

The reciprocal Mahler ensembles of random polynomials

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Abstract

We consider the roots of uniformly chosen complex and real reciprocal polynomials of degree N whose Mahler measure is bounded by a constant. After a change of variables this reduces to a generalization of Ginibre's complex and real ensembles of random matrices where the weight functions (on the eigenvalues of the matrices) is replaced by the exponentiated equilibrium potential of the interval $[-2, 2]$ on the real axis in the complex plane. In the complex (real) case the random roots form a determinantal (Pfaffian) point process, and in both cases the empirical measure on roots converges weakly to the arcsine distribution supported on $[-2, 2]$. Outside this region the kernels converge without scaling, implying among other things that there is a positive expected number of outliers away from $[-2, 2]$. These kernels, as well as the scaling limits for the kernels in the bulk $(-2, 2)$ and at the endpoints $\{-2, 2\}$ are presented. These kernels appear to be new, and we compare their behavior with related kernels which arise from the (non-reciprocal) Mahler measure ensemble of random polynomials as well as the classical Sine and Bessel kernels.

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1 Introduction

We study two ensembles of random polynomials/matrices related to Ginibre's real and complex ensembles of random matrices, but with weight functions which are not derived from 'classical' polynomial potentials, but rather from the equilibrium logarithmic potential of the interval $[-2, 2]$ on the real axis in the complex plane. This complements our study of similar ensembles formed from the equilibrium logarithmic potential of the unit disk [12, 13].

We introduce the complex ensemble first since it is simpler to define. Consider a joint density function of N complex random variables identified with $\mathbf{z} \in \mathbb{C}^N$ given by $P_N : \mathbb{C}^N \rightarrow [0, \infty)$ where

$$P_N(\mathbf{z}) = \frac{1}{Z_N} \left\{ \prod_{n=1}^N \phi(z_n) \right\} |\Delta(\mathbf{z})|^2, \quad \text{and} \quad \Delta(\mathbf{z}) = \prod_{m < n} (z_n - z_m). \quad (1.1)$$

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Here $\phi : \mathbb{C} \rightarrow [0, \infty)$ is the *weight function*, and Z_N is a constant necessary to make P_N a probability density. When $\phi(z) = e^{-|z|^2}$, P_N gives the joint density of eigenvalues of a matrix chosen randomly from Ginibre's ensemble of $N \times N$ complex matrices with i.i.d. complex standard normal entries [7]. When $\phi(z) = \max\{1, |z|\}^{-s}$ for $s = N + 1$, then P_N defines the joint density of the roots of random complex polynomials of degree N chosen uniformly from the set of such polynomials with Mahler measure at most 1 (Mahler measure is a measure of complexity of polynomials; see Section 1.1 below).

The ensembles (1.1) can be put in the context of two-dimensional electrostatics by envisioning the random variables z_1, \dots, z_N as a system of repelling particles confined to the plane, and placed in the presence of an attractive potential $V : \mathbb{C} \rightarrow [0, \infty)$ which keeps the particles from fleeing to infinity. When such a system is placed in contact with a heat reservoir at a particular temperature, the location of the particles is random, and the joint density of particles is given by P_N for $\phi(z) = e^{-V(z)}$. The connection between eigenvalues of random matrices and particle systems in 2D electrostatics is originally attributed to Dyson [4], and is central in the treatise [6]. From this perspective, it makes sense to investigate P_N for different naturally-arising confining potentials.

Real ensembles are different in that the roots/eigenvalues of the real polynomials/matrices are either real or come in complex conjugate pairs. Hence, the joint density for such ensembles is not defined on \mathbb{C}^N , but rather on

$$\bigcup_{L+2M=N} \mathbb{R}^L \times \mathbb{C}^M,$$

where the union is over all pairs of non-negative integers such that $L + 2M = N$. For each such pair, we specify a *partial* joint density $P_{L,M} : \mathbb{R}^L \times \mathbb{C}^M \rightarrow [0, \infty)$ given by

