\pm \setminus \bar{U}$ satisfies

$$(5.20) \quad \mathbf{I} + \rho_i^{-1}(s) \Phi^{-2n}(s) \mathbf{M}^{\text{reg}}(s) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} (\mathbf{M}^{\text{reg}})^{-1}(s) = \mathbf{I} + \mathcal{O}_\varepsilon(e^{-cn})$$

for some constant $c > 0$, where we also used the fact that the arcs Γ_i^\pm lie fixed distance away from Δ (again, the constant in $\mathcal{O}_\varepsilon(\cdot)$ is independent of n but depends on ε).

Now, if it holds that $\kappa_{i,j}(z) \equiv 1$, then $\mathbf{P}_0(z)$ is an exact parametrix and $\mathbf{Z}(z)$ does not have jumps on Δ_i^* . Below, we assume that all of these functions are not identically 1, the case where only some of them can be considered analogously. Let p be the largest integer such that $|1 - \kappa_{i,j}(s)| = \mathcal{O}(|s - a_0|^p)$ as $s \rightarrow a_0$ for all the pairs i, j . It necessarily holds that $p \geq 1$. It follows from **RHP- \mathbf{P}_0** (b) that the jump of $\mathbf{Z}(z)$ on Δ_i^* can be written as

$$(5.21) \quad \mathbf{I} + \left(\frac{1 - \kappa_{i,i-1}(s)}{\rho_{i+1+}(s)} + \frac{1 - \kappa_{i,i+1}(s)}{\rho_{i-1-}(s)} \right) \mathbf{P}_{0+}(s) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{P}_{0+}^{-1}(s) \\ = \mathbf{I} + \mathcal{O}(|s - a_0|^p) \mathbf{E}_0^{\text{reg}}(s) \Phi_{\alpha_0} \left(n^{2/3} \zeta(s); 0 \right) \mathbf{J} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{J}^{-1} \Phi_{\alpha_0}^{-1} \left(n^{2/3} \zeta(s); 0 \right) \mathbf{E}_0^{\text{reg}}(s)^{-1},$$

where we used estimates $|\rho_i(s)| \sim |s - a_0|^{\alpha_0} \sim |r(s)|^2$, see (5.12) and **RHP- \mathbf{P}_0** (b), as well as (5.14) (notice also that we do not need to take boundary values as the relevant entries of Φ_{α_0} are analytic across the considered arcs). Let $Z = n^{2/3} \zeta(s)$. We shall estimate the expression in (5.21) separately in two regimes: when $|Z| > 1$ and when $|Z| \leq 1$. Uniformity of the asymptotics in **RHP- Φ_α** (d) and (5.15) yield that

$$(5.22) \quad \Phi_{\alpha_0}(Z; 0) \mathbf{J} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{J}^{-1} \Phi_{\alpha_0}^{-1}(Z; 0) = \mathcal{O} \left(|Z|^{1/2} e^{-(4/3)|Z|^{3/2}} \right) = \mathcal{O}(|Z|^{-p})$$

uniformly for $|Z| > 1$, where one needs to observe that $Z^{3/2} > 0$ for $s \in \Delta_1^*$ and $Z^{3/2} < 0$ for $s \in \Delta_2^* \cup \Delta_3^*$, see (5.9) (recall also that $\det(\Phi_{\alpha_0}(Z; 0)) \equiv 1$, and notice that $|Z|^{p+1/2} e^{-(4/3)|Z|^{3/2}}$ is a bounded above positive function of $|Z|$ on $[1, \infty)$). On the other hand, we get from **RHP- Φ_α** (c) that

$$(5.23) \quad \Phi_{\alpha_0}(Z; 0) \mathbf{J} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{J}^{-1} \Phi_{\alpha_0}^{-1}(Z; 0) = \\ \begin{cases} \mathcal{O}(1) |Z|^{\alpha_0 \sigma_3/2} \mathcal{O}(1) |Z|^{-\alpha_0 \sigma_3/2} \mathcal{O}(1), & \alpha_0 \neq 0, \\ \mathcal{O}(1) \mathcal{O}(\log |Z|) \mathcal{O}(1) \mathcal{O}(\log |Z|) \mathcal{O}(1), & \alpha_0 = 0, \end{cases} = \begin{cases} \mathcal{O}(|Z|^{-|\alpha_0|}), & \alpha_0 \neq 0, \\ \mathcal{O}(\log^2 |Z|), & \alpha_0 = 0, \end{cases} = \mathcal{O}(|Z|^{-p})$$

uniformly for $|Z| \leq 1$ because $|Z|^{p-1|\alpha_0|}$ and $|Z|^p \log^2 |Z|$ are bounded positive functions of $|Z|$ on $[0, 1]$ since $p \geq 1$ and $|\alpha_0| \leq 1$. Recall that $|z - a_0|/|\zeta(z)|$ is bounded in U_0 and $\mathbf{E}_0^{\text{reg}}(z)$ is an analytic matrix function with the same determinant as $\mathbf{M}^{\text{reg}}(z)$. Definition (5.16), estimates (5.7), and the maximum modulus principle yield that $\mathbf{E}_0^{\text{reg}}(z) = \mathcal{O}_\varepsilon(1)n^{\sigma_3/6}$ and $\mathbf{E}_0^{\text{reg}}(z)^{-1} = \mathcal{O}_\varepsilon(1)n^{-\sigma_3/6}$. It now follows from (5.21)–(5.23) that the jump of $\mathbf{Z}(z)$ on $\cup \Delta_i^*$ can be estimated as

$$(5.24) \quad \mathbf{I} + \mathcal{O}_\varepsilon \left(n^{1/3} \max_{s \in \cup \Delta_i^*} |s - a_0|^p |Z|^{-p} \right) = \mathbf{I} + \mathcal{O}_\varepsilon(n^{-(2p-1)/3}),$$

where $Z = n^{2/3}\zeta(s)$. As $p \geq 1$, the error rate in (5.24) is not worse than $\mathcal{O}_\varepsilon(n^{-1/3})$.

Finally, by arguing as in [5, Theorem 7.103 and Corollary 7.108], see also [9, Theorem 8.1], we obtain from (5.19), (5.20), and (5.24) that the matrix $\mathbf{Z}(z)$ exists for all $n \in \mathbb{N}_\varepsilon^{\text{reg}}$ large enough and that

$$\|\mathbf{Z}_\pm - \mathbf{I}\|_{2,\Sigma} = \mathcal{O}_\varepsilon(n^{-1/3}).$$

Since the jumps of $\mathbf{Z}(z)$ on Σ are restrictions of holomorphic matrix functions, the standard deformation of the contour technique and the above estimate yield that

$$(5.25) \quad \mathbf{Z} = \mathbf{I} + \mathcal{O}_{\delta,\varepsilon}(n^{-1/3}) \quad \text{uniformly in } \overline{\mathbb{C}} \setminus \{a_0\}.$$

5.6. Proof of Theorem 2. Let $\mathbf{Z}(z)$ be a solution of RHP-Z, in which case (5.18) holds. Given a closed set $K \subset \overline{\mathbb{C}} \setminus \Delta$, the contour Σ can always be adjusted so that K lies in the exterior domain of Σ . Then it follows from (5.3) that $\mathbf{Y}(z) = \mathbf{X}(z)$ on K . Formulae (3.4) and (3.5) now follow immediately from (5.2), (5.4) and (5.25) since

$$w^{i-1}(z) [(\mathbf{Z}\mathbf{N}_*) (z)]_{1i} = (1 + v_{n1}(z))\Psi_n(z^{(i-1)}) + v_{n2}(z)\Psi_{n-1}(z^{(i-1)})$$

for $z \in K$, where $1 + v_{n1}(z), v_{n2}(z)$ are the first row entries of $\mathbf{Z}(z)$. Similarly, if K is a compact subset of Δ° , the lens Σ can be arranged so that K does not intersect \overline{U} . As before, we get from (5.2), (5.3), and (5.18) that

$$(5.26) \quad \gamma_n^{-1} \mathcal{Q}_n(z) = \left((1 + v_{n1}(z))\Psi_n(z) + v_{n2}(z)\Psi_{n-1}(z) \right) \pm (\rho_i w)^{-1}(z) \left((1 + v_{n1}(z))\Psi_n^*(z) + v_{n2}(z)\Psi_{n-1}^*(z) \right)$$

for $z \in \Omega_{i\pm} \setminus \overline{U}$. The top formula in (3.6) now follows by taking the trace of the right-hand side of the above equality on K and using the top relation in (2.3) (which yields that $\Psi_{n\pm}^*(s) = \Psi_{n\mp}(s)(\rho w_+)(s)$ for $s \in \Delta^\circ$). Since $R_n(z) = [\mathbf{X}]_{12}(z)$ for $z \in \Omega_{i\pm} \setminus \overline{U}$, the bottom relation in (3.6) is even simpler to derive.

6. PROOF OF THEOREM 3

The main difference between proofs of Theorem 2 and 3 is that we no longer can use the matrix $\Phi_{\alpha_0}(\zeta; 0)$ and shall substitute it by $\Phi_{\alpha_0}^{\text{sing}}(\zeta; 0)$, which has slightly different behavior at infinity. This change necessitates modifications in matrices $\mathbf{M}^{\text{reg}}(z)$ and $\mathbf{E}_0^{\text{reg}}(z)$.

6.1. On Matrices $\mathbf{E}_0^{\text{reg}}(z)$. To be able to introduce necessary modifications of $\mathbf{M}^{\text{reg}}(z)$ and $\mathbf{E}_0^{\text{reg}}(z)$, we need to discuss in more detail the behavior of $\mathbf{E}_0^{\text{reg}}(z)$ at a_0 . To this end, let us write

$$\mathbf{E}_0^{\text{reg}}(z) = \begin{pmatrix} E_{n,1}(z) & E_{n,2}(z) \\ E_{n,3}(z) & E_{n,4}(z) \end{pmatrix} n^{\sigma_3/6}.$$

It follows readily from (5.16) and (5.4) that

$$(6.1) \quad \begin{cases} E_{n,i}(z) & := (i\zeta^{1/2}(z))^{1-i} G(z) (T_n(z) - (-1)^i (HT_n^*)(z)), \\ E_{n,i+2}(z) & := (i\zeta^{1/2}(z))^{1-i} G(z) (\tilde{T}_{n-1}(z) - (-1)^i (\tilde{HT}_{n-1}^*)(z)), \end{cases} \quad z \in S_2,$$

for $i \in \{1, 2\}$, where we set $\tilde{T}_m(z) := (T_m/\Phi)(z)$, and

$$(6.2) \quad G(z) := \frac{(\zeta^{1/4} S_{\rho r})(z)}{\sqrt{2}}, \quad H(z) := \frac{1}{i((S_{\rho r})^2 w)(z)}, \quad z \in S_2.$$

Notice that the functions $G(z)$ and $H(z)$ remain bounded as $S_2 \ni z \rightarrow a_0$ by (2.14) and the very definitions of $r(z)$ in (5.12) and of $\zeta(z)$ in (5.8). We are interested in the quantity

$$(6.3) \quad D_n := E_{n,3}(a_0)E'_{n,1}(a_0) - E_{n,1}(a_0)E'_{n,3}(a_0).$$

Since $E_{n,i}(z)$ are analytic at a_0 and the functions $T_n(z)$ are analytic at a_0 , the lifts of $G(z)$ and $H(z)$ to $\pi^{-1}(S_2) \cap \mathfrak{S}^{(0)}$ admit analytic continuations to some neighborhood of a_0 . Thus, the quantities $G(a_0)$ and $H(a_0)$ are well-defined. In fact, since $E_{n,2}(a_0)$ and $E_{n,4}(a_0)$ are finite while $T_n(a_0) = T_n^*(a_0)$ and $\tilde{T}_n(a_0) = \tilde{T}_n^*(a_0)$, it must follow from (6.1) that

$$(6.4) \quad H(z) = 1 + h_1(z - a_0)^{1/2} + h_2(z - a_0) + \dots$$

(from now on all the stated expansions are assumed to hold for $z \in S_2 \cap U_0$), where we use the same determination of $(z - a_0)^{1/2}$ as in (2.28). It also clearly follows from (2.28) and (4.18) that

$$\tilde{T}_n(z) = t_0^{(n)} + t_1^{(n)}(z - a_0)^{1/2} + t_2^{(n)}(z - a_0) + \dots,$$

(that is, the first three coefficients in the Puiseux series of $\tilde{T}_n(z)$ and of $T_n(z)$ are the same). Observe also that

$$T_n^*(z) = t_0^{(n)} - t_1^{(n)}(z - a_0)^{1/2} + t_2^{(n)}(z - a_0) + \dots.$$

Let us write $G(z) = g_0 + g_1(z - a_0)^{1/2} + g_2(z - a_0) + \dots$. Notice that $g_0 \neq 0$ by (2.14). Then a straightforward computation shows that

$$(6.5) \quad E_{n,2k+1}(z) = 2g_0 t_0^{(n-k)} + \left((2g_2 + g_1 h_1 + g_0 h_2) t_0^{(n-k)} + g_0 \left(2t_2^{(n-k)} - h_1 t_1^{(n-k)} \right) \right) (z - a_0) + \dots$$

for $k \in \{0, 1\}$ (in particular, it must hold that $2g_1 + g_0 h_1 = 0$). Hence, we get from the definitions of the numbers D_n in (6.3) and the sequence $\mathbb{N}_{\varepsilon}^{\text{sing}}$ in (2.29) that

$$(6.6) \quad |D_n| = 2|g_0|^2 |\det(\mathbf{T}_n)| \geq 2|g_0|^2 \varepsilon > 0, \quad n \in \mathbb{N}_{\varepsilon}^{\text{sing}}.$$

Let us also justify the remark made after Proposition 5. Analysis similar to the one above yields that

$$E_{n,2k+2}(a_0) = -ig_0 \zeta_1 \left(2t_1^{(n-k)} - h_1 t_0^{(n-k)} \right), \quad \zeta_1 := \left(\lim_{z \rightarrow a_0} \frac{z - a_0}{\zeta(z)} \right)^{1/2},$$

for $k \in \{0, 1\}$. It further follows from (5.16) and the display after (5.4) that

$$(\gamma_n \gamma_{n-1}^*)^{-1} = \det(\mathbf{E}_0^{\text{reg}}(z)) = (E_{n,1} E_{n,4} - E_{n,2} E_{n,3})(z),$$

which is true for any $z \in U_0$. Therefore, taking $z = a_0$ in the above equality gives that

$$(6.7) \quad (\gamma_n \gamma_{n-1}^*)^{-1} = -4i \zeta_1 g_0^2 \left(t_0^{(n)} t_1^{(n-1)} - t_0^{(n-1)} t_1^{(n)} \right).$$

The desired claim now follows from (5.5)–(5.6).

6.2. Global and Local Parametrix. Recall the definition of $\mathbf{M}^{\text{reg}}(z)$ in (5.4). The new global parametrix is now defined as $N(z) = \mathbf{C} \mathbf{M}^{\text{sing}}(z) \Phi^{n\sigma_3}(z)$, where

$$(6.8) \quad \mathbf{M}^{\text{sing}}(z) := (\mathbf{I} + (z - a_0)^{-1} \mathbf{L}) \mathbf{M}^{\text{reg}}(z)$$

and the matrix \mathbf{L} is given by

$$(6.9) \quad \mathbf{L} := \frac{1}{D_n} \begin{pmatrix} -(E_{n,1} E_{n,3})(a_0) & E_{n,1}^2(a_0) \\ -E_{n,3}^2(a_0) & (E_{n,1} E_{n,3})(a_0) \end{pmatrix}.$$

Notice that $M^{\text{reg}}(\infty) = M^{\text{sing}}(\infty) = C^{-1}$ and, since L has zero trace and determinant, $\det(M^{\text{reg}}(z)) = \det(M^{\text{sing}}(z))$. Observe also that the just defined matrix $N(z)$ satisfies **RHP-N**(a,b) and **RHP-N**(c) around a_1, a_2, a_3 . However, its behavior at a_0 is obviously different. Most importantly for us, it is still true that

$$(6.10) \quad |(M^{\text{sing}})^{\pm 1}(z)| = \mathcal{O}_{K,\varepsilon}(1)$$

for $z \in K$ as $\mathbb{N}_\varepsilon^{\text{sing}} \ni n \rightarrow \infty$, where K is any compact set that avoids a_0, a_1, a_2, a_3 . Indeed, since $\mathbb{N}_\varepsilon^{\text{sing}} \subseteq \mathbb{N}_\varepsilon^{\text{reg}}$, estimates (5.7) still hold. It was computed in the previous subsection that

$$(6.11) \quad |E_{n,2k+1}(a_0)| = 2|g_0 t_0^{(n-k)}| \leq 2|g_0| M_R$$

for all $n \in \mathbb{N}$, where the last conclusion is a consequence of (2.26). Thus, (6.10) follows from (6.6).

Local parametrix $P_0(z)$ is now constructed similarly to (5.14) as

$$(6.12) \quad P_0(z) = E_0^{\text{sing}}(z) \Phi_{\alpha_0}^{\text{sing}}(n^{2/3} \zeta(z); 0) J(z) r^{-\sigma_3}(z),$$

where $\Phi_{\alpha_0}^{\text{sing}}(\zeta; x)$ is the solution of **RHP- $\Phi_\alpha^{\text{sing}}$** from Appendix B and $E_0^{\text{sing}}(z)$ is given by

$$(6.13) \quad E_0^{\text{sing}}(z) := \left(I + (z - a_0)^{-1} L \right) E_0^{\text{reg}}(z) (n^{2/3} \zeta(z))^{-\sigma_3}.$$

It follows from (5.16), (B.12), and (6.8) that (5.17) holds with $M^{\text{reg}}(z)$ replaced by $M^{\text{sing}}(z)$. Hence, provided $E_0^{\text{sing}}(z)$ is analytic in U_0 , the just constructed matrix $P_0(z)$ solves **RHP- P_0** with $M^{\text{reg}}(z)$ replaced by $M^{\text{sing}}(z)$ in **RHP- P_0** (d) (again, entries of L are bounded by (6.6) and (6.11) along $\mathbb{N}_\varepsilon^{\text{sing}}$). To show analyticity of $E_0^{\text{sing}}(z)$ in U_0 , observe that

$$E_0^{\text{sing}}(z) = \begin{pmatrix} 1 - \frac{(E_{n,1} E_{n,3})(a_0)}{(z-a_0)D_n} & \frac{E_{n,1}^2(a_0)}{(z-a_0)D_n} \\ -\frac{E_{n,3}^2(a_0)}{(z-a_0)D_n} & 1 + \frac{(E_{n,1} E_{n,3})(a_0)}{(z-a_0)D_n} \end{pmatrix} \begin{pmatrix} \frac{E_{n,1}(z)}{z-a_0} & (z-a_0)* \\ \frac{E_{n,3}(z)}{z-a_0} & (z-a_0)* \end{pmatrix} \left(\frac{n^{1/2} \zeta(z)}{z-a_0} \right)^{-\sigma_3}$$

by (6.9). Recall that $\zeta(z)$ has a simple zero at a_0 and therefore the last matrix above is analytic at a_0 . Thus, we only need to investigate analyticity of the first column of the product of the first two matrices above. The (1, 1)-entry of this product is equal to

$$\frac{1}{z-a_0} \left(E_{n,1}(z) - E_{n,1}(a_0) \frac{E_{n,3}(a_0) E_{n,1}(z) - E_{n,1}(a_0) E_{n,3}(z)}{(z-a_0)D_n} \right),$$

which is indeed analytic at a_0 by the very choice of D_n in (6.3). The fact that (2, 1)-entry of the product is analytic at a_0 can be checked analogously.

6.3. A Discussion. In this subsection we briefly digress from the proof of Theorem 3 and provide a broader view of the construction of global and local parametrices presented in this and the previous sections.

In this section we are forced to use matrix function $\Phi_{\alpha_0}^{\text{sing}}(\zeta; 0)$ instead of $\Phi_{\alpha_0}(\zeta; 0)$ as the latter does not exist in the considered case due to polar singularities at $x = 0$ of the functions $U_k(x)$ from (B.1). However, one can clearly see from (B.10) that $\Phi_{\alpha_0}^{\text{sing}}(\zeta; x)$ is actually well-defined around zero as long as $U_1(0) \neq 0$. A natural question arises whether one could use $\Phi_{\alpha_0}^{\text{sing}}(\zeta; 0)$ instead of $\Phi_{\alpha_0}(\zeta; 0)$ in all cases leading to $U_1(0) \neq 0$. Let us explain that this is indeed possible.

If we want to use $\Phi_{\alpha_0}^{\text{sing}}(\zeta; 0)$ as model local parametrix, we do need to introduce modified global parametrix $M^{\text{sing}}(z)$ via (6.8) with L given by (6.9), but differently defined constants D_n . Indeed, if $U_1(0) \neq 0$, then (B.12) no longer holds and needs to be replaced by

$$\left(I - \frac{1}{\zeta} \begin{pmatrix} 0 & 0 \\ 2^{-1/3} i U_1^{-1}(0) & 0 \end{pmatrix} \right) \Phi_{\alpha}^{\text{sing}}(\zeta; 0) = \frac{\zeta^{3\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \left(I + \mathcal{O}(\zeta^{-1/2}) \right) \exp \left\{ -\frac{2}{3} \zeta^{3/2} \sigma_3 \right\}$$

(in fact, the above formula extends (B.12) since $U_1^{-1}(0) = 0$ for weights in W^{sing}). This change necessitates the following modification in the definition of $\mathbf{E}_0^{\text{sing}}(z)$:

$$\begin{aligned} \mathbf{E}_0^{\text{sing}}(z) &:= \left(\mathbf{I} + \frac{\mathbf{L}}{z - a_0} \right) \mathbf{E}_0^{\text{reg}}(z) (n^{2/3} \zeta(z))^{-\sigma_3} \left(\mathbf{I} - \frac{2^{-1/3} i U_1^{-1}(0)}{n^{2/3} \zeta(z)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) \\ &= \left(\mathbf{I} + \frac{\mathbf{L}}{z - a_0} \right) \left[\begin{pmatrix} \frac{E_{n,1}(z)}{z - a_0} & (z - a_0) E_{n,2}(z) \\ \frac{E_{n,3}(z)}{z - a_0} & (z - a_0) E_{n,4}(z) \end{pmatrix} + n^{1/3} \Lambda(z) \begin{pmatrix} E_{n,2}(z) & 0 \\ E_{n,4}(z) & 0 \end{pmatrix} \right] \left(\frac{n^{1/2} \zeta(z)}{z - a_0} \right)^{-\sigma_3}, \end{aligned}$$

where $\Lambda(z) := -2^{-1/3} i U_1^{-1}(0) (\zeta(z)/(z - a_0))^2$. Therefore, $\mathbf{E}_0^{\text{sing}}(z)$ is analytic at a_0 if

$$\frac{\mathbf{L}}{(z - a_0)^2} \begin{pmatrix} E_{n,1}(z) \\ E_{n,3}(z) \end{pmatrix} + \frac{1}{z - a_0} \begin{pmatrix} E_{n,1}(z) \\ E_{n,3}(z) \end{pmatrix} + \frac{n^{1/3} \Lambda(z) \mathbf{L}}{z - a_0} \begin{pmatrix} E_{n,2}(z) \\ E_{n,4}(z) \end{pmatrix}$$

is analytic at a_0 . To achieve the latter, one needs to take \mathbf{L} as in (6.9) with

$$D_n = n^{1/3} \Lambda(0) (E_{n,3} E_{n,2} - E_{n,1} E_{n,4})(a_0) + (E_{n,3} E'_{n,1} - E_{n,1} E'_{n,3})(a_0).$$

Observe that when $U_1^{-1}(0) = 0$, $\Lambda(0) = 0$ and we recover the original definition of D_n in (6.3). Hence, when $U_1^{-1}(0) \neq 0$, it follows from (6.7) and the display before that D_n is non-zero for all $n \in \mathbb{N}_\varepsilon^{\text{reg}}$ large enough and the above construction goes through. It also holds in this case that $\mathbf{L} = \mathcal{O}(n^{-1/3})$ and therefore matrices $\mathbf{M}^{\text{reg}}(z)$ and $\mathbf{M}^{\text{sing}}(z)$ are asymptotically the same.

6.4. Proof of Theorem 3. Consider now **RHP-Z** with $N_*(z) = \mathbf{M}^{\text{sing}}(z) \Phi^{n\sigma_3}(z)$ and $\mathbf{P}_0(z)$ given by (6.12). Again, let us estimate the size of the jump matrices in **RHP-Z**(b). It follows from (6.10) that estimates (5.19)–(5.20) remain valid in the present case. Using (B.12) instead of **RHP- Φ_α** (d) gives the same estimate as in (5.22), one only needs to use boundedness of $|Z|^{p+3/2} e^{-(4/3)|Z|^{3/2}}$. Since **RHP- $\Phi_\alpha^{\text{sing}}$** (c) is the same as **RHP- Φ_α** (c), the estimate (5.23) remain unchanged. It follows from the definition of $\mathbf{E}_0^{\text{sing}}(z)$ in (6.13), its holomorphy in U_0 , the maximum modulus principle, and (6.10) that $\mathbf{E}_0^{\text{sing}}(z) = \mathcal{O}_\varepsilon(1) n^{-\sigma_3/2}$ and $\mathbf{E}_0^{\text{sing}}(z)^{-1} = \mathcal{O}_\varepsilon(1) n^{\sigma_3/2}$. Therefore, (5.24) now is replaced by

$$\mathbf{I} + \mathcal{O}_\varepsilon(n \max |s - a_0|^p |Z|^{-p}) = \mathbf{I} + \mathcal{O}_\varepsilon(n^{-(2p-3)/3}).$$

Recall that p is the largest integer such that $|1 - \kappa_{i,j}(s)| = \mathcal{O}(|s - a_0|^p)$ as $s \rightarrow a_0$. It can be readily checked that condition (3.7) implies that $p \geq 2$ and therefore $(2p - 3)/3 \geq 1/3$. Thus, the jump matrices in **RHP-Z**(b) still satisfy the estimate $\mathbf{I} + \mathcal{O}_\varepsilon(n^{-1/3})$. Hence, we again can conclude that $\mathbf{Z}(z)$ exists for all $n \in \mathbb{N}_\varepsilon^{\text{sing}}$ large enough and satisfies (5.25).