$$P_{L,M}(\mathbf{x}, \mathbf{z}) = \frac{1}{Z_N} \left\{ \prod_{\ell=1}^L \phi(x_\ell) \right\} \left\{ \prod_{m=1}^M \phi(z_m) \phi(\bar{z}_m) \right\} |\Delta(\mathbf{x} \vee \mathbf{z} \vee \bar{\mathbf{z}})|, \quad (1.2)$$

where $\mathbf{x} \vee \mathbf{z} \vee \bar{\mathbf{z}}$ is the vector formed by joining to \mathbf{x} all the z_m and their complex conjugates, and Z_N is the normalization constant given by

$$Z_N = \sum_{(L,M)} \frac{1}{L!M!} \int_{\mathbb{R}^L} \int_{\mathbb{C}^M} P_{L,M}(\mathbf{x}, \mathbf{z}) d\mu_{\mathbb{R}}^L(\mathbf{x}) d\mu_{\mathbb{C}}^M(\mathbf{z}).$$

(Here $\mu_{\mathbb{R}}^L$ and $\mu_{\mathbb{C}}^M$ are Lebesgue measure on \mathbb{R}^L and \mathbb{C}^M respectively). Note that we may assume that $\phi(z) = \phi(\bar{z})$, since otherwise we could replace $\phi(z)$ with $\sqrt{\phi(z)\phi(\bar{z})}$, without changing the partial joint densities.

In this work we focus on the case $\phi(z) = e^{-V(z)}$, where the confining potential $V(z)$ is the scaled logarithmic equilibrium potential of the interval $[-2, 2]$, see Section 1.2. That is,

$$V(z) = s \log \frac{|z + \sqrt{z^2 - 4}|}{2}, \quad s > N, \quad (1.3)$$

where we take the principal branch of the square root. We view $s > N$ as a parameter of the system and we will call the point process on \mathbb{C} whose joint density is given by (1.1) and (1.3) the *complex reciprocal Mahler ensemble*. Similarly, the joint densities given by (1.2) and (1.3) define a process on roots of real polynomials, and we will call this process the *real reciprocal Mahler process*. The reason we call these point processes Mahler ensembles as they can be interpreted as choosing polynomials uniformly at random from starbodies of Mahler measure when viewed as distance functions (in the sense of the geometry of numbers) on coefficient vectors of reciprocal polynomials.

1.1 Mahler measure

The Mahler measure of a polynomial $f(z) \in \mathbb{C}[z]$ is given by

$$M\left(a \prod_{n=1}^N (z - \alpha_n)\right) = |a| \prod_{n=1}^N \max\{1, |\alpha_n|\}.$$

This is an example of a *height* or measure of complexity of polynomials, and arises in number theory when restricted to polynomials with integer (or otherwise algebraic) coefficients. There are many examples of heights (for instance norms of coefficient vectors), but Mahler measure has the attractive feature that it is multiplicative: $M(fg) = M(f)M(g)$.

There are many open (and hard) questions revolving around the range of Mahler measure restricted to polynomials with integer coefficients. Perhaps the most famous is *Lehmer's question* which asks whether 1 is a limit point in the set of Mahler measures of integer polynomials. It is clear that, since cyclotomic polynomials all have Mahler measure equal to 1, that 1 is in this set; it is unknown whether there is a 'next smallest' Mahler measure, though to date the current champion in this regard is

$$z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1, \quad \text{whose Mahler measure is approximately 1.17.}$$

A reciprocal polynomial is a polynomial whose coefficient vector is palindromic; that is $f(z)$ is a degree N *reciprocal* polynomial if $z^N f(1/z) = f(z)$. Clearly if α is a root of f , then so too is $f(1/\alpha)$, from which the name 'reciprocal' arises. Reciprocal polynomials arise in the number theoretic investigation of Lehmer's conjecture since, as was shown by Smyth in the early 1970s, the Mahler measure of a non-reciprocal integer polynomial is necessarily greater than approximately 1.3 [14]. Thus, many questions regarding the range of Mahler measure reduce to its range when restricted to reciprocal integer polynomials.