As in the proof of Theorem 2, given a closed set $K \subset \mathbb{C} \setminus \Delta$, the contour Σ can always be adjusted so that K lies in the exterior domain of Σ . Then we get from (5.3) and (5.18) that

$$\mathbf{Y}(z) = \mathbf{X}(z) = \mathbf{C} \mathbf{M}^{\text{sing}}(z) \Phi^{n\sigma_3}(z), \quad z \in K.$$

Therefore, we get from (5.2) that

$$\begin{aligned} \gamma_n^{-1} Q_n(z) &= \left(1 + v_{n1}^{\text{sing}}(z) \right) \left(\left(1 + \frac{[\mathbf{L}]_{11}}{z - a_0} \right) \Psi_n(z) + \frac{[\mathbf{L}]_{12}}{z - a_0} \Psi_{n-1}(z) \right) \\ &\quad + v_{n2}^{\text{sing}}(z) \left(\left(1 + \frac{[\mathbf{L}]_{22}}{z - a_0} \right) \Psi_{n-1}(z) + \frac{[\mathbf{L}]_{21}}{z - a_0} \Psi_n(z) \right), \quad z \in K, \end{aligned}$$

where $1 + v_{n1}^{\text{sing}}(z)$, $v_{n2}^{\text{sing}}(z)$ are the first row entries of $\mathbf{Z}(z)$. For $i \in \{1, 2\}$, let

$$(6.14) \quad v_{ni}(z) := v_{ni}^{\text{sing}}(z) \left(1 + \frac{[\mathbf{L}]_{ii}}{z - a_0} \right) + v_{n3-i}^{\text{sing}}(z) \frac{[\mathbf{L}]_{3-ii}}{z - a_0}.$$

It follows from (6.9), (6.11), and (6.6) that $v_{ni}(z)$ satisfy bounds as in (3.5). Moreover, we have that

$$\gamma_n^{-1} Q_n(z) = \left(1 + \frac{[\mathbf{L}]_{11}}{z - a_0} + v_{n1}(z) \right) \Psi_n(z) + \left(\frac{[\mathbf{L}]_{12}}{z - a_0} + v_{n2}(z) \right) \Psi_{n-1}(z), \quad z \in K.$$

We further get from (6.9), (6.5), and (6.6) that

$$[\mathbf{L}]_{11} = -\frac{(E_{n,1}E_{n,3})(a_0)}{D_n} = -\frac{2t_0^{(n-1)}t_0^{(n)}}{\det(\mathbf{T}_n)} \quad \text{and} \quad [\mathbf{L}]_{12} = \frac{E_{n,1}^2(a_0)}{D_n} = \frac{2(t_0^{(n)})^2}{\det(\mathbf{T}_n)}.$$

This finishes the proof of the asymptotic formula for $Q_n(z)$ while the formula for $(wR_n)(z)$ can be shown absolutely analogously. Similarly, we get from (5.2), (5.3), and (5.18) with $\mathbf{M}^{\text{reg}}(z)$ replaced by $\mathbf{M}^{\text{sing}}(z)$ that

$$\begin{aligned} \gamma_n^{-1}Q_n(z) &= \left(\left(1 + \frac{[\mathbf{L}]_{11}}{z - a_0} + v_{n1}(z) \right) \Psi_n(z) + v_{n2}(z)\Psi_{n-1}(z) \right) \\ &\quad \pm (\rho_i w)^{-1}(z) \left(\left(1 + \frac{[\mathbf{L}]_{11}}{z - a_0} + v_{n1}(z) \right) \Psi_n^*(z) + v_{n2}(z)\Psi_{n-1}^*(z) \right) \end{aligned}$$

for $z \in \Omega_i^\pm \setminus \bar{U}$ with $v_{ni}(z)$ as in (6.14). After that the proof of the analog of the first relation in (3.6) proceeds exactly as after (5.26). The proof of the second relation in (3.6) can be obtained analogously.

APPENDIX A. RIEMANN-HILBERT PROBLEM FOR PAINLEVÉ II

A.1. Lax Pair and the Corresponding Riemann-Hilbert Problem. The material of this section originates in [8]. However, we essentially follow the presentation in [9, Section 5.0], see also [9, Section 1.0] for the relevant facts of the general theory of differential equations. Let $q(s)$ be a solution of Painlevé II equation

$$(A.1) \quad q''(s) = sq(s) + 2q^3(s) - v, \quad v > -1/2.$$

It is known that $q(s)$ is a meromorphic function in the entire complex plane. For each s , which is not a pole of q , consider the following system of differential equations:

$$(A.2) \quad \partial_\tau \Psi(\tau; s) = \mathbf{A}^{FN}(\tau; s)\Psi(\tau; s),$$

where

$$(A.3) \quad \mathbf{A}^{FN}(\tau; s) := \begin{pmatrix} -i(4\tau^2 + s + 2q^2(s)) & 4\tau q(s) + 2iq'(s) + v/\tau \\ 4\tau q(s) - 2iq'(s) + v/\tau & i(4\tau^2 + s + 2q^2(s)) \end{pmatrix}$$

and

$$(A.4) \quad \partial_s \Psi(\tau; s) = \mathbf{U}^{FN}(\tau; s)\Psi(\tau; s), \quad \mathbf{U}^{FN}(\tau; s) := \begin{pmatrix} -i\tau & q(s) \\ q(s) & i\tau \end{pmatrix}.$$

In general, such a system would be overdetermined, but the compatibility condition

$$(A.5) \quad \partial_s \mathbf{A}^{FN}(\tau; s) - \partial_\tau \mathbf{U}^{FN}(\tau; s) + [\mathbf{A}^{FN}(\tau; s), \mathbf{U}^{FN}(\tau; s)] = 0$$

of (A.2)–(A.4) exactly reduces to the fact that $q(s)$ solves (A.1). The general theory of differential equations implies that equation (A.2)–(A.3) has a set of canonical solutions $\Psi_k(\tau; s)$, $k \in \{1, \dots, 6\}$, that are uniquely determined by the conditions $\det(\Psi_k(\tau; s)) \equiv 1$ and

$$(A.6) \quad \Psi_k(\tau; s) = (\mathbf{I} + \mathcal{O}(\tau^{-1})) e^{-i(\frac{4}{3}\tau^3 + s\tau)\sigma_3} \quad \text{as} \quad \tau \rightarrow \infty, \quad \arg(\tau) \in \left(\frac{(k-2)\pi}{3}, \frac{k\pi}{3} \right).$$

Given any solution $\Psi(\tau; s)$ of (A.2)–(A.3) one can readily check by using (A.5) that

$$\partial_s \Psi(\tau; s) - \mathbf{U}^{FN}(\tau; s)\Psi(\tau; s)$$

must also satisfy (A.2)–(A.3). Hence, this difference is equal to $\Psi(\tau; s)\mathbf{S}(s)$, where $\mathbf{S}(s)$ is a matrix that does not depend on τ and can be expressed as

$$(A.7) \quad \mathbf{S}(s) = \Psi^{-1}(\tau; s)\partial_s \Psi(\tau; s) - \Psi^{-1}(\tau; s)\mathbf{U}^{FN}(\tau; s)\Psi(\tau; s).$$

A further straightforward computation leads to the fact that the whole system (A.2)–(A.4) is solved by

$$(A.8) \quad \Psi(\tau; s)\mathbf{Q}(s), \quad \partial_s \mathbf{Q}(s) = -(\mathbf{S}\mathbf{Q})(s).$$

By explicitly computing the term next to τ^{-1} in (A.6) (see (A.17) and (A.18) further below and note that the asymptotic expansion of $\Psi_k(\tau; s)$ holds in the whole Stokes sector $((k-2)\pi/3, k\pi/3)$), plugging this asymptotic formula into (A.7), and using the fact that (A.7) must be independent of τ , one can deduce that $S(s)$ corresponding to any $\Psi_k(\tau; s)$ is identically zero, i.e., we can take $Q(s) = I$ in (A.8), and therefore $\Psi_k(\tau; s)$ satisfies (A.4) as well (see the paragraph containing [9, Equation (5.0.7)]).

Define the rays Σ_k^{FN} and sectors S_k^{FN} by

$$\Sigma_k^{FN} := \left\{ \tau : \arg(\tau) = (2k-1)\pi/6 \right\} \quad \text{and} \quad S_k^{FN} := \left\{ z : \arg(z) \in \left(\frac{(2k-3)\pi}{6}, \frac{(2k-1)\pi}{6} \right) \right\},$$

where the rays are oriented away from the origin. Further, put

$$(A.9) \quad \Psi_{\nu}^{FN}(\tau; s) := \Psi_k(\tau; s), \quad \tau \in S_k^{FN}.$$

Since the matrices $\Psi_k(\tau; s)$ solve the same differential equation, they are related to each other through the right multiplication by a constant matrix. By studying their behavior at infinity, one can conclude that

$$(A.10) \quad \Psi_{\nu+}^{FN}(\tau; s) = \Psi_{\nu-}^{FN}(\tau; s) \begin{cases} \begin{pmatrix} 1 & 0 \\ s_{2j-1} & 1 \end{pmatrix}, & \tau \in \Sigma_{2j-1}^{FN}, \\ \begin{pmatrix} 1 & s_{2j} \\ 0 & 1 \end{pmatrix}, & \tau \in \Sigma_{2j}^{FN}, \end{cases}$$

for some constants s_1, \dots, s_6 (Stokes parameters). Since $\sigma_1 \mathbf{A}^{FN}(-\tau; s) \sigma_1 = -\mathbf{A}^{FN}(\tau; s)$ and due to the uniqueness of the solution of (A.2)–(A.3) satisfying (A.6) in a given sector, it holds that

$$(A.11) \quad \Psi_{\nu}^{FN}(\tau; s) = \sigma_1 \Psi_{\nu}^{FN}(-\tau; s) \sigma_1, \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

This implies that there are only three independent Stokes parameters and $s_1 = s_4, s_2 = s_5, s_3 = s_6$.

On the other hand, since the residue of $\mathbf{A}^{FN}(\tau; s)$ at $\tau = 0$ is equal to

$$\begin{pmatrix} 0 & \nu \\ \nu & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \nu & 0 \\ 0 & -\nu \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

where all matrices have unit determinants, it holds that the canonical solution of (A.2)–(A.3) at the origin that has unit determinant³ is of the form

$$\Psi^{(0)}(\tau; s) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \left(\mathbf{I} + \sum_{k=1}^{\infty} \Psi_{0,k}(s) \tau^k \right) \tau^{\nu\sigma_3} \begin{pmatrix} 1 & \chi_{\nu}(\Pi(s) + \kappa(s) \log \tau) \\ 0 & 1 \end{pmatrix},$$

where $\chi_{\nu} = 0$ in the non-resonant cases $\nu+1/2 \notin \mathbb{N}$ and $\chi_{\nu} = 1$ in the resonant cases $\nu+1/2 \in \mathbb{N}$. The matrices $\Psi_{0,k}(s)$ and the number $\kappa(s)$ are uniquely determined by (A.2)–(A.3) under the condition that we set (1, 2)-entry of $\Psi_{0,2\nu}(s)$ to be zero in the resonant cases, while $\Pi(s)$ is a free parameter for each fixed s (equivalently, we could set $\Pi(s) \equiv 0$ and say that (1, 2)-entry of $\Psi_{0,2\nu}(s)$ is a free parameter, whose choice necessarily affects the values of the second column of every $\Psi_{0,k}(s)$, $k > 2\nu$). Let us put $\eta(\tau; s) := \chi_{\nu}(\Pi(s) + \kappa(s) \log \tau)$. Then we get that

$$(\Psi^{(0)})^{-1} \partial_s \Psi^{(0)} = \begin{pmatrix} 1 & -\eta \\ 0 & 1 \end{pmatrix} \tau^{-\nu\sigma_3} \mathcal{O}(\tau) \tau^{\nu\sigma_3} \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \partial_s \eta \\ 0 & 0 \end{pmatrix}.$$

Similarly, we obtain from (A.4) that

$$(\Psi^{(0)})^{-1} \mathbf{U}^{FN} \Psi^{(0)} = q\sigma_3 + \begin{pmatrix} 0 & 2q\eta \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -\eta \\ 0 & 1 \end{pmatrix} \tau^{-\nu\sigma_3} \mathcal{O}(\tau) \tau^{\nu\sigma_3} \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}.$$

Plugging the above expressions into (A.7) gives

$$S(s) = -q\sigma_3 + \begin{pmatrix} 0 & \partial_s \eta - 2q\eta \\ 0 & 0 \end{pmatrix} + \tau \begin{pmatrix} O_1 - \eta\tau^{2\nu} O_3 & -\eta^2 \tau^{2\nu} O_3 + \eta O_1 - \eta O_4 + \tau^{-2\nu} O_2 \\ \tau^{2\nu} O_3 & O_4 + \eta \tau^{2\nu} O_3 \end{pmatrix},$$

³Of course, $-\Psi^{(0)}(\tau; s)$ also solves (A.2)–(A.3) and has unit determinant.

where $O_l(\tau; s)$ are some analytic functions of τ coming from the terms $\tau^{-\nu\sigma_3}\mathcal{O}(\tau)\tau^{\nu\sigma_3}$ in the preceding two relations. Since the above expression must be independent of τ we see immediately that $O_1 = O_3 = O_4 \equiv 0$ and $O_2(\tau; s) = O(s)\tau^{2\nu-1}$ for some suitable function $O(s)$, in which case

$$\mathbf{S}(s) = -q(s)\sigma_3 + \begin{pmatrix} 0 & (\kappa'(s) - 2(q\kappa)(s)) \log \tau + \Pi'(s) - 2(q\Pi)(s) + O(s) \\ 0 & 0 \end{pmatrix}.$$

Let $Q(s)$ be the exponential of some fixed antiderivative of $q(s)$. The independence from τ yields that $\kappa(s) = \kappa_\nu Q^2(s)$ for some constant κ_ν . Further, let $Q_*(s)$ be a function (particular solution) such that $Q'_*(s) = (Q_*q)(s) - (OQ^{-1})(s)$. Then the differential equation in (A.8) for $\mathbf{S}(s)$ as above is solved by

$$\mathbf{Q}(s) = \begin{pmatrix} Q(s) & hQ(s) - \Pi(s)Q^{-1}(s) + Q_*(s) \\ 0 & Q^{-1}(s) \end{pmatrix},$$

where h is an arbitrary constant (we look for solutions of determinant one and therefore use exponential of the same antiderivative of $q(s)$ in the diagonal entries). Therefore, the solution of the whole system (A.2)–(A.4) around the origin assumes the form

$$(A.12) \quad \Psi_0(\tau; s) = \frac{\mathbf{I} - i\sigma_2}{\sqrt{2}} (\mathbf{I} + \mathcal{O}(\tau)) \tau^{\nu\sigma_3} \begin{pmatrix} 1 & \chi_\nu((\kappa_\nu \log \tau + h)Q^2(s) + (Q_*Q)(s)) \\ 0 & 1 \end{pmatrix} Q^{\sigma_3}(s).$$

Since the matrices $\Psi_0(\tau; s)$ and $\Psi_1(\tau; s)$ solve the same system of differential equations, namely, (A.2)–(A.4), they are connected via the right multiplication by a constant matrix. Thus, we can write

$$(A.13) \quad \Psi_1(\tau; s) = \Psi_0(\tau; s)E$$

in some disk around the origin. Notice that $\Psi_0(\tau; s)$ must also obey the symmetry in (A.11). Hence, $\sigma_1 \Psi_0(-\tau; s) \sigma_1$ is connected to $\Psi_0(\tau; s)$ through the right multiplication by a constant matrix as well. In fact, it is not hard to compute that

$$\sigma_1 \Psi_0(-\tau; s) \sigma_1 = \Psi_0(\tau; s) e^{\pi i \nu \sigma_3} \begin{pmatrix} \pi i \kappa_\nu & 1 \\ -1 & 0 \end{pmatrix}.$$

This relation together with (A.10) and (A.13) yields that

$$E = e^{\pi i \nu \sigma_3} \begin{pmatrix} \pi i \kappa_\nu & 1 \\ -1 & 0 \end{pmatrix} \sigma_1 E \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & s_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_3 & 1 \end{pmatrix} \sigma_1,$$

see [9, Equation (5.0.14)]. Solving the above system of relations gives that

$$(A.14) \quad E = \begin{pmatrix} 1 & \chi_\nu p \\ 0 & 1 \end{pmatrix} d^{\sigma_3} A_1^{FN}$$

where $d \neq 0, p$ are some constants and the matrix A_1^{FN} is explicitly known and involves expressions depending on s_1, s_2, s_3 and ν , see [9, Equations (5.0.17)–(5.0.18)] excluding the exceptional resonant cases $\nu + 1/2 \in \mathbb{N}$ and $s_1 = s_2 = s_3 = i(-1)^{\nu+1/2}$ in which it is a lower-triangular matrix with one's on the main diagonal (notice that we renamed $\sqrt{-J_+}$ from [9, Equations (5.0.18)] as d here). Moreover, this computation also yields that

$$(A.15) \quad s_1 + s_2 + s_3 + s_1 s_2 s_3 = -2i \sin(\nu\pi),$$

and that $\kappa_\nu = (-1)^{\nu+1/2}/\pi$ in the non-exceptional resonant cases and $\kappa_\nu = 0$ otherwise. Combining (A.12) with (A.13) then gives that in the neighborhood of $\tau = 0$ the function $\Psi_\nu^{FN}(\tau; s)$ admits a representation

$$(A.16) \quad \Psi_\nu^{FN}(\tau; s) = \mathbf{H}^{FN}(\tau; s) \tau^{\nu\sigma_3} \begin{pmatrix} 1 & \kappa_\nu \log \tau \\ 0 & 1 \end{pmatrix} A_k^{FN}, \quad \tau \in S_k^{FN},$$

where $\mathbf{H}^{FN}(\tau; s)$ is a holomorphic matrix function of τ around the origin while the matrices A_k^{FN} are connected to each other via the jump relations (A.10) with A_1^{FN} as in (A.14).

Altogether, we see that $\Psi_\nu^{FN}(\tau; s)$ solves the following Riemann-Hilbert problem (RHP- Ψ_ν):

- (a) $\Psi_\nu(\tau; s)$ is holomorphic in $\mathbb{C} \setminus (\{0\} \cup \Sigma^{FN})$, $\Sigma^{FN} := \cup_{k=1}^6 \Sigma_k^{FN}$;
- (b) $\Psi_\nu(\tau; s)$ has continuous traces on Σ^{FN} that satisfy (A.10);

- (c) $\Psi_\nu(\tau; s)$ satisfies (A.16) as $\tau \rightarrow 0$;
(d) $\Psi_\nu(\tau; s)$ satisfies $\Psi_\nu(\tau; s) = (\mathbf{I} + \mathcal{O}(\tau^{-1})) e^{-i(\frac{4}{3}\tau^3 + s\tau)\sigma_3}$ uniformly in $\{|\tau| \geq 1\} \setminus \Sigma^{FN}$.

Conversely, it follows from [9, Theorem 5.1] that given $\nu > -1/2$, parameters s_1, s_2, s_3 satisfying (A.15), and, in the exceptional resonant cases, the value of the free parameter of the matrix \mathbf{A}_1^{FN} from (A.14), a solution of RHP- Ψ_ν uniquely exists as a meromorphic function of s and

$$q(s) = 2i \lim_{\mathbb{R} \ni \tau \rightarrow \infty} \tau \left[\Psi_\nu(\tau; s) e^{i(\frac{4}{3}\tau^3 + s\tau)\sigma_3} \right]_{12}$$

solves (A.1), where $[\cdot]_{12}$ is the (1,2)-entry of the corresponding matrix. In particular, Stokes parameters s_1, s_2, s_3 uniquely parametrize solutions of (A.1) excluding the exceptional resonant cases each of which corresponds to a family of solutions parametrized by the free parameter of \mathbf{A}_1^{FN} .

A.2. Asymptotic Expansion. Later we shall need the first four terms of the full asymptotic expansion of $\Psi_\nu^{FN}(\tau; s)$ at infinity (the first is the identity matrix, so, we need to find the other three). Following classical ideas, see [9, Proposition 1.1], one can write the expansion of $\Psi_\nu^{FN}(\tau; s)$ as a series in $1/\tau$ with off-diagonal coefficients times the exponential of a series with diagonal coefficients. Symmetry (A.11) yields that

$$\Psi_\nu^{FN}(\tau; s) = \left(\mathbf{I} + \sum_{n=1}^{\infty} \tau^{-n} \psi_n(s) \Sigma_n \right) \exp \left\{ \sum_{n=-3, n \neq 0}^{\infty} \frac{\tau^{-n}}{n} \lambda_n(s) \Lambda_n \right\},$$

where $\Sigma_{2m} = \sigma_1$ and $\Sigma_{2m+1} = i\sigma_2$, while $\Lambda_{2m} = \mathbf{I}$ and $\Lambda_{2m+1} = \sigma_3$. Plugging the above expansion into (A.2)–(A.3) gives

$$\begin{aligned} & - \sum_{n=2}^{\infty} (n-1) \tau^{-n} \psi_{n-1} \Sigma_{n-1} - \left(\mathbf{I} + \sum_{n=1}^{\infty} \tau^{-n} \psi_n \Sigma_n \right) \sum_{n=-2, n \neq 1}^{\infty} \tau^{-n} \lambda_{n-1} \Lambda_{n-1} = \\ & (-4i\tau^2 \sigma_3 + 4q\tau \sigma_1 - i(s + 2q^2)\sigma_3 - 2q'\sigma_2 + \nu\tau^{-1}\sigma_1) \left(\mathbf{I} + \sum_{n=1}^{\infty} \tau^{-n} \psi_n \Sigma_n \right). \end{aligned}$$

From the coefficients next to τ^2 and τ we get that

$$\begin{cases} \lambda_{-3}(s) &= 4i, \\ \psi_1(s) &= -iq(s)/2, \\ \lambda_{-2}(s) &\equiv 0. \end{cases}$$

By comparing the constant terms on both sides as well as the terms next to τ^{-1} , we get

$$\begin{cases} \lambda_{-1}(s) &= is, \\ \psi_2(s) &= q'(s)/4, \\ \psi_3(s) &= i(q^3(s) + sq(s) - \nu)/8. \end{cases}$$

By equating the terms next to τ^{-2} , we obtain

$$\begin{cases} \lambda_1(s) &= i(q^4(s) + sq^2(s) - 2\nu q(s) - (q'(s))^2)/2, \\ \psi_4(s) &= -(sq'(s) + q^2(s)q'(s) + q(s))/16. \end{cases}$$

Lastly, the coefficients next τ^{-3} and τ^{-4} give us

$$\begin{cases} \lambda_2(s) &= q^2(s)/4, \\ \psi_5(s) &= -i(2q'(s) + q(s)H(s) + (q^2(s) + s)(q^3(s) + sq(s) - \nu))/32, \\ \lambda_3(s) &= -i(q'(s)q(s) - sH(s) + \nu^2)/8, \end{cases}$$

where we denote by $H(s) := (q'(s))^2 - sq^2(s) - q^4(s) + 2vq(s)$ the Hamiltonian of Painlevé II equation. Now, we would like to rewrite the expansion of $\Psi_v^{FN}(\tau; s)$ at infinity as (A.17)

$$\Psi_v^{FN}(\tau; s) = \left(\mathbf{I} + \sum_{n=1}^{\infty} \frac{1}{(-i\tau)^n} \begin{pmatrix} A_n(s) & B_n(s) \\ (-1)^n B_n(s) & (-1)^n A_n(s) \end{pmatrix} \right) \exp \left\{ -i \left(\frac{4}{3} \tau^3 + s\tau \right) \sigma_3 \right\}$$

for some functions $A_n(s), B_n(s)$, $n \in \mathbb{N}$, where the form of the coefficients follows from (A.11). Since

$$\exp \left\{ \sum_{n=-3, n \neq 0}^{\infty} \frac{\tau^{-n}}{n} \lambda_n(s) \Lambda_n \right\} = \left(\mathbf{I} + \frac{\lambda_1(s)}{\tau} \sigma_3 + \frac{\lambda_2(s) + \lambda_1^2(s)}{2\tau^2} \mathbf{I} + \frac{2\lambda_3(s) + 3(\lambda_1\lambda_2)(s) + \lambda_1^3(s)}{6\tau^3} \sigma_3 + \mathcal{O} \left(\frac{1}{\tau^4} \right) \right) \times \exp \left\{ -i \left(\frac{4}{3} \tau^3 + s\tau \right) \sigma_3 \right\},$$

we get that

$$\begin{cases} iA_1(s)\sigma_3 - B_1(s)\sigma_2 & = \lambda_1(s)\sigma_3 + i\psi_1(s)\sigma_2, \\ -A_2(s)\mathbf{I} - B_2(s)\sigma_1 & = (\lambda_2 + \lambda_1^2)(s)\mathbf{I}/2 + (\psi_2 - \psi_1\lambda_1)(s)\sigma_1, \\ -iA_3(s)\sigma_3 + B_3(s)\sigma_2 & = (2\lambda_3 + 3\lambda_2\lambda_1 + \lambda_1^3)(s)\sigma_3/6 + \\ & \quad i(2\psi_3 - 2\psi_2\lambda_1 + \psi_1(\lambda_2 + \lambda_1^2))(s)\sigma_2/2. \end{cases}$$

Then it holds that

$$(A.18) \quad \begin{cases} -2B_1(s) & = q(s), & \begin{cases} -4B_2(s) & = q'(s) + q(s)H(s), \\ -8A_2(s) & = q^2(s) - H^2(s), \end{cases} \\ -2A_1(s) & = H(s), \end{cases}$$

and, by recalling (A.1), that

$$\begin{cases} -16B_3(s) & = 2q''(s) - 3q^3(s) + 2q'(s)H(s) + q(s)H^2(s), \\ 48A_3(s) & = 2q'(s)q(s) + (3q^2(s) - 2s)H(s) - H^3(s) + 2v^2. \end{cases}$$

One also can verify either directly or by using (A.4) that $A'_n(s) = (-1)^n q(s)B_n(s)$.