Another question regarding the range of Mahler measure concerns estimating the number of integer polynomials (or the number of reciprocal integer polynomials) of fixed degree N and Mahler measure bounded by $T > 0$ as $T \rightarrow \infty$. Such questions were considered in [3] and [11], and the main term in this estimate depends on the volume of the set of real degree N polynomials (or real reciprocal polynomials) whose Mahler measure is at most 1. This set is akin to a unit ball, though it is not convex. It is from here that our interest in the zeros of random polynomials chosen uniformly from these sorts of 'unit balls' arose. The (non-reciprocal) case was studied in [12] and [13], and here we take up the reciprocal case.

In order to be precise we need to specify exactly what we mean by choosing a reciprocal polynomial *uniformly* from those with Mahler measure at most 1. For $\lambda \in [0, \infty)$ we define the λ -homogeneous Mahler measure by

$$M_\lambda\left(a \prod_{n=1}^N (z - \alpha_n)\right) = |a|^\lambda \prod_{n=1}^N \max\{1, |\alpha_n|\}. \tag{1.4}$$

To treat polynomials with complex and real coefficients simultaneously, we shall write \mathbb{K} to mean \mathbb{C} or \mathbb{R} depending on the considered case. Identifying the coefficient vectors of degree N polynomials with \mathbb{K}^{N+1} , we also view M_λ as a function on \mathbb{K}^{N+1} , and define

$$B_\lambda(\mathbb{K}) = \{\mathbf{a} \in \mathbb{K}^{N+1} : M_\lambda(\mathbf{a}) \leq 1\}.$$

These are the degree N *unit-starbodies* for Mahler measure. We can then define the reciprocal unit starbody as the intersection of the subspace of reciprocal polynomials with the B_λ . However, as the set of reciprocal polynomials has Lebesgue measure zero in \mathbb{K}^{N+1} , this is not

an optimal definition for the purposes of selecting a polynomial uniformly from this set. In order to overcome this difficulty we need some natural parametrization of the set of reciprocal polynomials.

If N is odd, and f is reciprocal, then $-f(-1) = f(-1)$. That is, -1 is always a root of an odd reciprocal polynomial, and $f(z)/(z+1)$ is an even degree reciprocal polynomial. Thus, when considering the roots of random reciprocal polynomials, it suffices to study even degree reciprocal polynomials. By declaring that $M_\lambda(z^{-1}) = 1$ and demanding that M_λ be multiplicative, we can extend Mahler measure to the algebra of Laurent polynomials $\mathbb{C}[z, z^{-1}]$, and we define a *reciprocal Laurent* polynomial to be one satisfying $f(z^{-1}) = f(z)$. Notice that reciprocal Laurent polynomials form a sub-algebra of Laurent polynomials. We can map the set of degree $2N$ reciprocal polynomials on a set of reciprocal Laurent polynomials via the map $f(z) \mapsto z^{-N}f(z)$, and the λ -homogeneous Mahler measure is invariant under this map. We will call the image of this map the set of degree $2N$ reciprocal Laurent polynomials (the leading monomial is z^N , but there are $2N$ zeros).

Now observe that if $f(z) \in \mathbb{C}[z]$ is a degree N polynomial, then $f(z+1/z)$ is a degree $2N$ reciprocal Laurent polynomial and conversely, any reciprocal Laurent polynomial is an algebraic polynomial in $z+1/z$. Hence, we define the *reciprocal Mahler* measure of $f(z)$ to be the Mahler measure of $f(z+1/z)$. Specifically, let $M_\lambda^{\text{rec}} : \mathbb{C}[z] \rightarrow [0, \infty)$ be defined by $M_\lambda^{\text{rec}}(f(z)) := M_\lambda(f(z+1/z))$. As before, if we identify the set of degree N polynomials with \mathbb{K}^{N+1} and define the reciprocal starbodies to be

$$B_\lambda^{\text{rec}}(\mathbb{K}) = \{\mathbf{a} \in \mathbb{K}^{N+1} : M_\lambda^{\text{rec}}(\mathbf{a}) \leq 1\}.$$