Our primary interest is the behavior of $A_n(s), B_n(s)$ around a pole of $q(s)$. It can be readily checked that if $q(s)$ has a pole at s_0 , then

$$(A.19) \quad q(s) = \frac{q_{-1}}{s - s_0} + \mathcal{O}(s - s_0), \quad q_{-1} \in \{-1, 1\}, \quad \text{and} \quad H(s) = \frac{1}{s - s_0} + \mathcal{O}(1)$$

as $s \rightarrow s_0$. We shall need more precise behavior of $q(s)$ and $H(s)$ around $s_0 = 0$ when $q_{-1} = -1$. It is known, see [13, Equation (17.1)], or can be obtained directly from (A.1), that

$$(A.20) \quad \begin{cases} q(s) & = -\frac{1}{s} + \frac{v+1}{4}s^2 + \frac{q}{5}s^3 + \mathcal{O}(s^5), \\ H(s) & = \frac{1}{s} + 2q + \frac{v+1}{4}s^2 + \frac{2q}{15}s^3 + \mathcal{O}(s^5), \end{cases}$$

where q is a free parameter and we used the identity $H'(s) = -q^2(s)$ to find the expansion of $H(s)$ with the constant term found through its original definition. Hence, in this case it is also true that

$$\begin{cases} B_2(s) & = \frac{q}{2s} - \frac{v+1}{8}s + \mathcal{O}(s^2), \\ A_2(s) & = \frac{q}{2s} + \frac{q^2}{2} + \frac{v+1}{8}s + \mathcal{O}(s^2), \end{cases}$$

and

$$(A.21) \quad B_3(s) = \frac{q^2}{4s} + \mathcal{O}(s) \quad \text{and} \quad A_3(s) = -\frac{q^2}{4s} + \mathcal{O}(1).$$

APPENDIX B. RIEMANN-HILBERT PROBLEM FOR PAINLEVÉ XXXIV

The primary source for the material of this appendix is [15].

B.1. Riemann-Hilbert Problem. We retain the notation of the previous appendix. In what follows, we always assume that

$$\tau = i2^{-1/3}\zeta^{1/2} \quad \text{and} \quad s = -2^{1/3}x,$$

where $\arg(\zeta) \in (-\pi, \pi)$ and the square root is principal (that is, $\zeta_{\pm}^{1/2} = \pm i|\zeta|^{1/2}$ for $\zeta \in (-\infty, 0)$). Set

$$(B.1) \quad \begin{cases} U_n(x) := A_n(s) + (-1)^n B_n(s), \\ V_n(x) := A_n(s) - (-1)^n B_n(s). \end{cases}$$

Given $\alpha > -1$, let q be a solution of (A.1) with $\nu = \alpha + 1/2$ and x such that $q(s)$ is finite. Let $\Psi_{\alpha+1/2}^{FN}(\tau; s)$ be given by (A.9) and s_1, s_2, s_3 be the corresponding Stokes parameters satisfying (A.15). Set

$$(B.2) \quad \Phi_{\alpha}(\zeta; x) := \begin{pmatrix} 1 & 0 \\ -2^{1/3}iV_1(x) & 1 \end{pmatrix} \frac{\zeta^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{\pi i \sigma_3/4} \Psi_{\alpha+1/2}^{FN}(\tau; s) e^{-\pi i \sigma_3/4}.$$

Then $\Phi_{\alpha}(\zeta; x)$ solves the following Riemann-Hilbert problem (RHP- Φ_{α}):

- (a) $\Phi_{\alpha}(\zeta; x)$ is holomorphic in $\mathbb{C} \setminus (\Sigma_2 \cup \Sigma_3 \cup (-\infty, \infty))$, where

$$\Sigma_k := \left\{ e^{2(k-1)\pi i/3} x : x \in (0, \infty) \right\}, \quad k \in \{2, 3\},$$

are oriented towards the origin (the real line and its subsets are oriented from left to right as standard);

- (b) $\Phi_{\alpha}(\zeta; x)$ has continuous traces on $(-\infty, 0) \cup (0, \infty) \cup \Sigma_2 \cup \Sigma_3$ that satisfy

$$\Phi_{\alpha+}(\tau; x) = \Phi_{\alpha-}(\tau; x) \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \tau \in (-\infty, 0), \\ \begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix}, & \tau \in (0, \infty), \\ \begin{pmatrix} 1 & 0 \\ b_k & 1 \end{pmatrix}, & \tau \in \Sigma_k, k \in \{2, 3\}, \end{cases}$$

where $b_1 := is_2$, $b_2 = is_3$, and $b_3 = is_1$ ($b_1 + b_2 + b_3 - b_1 b_2 b_3 = 2 \cos(\alpha\pi)$);

- (c) as $\zeta \rightarrow 0$ it holds that

$$\Phi_{\alpha}(\zeta; x) = \mathbf{H}(\zeta; x) \zeta^{\alpha\sigma_3/2} \begin{pmatrix} 1 & i\kappa_{\alpha+1/2} \log \zeta \\ 0 & 1 \end{pmatrix} \mathbf{A}_j, \quad \zeta \in Q_j,$$

where Q_j is the connected component of $\mathbb{C} \setminus (\Sigma_2 \cup \Sigma_3 \cup (-\infty, \infty))$ contained in or containing the j -th quadrant, $\mathbf{H}(\zeta; x)$ is holomorphic around $\zeta = 0$, and $\mathbf{A}_j = e^{\pi i \sigma_3/4} \mathbf{A}_j^{FN} e^{-\pi i \sigma_3/4}$;

- (d) $\Phi_{\alpha}(\zeta; x)$ has the following asymptotic expansion near ∞ :

$$\Phi_{\alpha}(\zeta; x) = \left(\mathbf{I} + \sum_{n=1}^{\infty} \frac{2^{2n/3}}{\zeta^n} \begin{pmatrix} U_{2n}(x) & -2^{-1/3}iU_{2n-1}(x) \\ 2^{1/3}i(V_{2n+1} - U_{2n}V_1)(x) & (V_{2n} - V_1U_{2n-1})(x) \end{pmatrix} \right) \\ \times \frac{\zeta^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \exp \left\{ -\frac{2}{3} \left(\zeta^{3/2} + x\zeta^{1/2} \right) \sigma_3 \right\},$$

which holds uniformly in $\{|\zeta| \geq 1\} \setminus (\Sigma_2 \cup \Sigma_3 \cup (-\infty, \infty))$.

The only claim that is not contained in [15] is the explicit form of the series in RHP- Φ_{α} (d). The latter follows easily from (A.17) since

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} e^{\pi i \sigma_3/4} \begin{pmatrix} A_n(s) & B_n(s) \\ (-1)^n B_n(s) & (-1)^n A_n(s) \end{pmatrix} e^{-\pi i \sigma_3/4} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \\ = \frac{1}{2} \begin{pmatrix} (1 + (-1)^n)U_n(x) & -i(1 - (-1)^n)U_n(x) \\ i(1 - (-1)^n)V_n(x) & (1 + (-1)^n)V_n(x) \end{pmatrix}$$

and for $n = 2m$ it holds that

$$\zeta^{-\sigma_3/4} \frac{2^{2m/3}}{\zeta^m} \begin{pmatrix} U_{2m}(x) & 0 \\ 0 & V_{2m}(x) \end{pmatrix} \zeta^{\sigma_3/4} = \frac{2^{2m/3}}{\zeta^m} \begin{pmatrix} U_{2m}(x) & 0 \\ 0 & V_{2m}(x) \end{pmatrix}$$

while for $n = 2m - 1$ it holds that

$$\begin{aligned} \zeta^{-\sigma_3/4} \frac{2^{(2m-1)/3}}{\zeta^{m-1/2}} \begin{pmatrix} 0 & -iU_{2m-1}(x) \\ iV_{2m-1}(x) & 0 \end{pmatrix} \zeta^{\sigma_3/4} = \\ \frac{2^{2(m-1)/3}}{\zeta^{m-1}} \begin{pmatrix} 0 & 0 \\ i2^{1/3}V_{2(m-1)+1}(x) & 0 \end{pmatrix} + \frac{2^{2m/3}}{\zeta^m} \begin{pmatrix} 0 & -i2^{-1/3}U_{2m-1}(x) \\ 0 & 0 \end{pmatrix} \end{aligned}$$

(notice that the identity matrix plus the first matrix on the right-hand side of the last equality for $m = 1$ is exactly the inverse of the first matrix in (B.2)).

B.2. Lax Pair. Define

$$(B.3) \quad \begin{cases} \mathcal{U}(x) := 2^{-2/3}(H - q)(s), \\ u(x) := \mathcal{U}'(x) - x/2 = 2^{-1/3}(q'(s) + q^2(s) + s/2). \end{cases}$$

Then it can be verified that $u(x)$ is a solution of Painlevé XXXIV equation with parameter α^2 :

$$(B.4) \quad u''(x) = 4u^2(x) + 2xu(x) + \frac{(u'(x))^2 - \alpha^2}{2u(x)}.$$

It has been shown in [15, Lemma 3.3] that $\Phi_\alpha(\zeta; x)$ satisfies

$$(B.5) \quad \begin{cases} \partial_\zeta \Phi_\alpha(\zeta; x) = \mathbf{A}(\zeta; x) \Phi_\alpha(\zeta; x), \\ \partial_x \Phi_\alpha(\zeta; x) = \mathbf{U}(\zeta; x) \Phi_\alpha(\zeta; x), \end{cases}$$

in each sector Q_j , where

$$(B.6) \quad \mathbf{A}(\zeta; x) = \begin{pmatrix} \mathcal{U}(x) & i \\ -i(\zeta + u(x) + x - \mathcal{U}^2(x)) & -\mathcal{U}(x) \end{pmatrix} + \frac{1}{\zeta} \begin{pmatrix} u'(x)/2 - (u\mathcal{U})(x) & -iu(x) \\ -i \left((\mathcal{U}^2 u)(x) - (\mathcal{U}u')(x) + \frac{(u'(x))^2 - \alpha^2}{4u(x)} \right) & (u\mathcal{U})(x) - u'(x)/2 \end{pmatrix}$$

and

$$(B.7) \quad \mathbf{U}(\zeta; x) := \begin{pmatrix} \mathcal{U}(x) & i \\ i(-\zeta + \mathcal{U}^2(x) - \mathcal{U}'(x)) & -\mathcal{U}(x) \end{pmatrix}.$$

Expressions (B.6) and (B.7) are not explicitly stated in [15, Lemma 3.3]. What follows from the lemma is that $\mathbf{A}(\zeta; x)$ is equal to

$$\begin{pmatrix} 1 & 0 \\ -2^{1/3}iV_1(x) & 1 \end{pmatrix} \begin{pmatrix} -2^{1/3}q(s) + \alpha/(2\zeta) & i(1 - u(x)/\zeta) \\ -i(\zeta - u(x) + 2^{2/3}q'(s)) & 2^{1/3}q(s) - \alpha/(2\zeta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2^{1/3}iV_1(x) & 1 \end{pmatrix}$$

and that $\mathbf{U}(\zeta; x)$ is equal to

$$\left[\begin{pmatrix} 0 & 0 \\ -2^{1/3}iV_1'(x) & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -2^{1/3}iV_1(x) & 1 \end{pmatrix} \begin{pmatrix} -2^{1/3}q(s) & i \\ -i\zeta & 2^{1/3}q(s) \end{pmatrix} \right] \begin{pmatrix} 1 & 0 \\ 2^{1/3}iV_1(x) & 1 \end{pmatrix},$$

which yield (B.6) and (B.7) (identity $\alpha - u'(x) = 2^{4/3}u(x)q(s)$ is useful in deriving (B.6); recall also that $-2V_1(x) = q(s) + H(s)$ by (A.18) and (B.1)).

As mentioned in the previous subsection, it follows from the general theory of Riemann-Hilbert problems that the solution of RHP- Φ_α is a meromorphic function of x . In fact, if $s = -2^{1/3}x$ is not a pole of q , then the solution of RHP- Φ_α exists and is given by (B.2). Moreover, if s_0 is a pole of q with the residue 1, that is, $q_{-1} = 1$ in (A.19), then (A.19) yields that the functions $\mathcal{U}(x)$ and $u(x)$

are, in fact, analytic at $x_0 = -2^{-1/3}s_0$. In this case, the matrices $A(\zeta; x)$ and $U(\zeta; x)$ are analytic at x_0 as well (notice that the fraction in (2, 1)-entry of $A(\zeta; x)$ is equal to $u''(x)/2 - 2u^2(x) - xu(x)$ by (B.4)). Then we get from the second relation in (B.5) that $\Phi_\alpha(\zeta; x)$ also must be analytic at x_0 . On the other hand, if $q_{-1} = -1$, then x_0 is a simple pole of $\mathcal{U}(x)$ and a double pole of $u(x)$. Moreover, the Riemann-Hilbert problem $\text{RHP-}\Phi_\alpha$ cannot be solvable since it follows from $\text{RHP-}\Phi_\alpha$ (d) that

$$\lim_{\zeta \rightarrow \infty} \zeta \left[\Phi_\alpha(\zeta; x) \exp \left\{ \frac{2}{3} \left(\zeta^{3/2} + x\zeta^{1/2} \right) \sigma_3 \right\} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \frac{\zeta^{\sigma_3/4}}{\sqrt{2}} \right]_{12} = -2^{1/3}iU_1(x) = i\mathcal{U}(x)$$

(analyticity of $\Phi_\alpha(\zeta; x)$ at x_0 implies analyticity of $\mathcal{U}(x)$ at x_0).

B.3. Modified Riemann-Hilbert Problem. Assume now that α and Stokes parameters b_1, b_2, b_3 from $\text{RHP-}\Phi_\alpha$ (b) are such that $x_0 = 0$ is a pole of $\Phi_\alpha(\zeta; x)$ (hence, $s_0 = 0$ is a pole of $q(s)$ with residue -1). Note that in this case

$$\begin{cases} U_1(x) &= 2^{-1/3}x^{-1} - \mathfrak{q} + \mathcal{O}(x^3), \\ U_2(x) &= -2^{-1/3}\mathfrak{q}x^{-1} + \mathfrak{q}^2/2 + \mathcal{O}(x^2), \\ U_3(x) &= 2^{-4/3}\mathfrak{q}^2x^{-1} + \mathcal{O}(1), \end{cases}$$

as evident by (A.20)–(A.21) and (B.1). Recall that $\mathcal{U}(x) = -2^{1/3}U_1(x)$ and set

$$\begin{cases} \mathcal{S}(x) &:= 2^{2/3}(U_2 + U_3U_1^{-1})(x), \\ \mathcal{W}(x) &:= (\mathcal{U}^2 - \mathcal{U}' - \mathcal{S})(x). \end{cases}$$

One can readily verify that

$$(B.9) \quad \begin{cases} \mathcal{U}(x) &= -x^{-1} + 2^{1/3}\mathfrak{q} + \mathcal{O}(x^3), \\ \mathcal{S}(x) &= -2^{1/3}\mathfrak{q}x^{-1} + 2^{2/3}\mathfrak{q}^2 + S_*x + \mathcal{O}(x^2), \\ \mathcal{W}(x) &= -2^{1/3}\mathfrak{q}x^{-1} - S_*x + \mathcal{O}(x^2), \end{cases}$$

as $x \rightarrow 0$ for some constant S_* that we can avoid computing. To get rid of the pole of $\Phi_\alpha(\zeta; x)$ at $x_0 = 0$, let us define

$$(B.10) \quad \Phi_\alpha^{\text{sing}}(\zeta; x) := \begin{pmatrix} \zeta - \mathcal{S}(x) & -i\mathcal{U}(x) \\ -i\mathcal{U}^{-1}(x) & 0 \end{pmatrix} \Phi_\alpha(\zeta; x).$$

Denote the prefactor above by $S(\zeta; x)$ (the way it can be found will become clear at the end of this subsection). Trivially,

$$(B.11) \quad \partial_x \Phi_\alpha^{\text{sing}}(\zeta; x) = \mathbf{U}^{\text{sing}}(\zeta; x) \Phi_\alpha^{\text{sing}}(\zeta; x), \quad \mathbf{U}^{\text{sing}} := \partial_x S S^{-1} + S \mathbf{U} S^{-1}.$$

Since the matrix $S(\zeta; x)$ has determinant identically equal to 1, it is a lengthy but straightforward computation to find that

$$\mathbf{U}^{\text{sing}}(\zeta; x) = i \begin{pmatrix} i(\zeta + \mathcal{W}(x))\mathcal{U}^{-1}(x) & \zeta^2 + \zeta(\mathcal{W} - \mathcal{S})(x) + (\mathcal{W}^2 + \mathcal{U}'\mathcal{W} - \mathcal{U}\mathcal{S}')(x) \\ \mathcal{U}^{-2}(x) & -i(\zeta + \mathcal{W}(x))\mathcal{U}^{-1}(x) \end{pmatrix}.$$

It readily follows from (B.9) that $\mathbf{U}^{\text{sing}}(\zeta; x)$ is in fact analytic at $x_0 = 0$. Hence, it follows from (B.11) that $\Phi_\alpha^{\text{sing}}(\zeta; x)$ is analytic at $x_0 = 0$ as well. It further follows from $\text{RHP-}\Phi_\alpha$, the definition of $\Phi_\alpha^{\text{sing}}(\zeta; x)$ in (B.10), the above proven analyticity in the parameter x in some neighborhood of 0, and the explanation further below that $\Phi_\alpha^{\text{sing}}(\zeta; x)$ solves the following Riemann-Hilbert problem ($\text{RHP-}\Phi_\alpha^{\text{sing}}$):

- (a,b,c) $\Phi_\alpha^{\text{sing}}(\zeta; x)$ satisfies $\text{RHP-}\Phi_\alpha$ (a,b,c);
- (d) $\Phi_\alpha^{\text{sing}}(\zeta; x)$ has the following behavior near ∞ :

$$\Phi_\alpha^{\text{sing}}(\zeta; x) = \left(\mathbf{I} + \mathcal{O}(\zeta^{-1}) \right) \frac{\zeta^{3\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \exp \left\{ -\frac{2}{3} \left(\zeta^{3/2} + x\zeta^{1/2} \right) \sigma_3 \right\}$$

uniformly in $\mathbb{C} \setminus (\Sigma_2 \cup \Sigma_3 \cup (-\infty, \infty))$.

Indeed, since $S(\zeta; x)$ is an analytic (in fact, linear) in ζ , **RHP- $\Phi_\alpha^{\text{sing}}$** (a,b,c) are obvious. It also holds that

$$\begin{aligned} & \begin{pmatrix} \zeta & 0 \\ 0 & 0 \end{pmatrix} \left(\mathbf{I} + \sum_{n=1}^{\infty} \frac{2^{2n/3}}{\zeta^n} \begin{pmatrix} * & -2^{-1/3}iU_{2n-1}(x) \\ * & * \end{pmatrix} \right) = \\ & \begin{pmatrix} \zeta & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} * & -2^{1/3}iU_1(x) \\ 0 & 0 \end{pmatrix} + \frac{1}{\zeta} \begin{pmatrix} * & -2iU_3(x) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \mathcal{O}(\zeta^{-2}) & \mathcal{O}(\zeta^{-2}) \\ 0 & 0 \end{pmatrix} = \\ & \begin{pmatrix} \zeta & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -2iU_3(x) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \mathcal{O}(\zeta^{-1}) & \mathcal{O}(\zeta^{-1}) \\ 0 & 0 \end{pmatrix} \Big) \zeta^{\sigma_3}. \end{aligned}$$

Furthermore, since $-U_2(x) = (V_2 - V_1U_1)(x)$, which can be verified using (A.18) and (B.1), it holds that

$$\begin{aligned} & \begin{pmatrix} -2^{2/3}(U_2 + U_3U_1^{-1})(x) & 2^{1/3}iU_1(x) \\ 2^{-1/3}iU_1^{-1}(x) & 0 \end{pmatrix} \left(\mathbf{I} + \sum_{n=1}^{\infty} \frac{2^{2n/3}}{\zeta^n} \begin{pmatrix} * & -2^{-1/3}iU_{2n-1}(x) \\ * & (V_{2n} - V_1U_{2n-1})(x) \end{pmatrix} \right) = \\ & \begin{pmatrix} * & 2^{1/3}iU_1(x) \\ 2^{-1/3}iU_1^{-1}(x) & 0 \end{pmatrix} + \frac{1}{\zeta} \begin{pmatrix} * & 2iU_1(x)(U_2 + U_3U_1^{-1} + V_2 - V_1U_1)(x) \\ * & 1 \end{pmatrix} + \mathcal{O}\left(\frac{1}{\zeta^2}\right) = \\ & \begin{pmatrix} \zeta & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2iU_3(x) \\ 0 & 1 \end{pmatrix} + \frac{1}{\zeta} \begin{pmatrix} * & * \\ 2^{-1/3}iU_1^{-1}(x) & * \end{pmatrix} + \mathcal{O}\left(\frac{1}{\zeta^2}\right) \Big) \zeta^{\sigma_3} \end{aligned}$$

(these are exactly the relations that define $S(\zeta; x)$). **RHP- $\Phi_\alpha^{\text{sing}}$** (d) now follows from **RHP- Φ_α** (d) and the definition of $S(\zeta; x)$ in (B.10). Since $U_1^{-1}(0) = 0$, the last two computations also show that

$$\begin{aligned} \text{(B.12)} \quad \Phi_\alpha^{\text{sing}}(\zeta; 0) &= \left(\mathbf{I} + \frac{1}{\zeta} \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} + \mathcal{O}(\zeta^{-2}) \right) \frac{\zeta^{3\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \exp \left\{ -\frac{2}{3} \zeta^{3/2} \sigma_3 \right\} \\ &= \frac{\zeta^{3\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \left(\mathbf{I} + \mathcal{O}(\zeta^{-1/2}) \right) \exp \left\{ -\frac{2}{3} \zeta^{3/2} \sigma_3 \right\} \end{aligned}$$

uniformly in $\{|\zeta| \geq 1\} \setminus (\Sigma_2 \cup \Sigma_3 \cup (-\infty, \infty))$.

APPENDIX C. AN EXAMPLE OF WEIGHTS IN W^{sing}

In this appendix we discuss a group of examples of weights in W^{sing} for which $\mathbb{N}_\varepsilon^{\text{sing}} = \emptyset$ and therefore our results do not apply. As follows from Proposition 5, this can happen only on Δ_{sym} .