The real/complex reciprocal Mahler ensemble is the point process on \mathbb{C} induced by choosing a polynomial uniformly from $B_\lambda^{\text{rec}}(\mathbb{K})$ for $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . It was observed by the first author that the joint density function of such a process is given by (1.1) and (1.3) in the complex case [10] and by (1.2) and (1.3) in the real case [11] with $s = (N+1)/\lambda$. Without going into the details we just mention that the factors $|\Delta(\mathbf{z})|^2$ and $|\Delta(\mathbf{x} \vee \mathbf{z} \vee \bar{\mathbf{z}})|$ in (1.1) and (1.2), respectively, come from the Jacobian of the change of variables from the coefficients of polynomials to their roots and $\phi(z) = e^{-V(z)}$ with $V(z)$ as in (1.3) appears because

$$f\left(z + \frac{1}{z}\right) = a \prod_{n=1}^N \left(z + \frac{1}{z} - \alpha_j\right) = a \prod_{n=1}^N \frac{1}{z} \left(z - \frac{\alpha_n + \sqrt{\alpha_n^2 - 4}}{2}\right) \left(z - \frac{\alpha_n - \sqrt{\alpha_n^2 - 4}}{2}\right),$$

where we take the principal branch of the square root, which necessarily yields that

$$M_\lambda^{\text{rec}}(f) = |a|^\lambda \prod_{n=1}^N \frac{|\alpha_n + \sqrt{\alpha_n^2 - 4}|}{2}. \quad (1.5)$$

1.2 Mahler Measures and Logarithmic Potentials

Mahler measure and the reciprocal Mahler measure can put into the more general framework of *multiplicative distance functions* formed with respect to a given compact set $K \subset \mathbb{C}$. Indeed, given a compact set K , it is known that the infimum of the logarithmic energies

$$I[\nu] := - \iint \log |z - w| d\nu(z) d\nu(w)$$

taken over all probability Borel measures supported on K is either infinite (K cannot support a measure with finite logarithmic energy; such sets are called *polar*) or is finite and is achieved by

a unique minimizer, say ν_K , which is called the *equilibrium distribution* on K . The logarithmic capacity of K is set to be zero in the former case and $e^{-I[\nu_K]}$ in the latter. It is further known that the function

$$V_K(z) := I[\nu_K] + \int \log |z - w| d\nu_K(w)$$

is positive and harmonic in $\mathbb{C} \setminus K$ and is zero on K except perhaps on a polar subset. Assume for convenience that K has capacity 1, i.e., $I[\nu_K] = 0$. Then we can define the multiplicative distance function with respect to K on algebraic polynomials by

$$M(f; K) = \exp \left\{ \int \log |f(z)| d\nu_K(z) \right\} = |a| \prod_{n=1}^N e^{-V_K(\alpha_n)}, \quad \text{where } f(z) = a \prod_{n=1}^N (z - \alpha_n).$$

When $\overline{\mathbb{C}} \setminus K$ is simply connected, $V_K(z)$ is in fact continuous in \mathbb{C} and is given by $\log |\Phi_K(z)|$ which is continued by 0 to K , where Φ_K is a conformal map of the complement of K to the complement of the closed unit disk such that $\Phi_K(\infty) = \infty$. Then

$$M(f; K) = |a| \prod_{\alpha_n \notin K} |\Phi_K(\alpha_n)|.$$

One can easily check now that the Mahler measure M_1 , see (1.4), and the reciprocal Mahler measure M_1^{rec} , see (1.5), are equal to $M(\cdot; \mathbb{D})$ and $M(\cdot; [-2, 2])$, respectively.

1.3 Determinantal and Pfaffian Point Processes

Everything in this section is standard, but we include it for completeness. The interested reader might consult [9, 8] and [2] to get a more detailed explanations of the complex and real cases, respectively.