C.1. Orthogonal Polynomials on a Segment. Let $\hat{\rho}(z)$ be an analytic and non-vanishing function in some neighborhood of $[-1, 0]$ and $\hat{Q}_n(z)$ be the non-identically zero monic polynomial of smallest degree, which is necessarily at most n , such that

$$\text{(C.1)} \quad \int_{-1}^0 x^k \hat{Q}_n(x) \hat{\rho}(x) dx = 0, \quad k \in \{0, \dots, n-1\}$$

(we could also introduce Jacobi-type singularities at -1 and 0 into the weight $\hat{\rho}(x)$, but choose not to for simplicity of the exposition). Such polynomials were studied in [18] using Riemann-Hilbert approach under an additional assumption of positivity on $[-1, 0]$ (this assumption is really not needed for the approach to work). To describe the asymptotics of these polynomials, let

$$\hat{\Phi}(z) := 2z + 1 + 2\hat{w}(z), \quad \hat{w}(z) := \sqrt{z(z+1)},$$

where the branch of the square root is chosen so that $\hat{w}(z)$ is analytic in $\mathbb{C} \setminus [-1, 0]$ and $\hat{w}(z) = z + \mathcal{O}(1)$ as $z \rightarrow \infty$, while $\hat{\Phi}(z)$ is nothing but the conformal map of $\overline{\mathbb{C}} \setminus [-1, 0]$ onto $\{|z| > 1\}$ such that

$\widehat{\Phi}(\infty) = \infty$ and $\widehat{\Phi}'(\infty) > 0$. Further, let

$$(C.2) \quad D_{\widehat{\rho}}(z) := \exp \left\{ \frac{\widehat{w}(z)}{2\pi i} \int_{-1}^0 \frac{\log \widehat{\rho}(x)}{z-x} \frac{dx}{\widehat{w}_+(x)} \right\}, \quad z \notin [-1, 0],$$

which is simply the Szegő function of $\widehat{\rho}(x)$ (one must choose a continuous determination of $\log \widehat{\rho}(x)$ on $[-1, 0]$; Szegő functions for different continuous determinations will either coincide or differ by a sign), i.e., it is a non-vanishing and holomorphic function in the domain of its definition with continuous traces on $[-1, 0]$ that satisfy $D_{\widehat{\rho}_+}(x)D_{\widehat{\rho}_-}(x) = 1/\widehat{\rho}(x)$. Lastly, let

$$D(z) := \left(\widehat{\Phi}(z)/\widehat{w}(z) \right)^{1/2}$$

be the branch holomorphic in $\overline{\mathbb{C}} \setminus [-1, 0]$ that is positive for $z > 0$ (this is the Szegő function of $\widehat{w}_+(x)$, that is, $D(z)$ is analytic and non-vanishing in $\overline{\mathbb{C}} \setminus [-1, 0]$ and $D_+(x)D_-(x) = 1/\widehat{w}_+(x)$ for $x \in (-1, 0)$). Then it holds that

$$(C.3) \quad \widehat{Q}_n(z) = (1 + O(1/n)) \widehat{\gamma}_n \widehat{\Phi}^n(z) (DD_{\widehat{\rho}})(z)$$

locally uniformly in $\overline{\mathbb{C}} \setminus [-1, 0]$, where $\widehat{\gamma}_n^{-1} = 4^n (DD_{\widehat{\rho}})(\infty)$ is the normalizing constant that makes the right-hand side of (C.3) behave like $z^n + O(z^{n-1})$ around infinity. The reader should see clear parallels between (C.3) and (2.20), (3.4) (the product $(DD_{\widehat{\rho}})(z)$ also can be defined via an integral representation (C.2) with $\widehat{\rho}(x)$ replaced by $(\widehat{\rho}\widehat{w}_+)(x)$).

C.2. Rotationally Symmetric Weights. Let Δ_{sym} be given by (2.23). In this case we let $a_1 := -1$, $a_2 = -\eta$, and $a_3 = -\eta^2$, where $\eta := e^{2\pi i/3}$. Define a weight function $\rho(s)$ on Δ_{sym} by setting $\rho(s) := \widehat{\rho}(s^3)$, $s \in \Delta_{\text{sym}}$. Let $Q_n(s)$ be the minimal degree non-identically zero polynomial satisfying (1.5) with the just defined weight $\rho(s)$. Then

$$Q_{3n}(z) = Q_{3n+1}(z) = Q_{3n+2}(z) = \widehat{Q}_n(z^3).$$

Indeed, as all the legs of Δ_{sym} are oriented towards the origin, it holds for $k \leq 3n + 1$ that

$$\int_{\Delta_{\text{sym}}} s^k \widehat{Q}_n(s^3) \rho(s) ds = \int_{-1}^0 \left(1 + \eta^{k+1} + \eta^{2(k+1)} \right) x^k \widehat{Q}_n(x^3) \widehat{\rho}(x^3) dx.$$

The above expression is equal to 0 when $k \neq 3m + 2$ due to the sum $1 + \eta^{k+1} + \eta^{2(k+1)}$. When $k = 3m + 2$ we get that

$$\int_{\Delta_{\text{sym}}} s^{3m+2} \widehat{Q}_n(s^3) \rho(s) ds = 3 \int_{-1}^0 x^{3m+2} \widehat{Q}_n(x^3) \widehat{\rho}(x^3) dx = \int_{-1}^0 x^m \widehat{Q}_n(x) \widehat{\rho}(x) dx = 0,$$

where the last conclusion holds by (C.1) and one needs to observe that $3m + 2 = k \leq 3n + 1$ implies that $m \leq n - 1$. Thus, it follows from (C.3) that

$$(C.4) \quad Q_{3n}(z) = \widehat{\gamma}_n (1 + o(1)) \widehat{\Phi}^n(z^3) (DD_{\widehat{\rho}})(z^3)$$

locally uniformly in $\overline{\mathbb{C}} \setminus \Delta_{\text{sym}}$. Let \mathfrak{S} be the Riemann surface of $w(z) = \sqrt{z(z^3 + 1)}$ as defined in Section 2.1 and Δ be the lift of Δ_{sym} to this surface, where $w(z)$ is the branch from (2.2). As before, Δ is the boundary of the sheets $\mathfrak{S}^{(0)}$ and $\mathfrak{S}^{(1)}$. Set

$$\widetilde{\Psi}(z) := \left(\widehat{\Phi}^n(z^3) (DD_{\widehat{\rho}})(z^3) \right)^{(-1)^k}, \quad z \in \mathfrak{S}^{(k)} \setminus \Delta.$$

Then $\widetilde{\Psi}(z)$ is a sectionally meromorphic function on $\mathfrak{S} \setminus \Delta$ that has a pole of order $3n$ at $\infty^{(0)}$, a zero of order $3n$ at $\infty^{(1)}$, and otherwise is non-vanishing and finite. These are exactly the properties exhibited by $\widetilde{\Psi}_n(z)$ from Theorem 1. However, it holds that

$$\widetilde{\Psi}_+(s) = \widetilde{\Psi}_-(s) / (\widehat{\rho}\widehat{w}_+)(s^3) = \widetilde{\Psi}_-(s) / (s(\rho w_+)(s))$$

for $s \in \Delta$. That is the jump relations satisfied by $\widetilde{\Psi}(s)$ are different from the ones in (2.3). Hence, formula (C.4) is distinct from (3.4). In what follows we find \mathbb{N}^{reg} (as we explained in Proposition 3

in this case the sequence is 3-periodic and therefore is independent of ε), show that the above weight $\rho(s)$ belongs to W^{sing} , and that $\mathbb{N}^{\text{sing}} = \emptyset$. Thus, formulae (3.4) are indeed not applicable.

C.3. Sequence \mathbb{N}^{reg} . As before, we shall write $\Delta_{\text{sym}} = \Delta_1 \cup \Delta_2 \cup \Delta_3$, where Δ_i has endpoints $a_i = -\eta^{i-1}$ and 0. It can be readily checked that

$$(C.5) \quad w_+(x) = |w(x)| \begin{cases} e^{3\pi i/2}, & x \in \Delta_1, \\ e^{-7\pi i/6}, & x \in \Delta_2, \\ e^{\pi i/6}, & x \in \Delta_3, \end{cases} =: |w(x)|e^{a_w(x)}.$$

One can also check that Δ_1 , Δ_2 , and Δ_3 are homologous to $\alpha + \beta$, $-\beta$, and $-\alpha$, respectively, see Figure 2. Therefore,

$$\oint_{\alpha} \frac{ds}{w(s)} = -2 \int_{\Delta_3} \frac{ds}{w_+(s)} = -2e^{-\pi i/6} \int_{\Delta_3} \frac{ds}{|w(s)|} = 2e^{\pi i/6} \int_{-1}^0 \frac{dx}{|w(x)|},$$

where we used (C.5) for the second equality. Similarly, it holds that

$$\oint_{\beta} \frac{ds}{w(s)} = -2 \int_{\Delta_2} \frac{ds}{w_+(s)} = 2e^{\pi i/6} \int_{\Delta_2} \frac{ds}{|w(s)|} = 2e^{5\pi i/6} \int_{-1}^0 \frac{dx}{|w(x)|}.$$

That is, we have that $\mathbf{B} = e^{2\pi i/3}$, see (2.8). Moreover, we get from (4.11) and the fact that $I_i = 1/3$ for Δ_{sym} that $\omega = -\tau = 2/3$. In particular, $\Phi^3(z)$ must be a rational function on the whole surface \mathfrak{S} , see (2.6). In fact, it is not hard to see that

$$\Phi^3(z^{(k)}) = \widehat{\Phi}^{(-1)^k}(z^3) = (2z^3 + 1 + 2zw(z))^{(-1)^k}.$$

Define $\sqrt{w}(z)$ to be the branch holomorphic in $\mathbb{C} \setminus \Delta_{\text{sym}}$ satisfying $\sqrt{w}(z) = z + \mathcal{O}(1)$ as $z \rightarrow \infty$. It holds that

$$\sqrt{w}_{\pm}(s) = \sqrt{|w(s)|} \begin{cases} e^{\pm 3\pi i/4}, & s \in \Delta_1, \\ e^{-\pi i/3 \mp \pi i/4}, & s \in \Delta_2, \\ e^{\pi i/3 \mp \pi i/4}, & s \in \Delta_3. \end{cases}$$

Therefore, we have that $-i\sqrt{w}_+(s)\sqrt{w}_-(s) = w_+(s)$, $s \in \Delta_{\text{sym}}$, see (C.5). Recall (2.9). Let

$$(C.6) \quad S(z^{(k)}) := \exp\left\{-2\pi i(\tau + 2)a(z^{(k)})\right\} \Phi(z^{(k)}) \left(e^{-\pi i/4} \sqrt{w}(z)\right)^{-(-1)^k},$$

for $z \in \mathbb{C} \setminus (\Delta_{\text{sym}} \cup \pi(\alpha) \cup \pi(\beta))$, $k \in \{0, 1\}$. This function is holomorphic and non-vanishing in the domain of its definition. Notice that it extends holomorphically to $\infty^{(0)}$ and $\infty^{(1)}$ and has non-zero values there. We also get from (2.6) and (4.10) that $S(z)$ is holomorphic and non-vanishing $\mathfrak{S} \setminus (\Delta \cup \alpha)$ and satisfies

$$S_+(s) = S_-(s) \begin{cases} \exp\{2\pi i(\omega + \mathbf{B}\tau + 2\mathbf{B})\}, & s \in \alpha \setminus \mathbf{A}, \\ 1/w_+(s), & s \in \Delta \setminus \mathbf{A}. \end{cases}$$

One should observe as well that $S(z)S(z^*) \equiv 1$ for $z \in \mathfrak{S}$. Thus, if we can show that c_{ρ} , defined in (2.13), is equal to $\omega + \mathbf{B}\tau + 2\mathbf{B} = 2i/\sqrt{3}$, then we will get that

$$(C.7) \quad S_{\rho}(z^{(k)}) = S(z^{(k)})D_{\widehat{\rho}}^{(-1)^k}(z^3), \quad z \in \overline{\mathbb{C}} \setminus \Delta.$$

Indeed, it is straightforward to see that this function is holomorphic and non-vanishing in $\mathfrak{S} \setminus (\Delta \cup \alpha)$ and $S_{\rho}(z)S_{\rho}(z^*) \equiv 1$ there. Moreover, when $z \rightarrow s \in \Delta_+ \setminus \mathbf{A}$ and $t \rightarrow s \in \Delta_- \setminus \mathbf{A}$, it holds that $z \rightarrow s \in \Delta_{\text{sym}\pm}^{\circ}$ and $t \rightarrow s \in \Delta_{\text{sym}\mp}^{\circ}$. Therefore, with a slight abuse of notation, we can write

$$S_{\rho\pm}(s) = S_+(s)D_{\widehat{\rho}\pm}(s^3) = S_-(s)/(w_+(s)D_{\widehat{\rho}\mp}(s^3)\widehat{\rho}(s^3)) = S_{\rho-}(s)/(\rho w_+(s))$$

for $s \in \Delta \setminus \mathbf{A}$. Since the jumps of $S_{\rho}(s)$ and of $S(z)$ across $\alpha \setminus \mathbf{A}$ coincide, the desired claim follows from the uniqueness part of Proposition 1. Thus, to see that functions defined via (2.11) and (C.7)

coincide, we only need to show that $c_\rho = 2i/\sqrt{3}$. Of course, this claim relies on the choice of the branch of $\log(\rho w_+)(s)$. We shall assume that (2.11) uses the following determination:

$$\log(\rho w_+)(s) = \log \widehat{\rho}(s^3) + \log |w_+(s)| + a_w(s), \quad s \in \Delta^\circ,$$

for some continuous determination of $\log \widehat{\rho}(s)$, where $a_w(s)$ was defined in (C.5). Then

$$\begin{aligned} \frac{1}{2\pi i} \oint_{\Delta} \log(\rho w_+)(s) \frac{ds}{w(s)} &= \frac{1}{\pi i} \int_{\Delta_{\text{sym}}} a_w(s) \frac{ds}{w_+(s)} + \\ &\quad \frac{e^{-3\pi i/2} + e^{7\pi i/6} + e^{-\pi i/6}}{\pi i} \int_{-1}^0 \log(\widehat{\rho}(x^3)|w_+(x)|) \frac{dx}{|w(x)|}, \end{aligned}$$

where one can readily check that the constant in front of the last integral is equal to zero. Hence,

$$\begin{aligned} \frac{1}{2\pi i} \oint_{\Delta} \log(\rho w_+)(s) \frac{ds}{w(s)} &= \frac{3}{2} \int_{\Delta_1} \frac{ds}{w_+(s)} - \frac{7}{6} \int_{\Delta_2} \frac{ds}{w_+(s)} + \frac{1}{6} \int_{\Delta_3} \frac{ds}{w_+(s)} \\ &= \int_{-1}^0 \left(\frac{3}{2} e^{-3\pi i/2} - \frac{7}{6} e^{11\pi i/6} + \frac{1}{6} e^{7\pi i/6} \right) \frac{dx}{|w(x)|} = \frac{4\eta}{\sqrt{3}} \int_{-1}^0 \frac{dx}{|w(x)|}. \end{aligned}$$

Thus, we get from (2.13) and the first computation after (C.5) that indeed $c_\rho = 2i/\sqrt{3}$.

The Jacobi inversion problem (2.19) now reads

$$\mathbf{a}(z_n) = [(n + 1/2)(\omega + B\tau)] = [(2n + 1)(1 - B)/3].$$

From this we can conclude that $z_0 = \infty^{(0)}$, $z_1 = \mathbf{a}_0 = \mathbf{0}$, and $z_2 = \infty^{(1)}$. Indeed, the middle claim follows immediately from (2.15) and the unique solvability of (2.19). The other two equalities follows from the fact that

$$\mathbf{a}(\infty^{(0)}) = \left[\frac{\omega + B\tau}{2} \right] = \left[\frac{1}{3}(1 - B) \right] \quad \text{and} \quad \mathbf{a}(\infty^{(1)}) = \left[-\frac{\omega + B\tau}{2} \right] = \left[\frac{2}{3}(1 - B) \right],$$

which was shown in (4.8). Thus, $\mathbb{N}^{\text{reg}} = \mathbb{N} \setminus \{3\mathbb{N}\}$.

C.4. Sequence \mathbb{N}^{sing} . It readily follows from (3.2) that $b_i(\rho) = -2$. It was shown in [17] that in this case **RHP- Φ_α** with $\alpha = 0$ is not solvable for $x = 0$, that is, $\rho \in W^{\text{sing}}$. More precisely, in [17, Equations (18) and (19)] it was shown that the Stokes parameters⁴ $s_1 = s_2 = s_3 = 2i$ in **RHP- Ψ_ν** with $\nu = 1/2$ correspond to the solution $q(s)$ of (A.1) with a pole at the origin of residue -1 . The desired conclusion then follows from the explanation given before (B.8).

Due to the periodicity of z_n , we only need to compute $\det(\mathbf{T}_1)$ and $\det(\mathbf{T}_2)$. Since $z_1 = \mathbf{a}_0$, it holds that $t_0^{(1)} = 0$ and therefore

$$\det(\mathbf{T}_1) = t_0^{(0)} \left(2t_2^{(1)} - h_1 t_1^{(1)} \right) \quad \text{and} \quad \det(\mathbf{T}_2) = t_0^{(2)} \left(2t_2^{(1)} - h_1 t_1^{(1)} \right).$$

Since $t_0^{(0)} t_0^{(2)} \neq 0$, to show that $\mathbb{N}^{\text{sing}} = \emptyset$, we need to prove that $2t_2^{(1)} = h_1 t_1^{(1)}$. To this end, set

$$A(z) := \exp \{ -2\pi i(\tau + 2)a(z) \}$$

and, as usual, we shall write $A(z)$ for the pull-back of $A(z)$ from $\mathfrak{S}^{(0)}$. It now follows from (6.2), (5.12), (C.6), and (C.7) that

$$H^{-1}(z) = i(S_\rho r)^2(z)w(z) = (A\Phi)^2(z)(\widehat{\rho}D_\rho^2)(z^3).$$

Let U_r be two copies of $\{|z| < r\} \setminus [-r, 0]$, $r < 1$, glued crosswise across the cuts (double cyclic neighborhood of 0). It is quite easy to see that the function $(\widehat{\rho}D_\rho^2)(z)$ lifted to one of the copies

⁴A slightly different Lax pair than the one presented in (A.2)–(A.4) is used in [17] so that the Stokes parameters appearing there are equal to $-is_1, is_2, -is_3$.

of $\{|z| < r\} \setminus [-r, 0]$ extends analytically to U_r by lifting its reciprocal to the other copy. Hence, $(\widehat{\rho}D_{\widehat{\rho}}^2)(z^3)$ possesses a Puiseux series at zero and satisfies there

$$(\widehat{\rho}D_{\widehat{\rho}}^2)(z^3) = 1 + \mathcal{O}(z^{3/2}) \quad \text{as } z \rightarrow 0.$$

Recall that the same is true about $\Phi(z)$, see (4.18). Thus, we can deduce from (6.4) that if we develop $A(z)$ into a Puiseux series in the sector S_2 , then

$$A(z) = 1 - (h_1/2)z^{1/2} + \mathcal{O}(z) \quad \text{as } z \rightarrow 0.$$

On the other hand, it follows from (2.6), (2.18), (4.10), and the fact that $c_\rho = \omega + \mathbf{B}\tau + 2\mathbf{B}$ that $K(z) := (AT_1\Phi^2)(z)$ is a rational function on \mathfrak{S} with the zero/pole divisor $\mathbf{a}_0 + \infty^{(1)} - 2\infty^{(0)}$. The Puiseux series of $K(z)$ in the sector S_2 is equal to

$$(C.8) \quad K(z) = t_1^{(1)}z^{1/2} + \left(t_2^{(1)} - h_1t_1^{(1)}/2\right)z + \mathcal{O}\left(z^{3/2}\right) \quad \text{as } z \rightarrow 0.$$

It follows from the symmetries of Δ_{sym} that $K(\eta z)$ and $K(\bar{\eta}z)$, $\eta = e^{2\pi i/3}$, are rational functions on \mathfrak{S} as well. Moreover, they have the same zero/pole divisors as $K(z)$. Thus, they must be constant multiples of $K(z)$. By looking at their Puiseux series in S_2 we then can conclude that the coefficient next z in (C.8) must be equal to 0, which is exactly the desired claim.

APPENDIX D. EXAMPLES OF APPROXIMATED FUNCTIONS

In this appendix we show how to write functions (1.3) in the form (1.4). We also compute their Stokes parameters (3.2) and show that Theorem 3 is always applicable to these functions.

Logarithmic functions in (1.3) are the easiest example. Let c_1, c_2, c_3 be arbitrary non-zero complex numbers. Set $c_0 := -c_1 - c_2 - c_3$ and define $\rho_i(s) = \rho_{\Delta_i^\circ}(s) = 2\pi i c_i$, $i \in \{1, 2, 3\}$ (unless $c_1 = c_2 = c_3$, $\rho(s)$ is not well defined at a_0). Then it holds that

$$f(z) = \frac{1}{2\pi i} \int_{\Delta} \frac{\rho(s)ds}{s-z} = \sum_{i=1}^3 c_i \log(z-s) \Big|_{a_i}^{a_0} = \sum_{k=0}^3 c_k \log(z-a_k),$$

where the branches of the logarithms are chosen so that the right-hand side above is analytic in $\overline{\mathbb{C}} \setminus \Delta$. Since each function $\rho_i(s)$ is constant on Δ_i , condition (3.7) is satisfied automatically. In fact, it holds that $\kappa_{i,j}(z) \equiv 1$ and therefore **RHP- \mathcal{P}_0** is not an approximate but an exact parametrix for these functions. In particular, estimate (5.21) and an analogous estimate in the proof of Theorem 3 are not needed. This class of functions does include weights in both W^{reg} and W^{sing} . Indeed, it holds that

$$b_1(\rho) = -\frac{c_2 + c_3}{c_1}, \quad b_2(\rho) = -\frac{c_1 + c_3}{c_2}, \quad \text{and} \quad b_3(\rho) = -\frac{c_1 + c_2}{c_3}.$$

Therefore, if $-c_0 = c_1 + c_2 + c_3 = 0$, then $b_i(\rho) = 1$ and $\rho \in W^{\text{reg}}$ ($f(z)$ has no logarithmic singularity at a_0 and such functions are also covered by the results of [2]). On the other hand, if $c_1 = c_2 = c_3$, then $b_i(\rho) = -2$ and, as mentioned at the beginning of Section C.4, $\rho \in W^{\text{sing}}$.

Let $\alpha_k > -1$, $k \in \{0, 1, 2, 3\}$, be such that their sum is an integer necessarily bigger or equal to -2 . Recall our notation from Section 5.2, where we denoted by S_1, S_2, S_3 the connected components of $U_0 \setminus \Delta$ labeled in such a way that S_k does not border Δ_k° . Let L be any curve extending to infinity from a_0 that partially lies within S_1 . We take branches $(z - a_i)^{\alpha_i}$ that are analytic off $\Delta_i \cup L$, $i \in \{1, 2, 3\}$, and $(z - a_0)^{\alpha_0}$ that is analytic off L . Define

$$f(z) = \prod_{k=0}^3 (z - a_k)^{\alpha_k} - p(z),$$

where $p(z)$ is the polynomial part of the product at infinity. That is, $f(z)$ is analytic in $\overline{\mathbb{C}} \setminus \Delta$ and vanishes at infinity. Let $\rho(s) = f_+(s) - f_-(s)$, which is a smooth function on $\Delta \setminus \{a_0, a_1, a_2, a_3\}$. Let $C_\rho(z)$ be the right-hand side of (1.4). Then it follows from Plemelj-Sokhotski formulae that

$$C_{\rho+}(s) - f_+(s) = C_{\rho-}(s) + \rho(s) - f_+(s) = C_{\rho-}(s) - f_-(s), \quad s \in \Delta \setminus \{a_0, a_1, a_2, a_3\}.$$

Since $C_\rho(z) - f(z)$ has smooth traces on each Δ_i° , this function is analytic across each Δ_i° by Morera's theorem. As this difference vanishes at infinity and cannot have polar or essential singularities at $\{a_0, a_1, a_2, a_3\}$ (this follows from the known behavior of the Cauchy integrals around the endpoints of the contours of integration), it must hold that $f(z) \equiv C_\rho(z)$. Let

$$\varrho(z) = \prod_{i=1}^3 (z - a_i)^{\alpha_i} \begin{cases} 1, & z \in S_2, \\ e^{-2\pi i \alpha_1}, & z \in S_3, \end{cases} \quad \varrho(z) = \prod_{i=1}^3 (z - a_i)^{\alpha_i} \begin{cases} e^{-2\pi i(\alpha_1 + \alpha_2)}, & z \in S_{12}, \\ e^{2\pi i \alpha_3}, & z \in S_{13}, \end{cases}$$

where S_{1j} is the connected component of $S_1 \setminus L$ that borders Δ_j . It can be readily seen that $\varrho(z)$ is in fact analytic in U_0 . Recall (3.1). Then it follows from the first set of definitions above that

$$\varrho_1(s) = \frac{f_+(s) - f_-(s)}{(s - a_0)^{\alpha_0}} = \rho(s)(1 - e^{2\pi i \alpha_1}), \quad s \in \Delta_1^\circ.$$

Similarly, we get that

$$\varrho_2(s) = \varrho(s)e^{2\pi i \alpha_1}(1 - e^{2\pi i \alpha_2}) \quad \text{and} \quad \varrho_3(s) = \varrho(s)(e^{-2\pi i \alpha_3} - 1).$$

Hence, the weights $\varrho_i(s)$ are constant multiples of the same function analytic in U_0 and the condition (3.7) is satisfied (in fact, all the functions $\kappa_{i,j}(z) \equiv 1$ and **RHP- P_0** is an exact parametrix). Moreover, by plugging the above expressions into (3.2) we get that

$$b_i(\rho) = \frac{\sin(\alpha_0 + \alpha_i)\pi}{\sin \alpha_i \pi} = \cos \alpha_0 \pi + \sin \alpha_0 \pi \cot \alpha_i \pi$$

for each $i \in \{1, 2, 3\}$. Hence, if $\alpha_0 = 0$, then $b_i(\rho) = 1$ and $\rho \in W^{\text{reg}}$ ($f(z)$ has no singularity at a_0 and such functions are also covered by the results of [2]). On the other hand, for any other $\alpha_0 \neq 0$ fixed, every real solution of (3.3) can be expressed in the above form with an appropriate choice of $\alpha_1, \alpha_2, \alpha_3 \in (-1, 0)$. It is an interesting questions whether the class of real Stokes parameters contains some that lead to the matrices Ψ_{α_0} that have a pole at $x = 0$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, 530 CHURCH STREET, ANN ARBOR, MI 48109, USA
Email address: barhoumi@umich.edu

DEPARTMENT OF MATHEMATICAL SCIENCES, INDIANA UNIVERSITY-PURDUE UNIVERSITY INDIANAPOLIS, 402 NORTH BLACKFORD STREET, INDIANAPOLIS, IN 46202, USA
Email address: maxyatts@iu.edu