Given a Borel set $B \subset \mathbb{C}$ and a random vector \mathbf{z} chosen according to (1.1), we define the random variable N_B to be the number of points in \mathbf{z} that belong to B . The n th *intensity* or *correlation* function of the ensemble (1.1) is a function $R_n : \mathbb{C}^n \rightarrow [0, \infty)$ such that for disjoint Borel sets B_1, \dots, B_n ,

$$E[N_{B_1} \cdots N_{B_n}] = \int_{B_1 \times \cdots \times B_n} R_n(\mathbf{z}) d\mu_{\mathbb{C}}^n(\mathbf{z}). \tag{1.6}$$

Correlation functions are the basic objects from which probabilities of interest are computed. Ensembles with joint densities of the form (1.1) are *determinantal*, that is, there exists a *kernel* $K_N : \mathbb{C} \times \mathbb{C} \rightarrow [0, \infty)$ such that for all n ,

$$R_n(\mathbf{z}) = \det [K_N(z_j, z_k)]_{j,k=1}^n. \tag{1.7}$$

Theorem 1.1. *The kernel for the ensemble (1.1) is given by*

$$K_N(z, w) = \phi(z)\phi(w) \sum_{n=0}^{N-1} \pi_n(z) \overline{\pi_n(w)}$$

where π_n are orthonormal polynomials w.r.t. ϕ^2 , i.e., $\int \pi_n(z) \overline{\pi_m(z)} \phi(z)^2 d\mu_{\mathbb{C}}(z) = \delta_{nm}$.

The situation is a bit more complicated for real ensembles with partial joint densities given by (1.2) since the expected number of real roots is positive. In this case, we define the (ℓ, m) -intensity or correlation function to be $R_{\ell, m} : \mathbb{R}^{\ell} \times \mathbb{C}^m \rightarrow [0, \infty)$ such that for disjoint Borel

subsets A_1, \dots, A_ℓ of the real line and disjoint Borel subsets B_1, \dots, B_m of the open upper half plane,

$$E[N_{A_1} \cdots N_{A_\ell} N_{B_1} \cdots N_{B_m}] = \int_{A_1 \times \cdots \times A_\ell} \int_{B_1 \times \cdots \times B_m} R_{\ell, m}(\mathbf{x}, \mathbf{z}) d\mu_{\mathbb{R}}^\ell(\mathbf{x}) d\mu_{\mathbb{C}}^m(\mathbf{z}). \quad (1.8)$$

Ensembles with partial joint densities given as in (1.2) are *Pfaffian*: there exists a 2×2 matrix kernel $\mathbf{K}_N : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ such that

$$R_{\ell, m}(\mathbf{x}, \mathbf{z}) = \text{Pf} \begin{bmatrix} [\mathbf{K}_N(x_i, x_j)]_{i, j=1}^\ell & [\mathbf{K}_N(x_i, z_n)]_{i, n=1}^{\ell, m} \\ -[\mathbf{K}_N^\top(z_k, x_j)]_{k, j=1}^{m, \ell} & [\mathbf{K}_N(z_k, z_n)]_{k, n=1}^m \end{bmatrix}. \quad (1.9)$$

The formula for the kernel depends on the *species* (real or complex) of the argument. This kernel takes the form

$$\mathbf{K}_N(z, w) = \begin{bmatrix} \kappa_N(z, w) & \kappa_N \epsilon(z, w) \\ \epsilon \kappa_N(z, w) & \epsilon \kappa_N \epsilon(z, w) + \frac{1}{2} \text{sgn}(z - w) \end{bmatrix}, \quad (1.10)$$

where $\text{sgn}(\cdot)$ of a non-real number or zero is set to be zero, κ_N is an *orto-kernel* $\mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$, and

$$\epsilon f(z) := \begin{cases} \frac{1}{2} \int f(t) \text{sgn}(t - z) d\mu_{\mathbb{R}}(t) & \text{if } z \in \mathbb{R}, \\ \iota(z) f(\bar{z}) & \text{if } z \in \mathbb{C} \setminus \mathbb{R}, \end{cases} \quad (1.11)$$

where $\iota(z) := i \text{sgn}(\text{im}(z))$ and when written on the left, as in $\epsilon \kappa_N(z, w)$, ϵ acts on κ_N as a function of z , and when written on the right it acts on κ_N as a function of w .

Theorem 1.2. *Let $N = 2J$ be even. Then the orto-kernel $\kappa_N(z, w)$ for ensemble (1.2) is given by*

$$\kappa_N(z, w) = 2\phi(z)\phi(w) \sum_{j=0}^{J-1} (\pi_{2j}(z)\pi_{2j+1}(w) - \pi_{2j}(w)\pi_{2j+1}(z)), \quad (1.12)$$

where polynomials π_n , $\deg(\pi_n) = n$, are skew-orthonormal, that is, $\langle \pi_{2n} | \pi_{2m} \rangle = \langle \pi_{2n+1} | \pi_{2m+1} \rangle = 0$ and $\langle \pi_{2n} | \pi_{2m+1} \rangle = -\langle \pi_{2m+1} | \pi_{2n} \rangle = \delta_{nm}$, w.r.t. the skew-symmetric inner product

$$\langle f | g \rangle := \int [(f\phi)(z)\epsilon(g\phi)(z) - \epsilon(f\phi)(z)(g\phi)(z)] d(\mu_{\mathbb{R}} + \mu_{\mathbb{C}})(z). \quad (1.13)$$

Note that the skew-orthogonal polynomials are not uniquely defined since one may replace π_{2m+1} with $\pi_{2m+1} + c\pi_{2m}$ without disturbing skew-orthogonality. Moreover, if polynomials π_{2n} and π_{2n+1} are skew-orthonormal, then so are $c\pi_{2n}$ and π_{2n+1}/c . However, neither of these changes alters the expression (1.12) for the orto-kernel κ_N .

When N is odd, there is a formula for the orto-kernel similar to that given in Theorem 1.2. We anticipate that the scaling limits of the odd N case will be the same as those for the even N case (reported below), but due to the extra complexity (with little additional gain) we concentrate only on the even N case here.

2 Main Results

Henceforth, we always assume that s and $N \leq \lfloor s - 1 \rfloor$ are such that the limits

$$c := \lim_{N \rightarrow \infty} (s - N) \in [1, \infty] \quad \text{and} \quad \lambda := \lim_{N \rightarrow \infty} Ns^{-1} \in [0, 1] \quad (2.1)$$

exist. Furthermore, we set $\phi(z) := |\Phi(z)|^{-s}$, $\Phi(z) = (z + \sqrt{z^2 - 4})/2$. Notice that $\phi(z)$ is well defined in the whole complex plane.

2.1 Orthogonal and Skew-Orthogonal Polynomials

Denote by $K_{N,s}(z, w)$ and $\kappa_{N,s}(z, w)$ the kernels introduced in (1.7) and (1.10), respectively, for ϕ as above. It follows from Theorem 1.1 and [12, Proposition 2] that

$$K_{N,s}(z, w) = \phi(z)\phi(w) \sum_{n=0}^{N-1} \frac{s^2 - (n+1)^2}{2\pi s} U_n(z) \overline{U}_n(w) \quad (2.2)$$

where $U_n(z)$ is the n -th monic Chebyshev polynomial of the second kind for $[-2, 2]$, i.e.,

$$U_n(z) := \Phi^n(z) \Phi'(z) (1 - \Phi^{-2(n+1)}(z)) = \frac{\Phi^{n+1}(z) - \Phi^{-n-1}(z)}{\sqrt{z^2 - 4}}. \quad (2.3)$$

Similarly, we know from Theorem 1.2 that the ortho-kernel $\kappa_{N,s}(z, w)$ is expressible via skew-orthogonal polynomials.

Theorem 2.1. *Polynomials skew-orthonormal w.r.t. skew-inner product (1.13) with ϕ as above are given by*

$$\begin{cases} \pi_{2n}(z) &= \frac{4n+3}{16} C_{2n}^{(3/2)}\left(\frac{z}{2}\right), \\ \pi_{2n+1}(z) &= \left(1 - \frac{(2n+1)^2}{s^2}\right) C_{2n+1}^{(1/2)}\left(\frac{z}{2}\right) - \frac{1}{s^2} C_{2n+1}^{(3/2)}\left(\frac{z}{2}\right), \end{cases} \quad (2.4)$$

where $C_m^{(\alpha)}(x)$ is the classical ultraspherical polynomial of degree m , i.e., it is orthogonal to all polynomials of smaller degree w.r.t. the weight $(1-x^2)^{\alpha-1/2}$ on $[-1, 1]$ having $\frac{2^m \Gamma(m+\alpha)}{\Gamma(\alpha)\Gamma(m+1)}$ as the leading coefficient.

2.2 Exterior Asymptotics

We start with the asymptotic behavior of the kernels in $\overline{\mathbb{C}} \setminus [-2, 2]$.

Theorem 2.2. *Assuming (2.1), it holds that*

$$\lim_{N \rightarrow \infty} \frac{|\Phi(z)\Phi(w)|^s}{(\Phi(z)\overline{\Phi}(w))^N} \frac{K_{N,s}(z, w)}{s-N} = \frac{1+\lambda}{2\pi} \left[1 + \frac{c^{-1}}{\Phi(z)\overline{\Phi}(w) - 1} \right] \frac{\Phi'(z)\overline{\Phi}'(w)}{\Phi(z)\overline{\Phi}(w) - 1} \quad (2.5)$$

locally uniformly for $z, w \notin [-2, 2]$.

It follows from (2.5) that the limit of $K_{N,s}(z, w)$ is equal to zero when $c = \infty$, while

$$\lim_{N \rightarrow \infty} K_{N,s}(z, z) = \frac{1}{\pi} \frac{1}{|\Phi(z)|^{2c}} \left[c + \frac{1}{|\Phi(z)|^2 - 1} \right] \frac{|\Phi'(z)|^2}{|\Phi(z)|^2 - 1}$$

when $c < \infty$. Hence, the expected number of zeros of random polynomials in each open subset of $\overline{\mathbb{C}} \setminus [-2, 2]$ is positive and finite in this case, see (1.7).

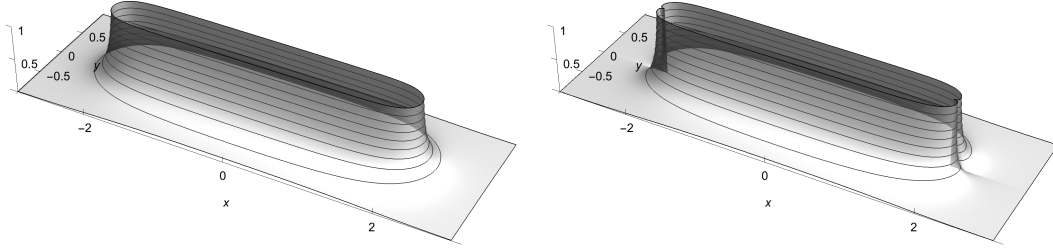


Figure 1: The unscaled limiting spatial density of complex roots near $[-2, 2]$ for the complex ensemble on the left and the real ensemble on the right. Notice the cleft along the real axis for the real ensemble. This is due to the fact that roots repel. When the roots come in complex conjugate pairs, this repulsion introduces a paucity of complex zeros near the real axis.

Theorem 2.3. *Let N be even. Assuming (2.1), it holds that*

$$\lim_{N \rightarrow \infty} \frac{|\Phi(z)\Phi(w)|^s \kappa_{N,s}(z, w)}{(\Phi(z)\Phi(w))^N s - N} = \frac{\lambda(1 + \lambda)}{2\pi} \left[1 + \frac{c^{-1}}{\Phi(z)\Phi(w) - 1} \right] \frac{\Phi'(z)\Phi'(w)}{\Phi(z)\Phi(w) - 1} \times \frac{\Phi(w) - \Phi(z)}{\sqrt{\Phi^2(z) - 1}\sqrt{\Phi^2(w) - 1}} \quad (2.6)$$

locally uniformly for $z, w \notin [-2, 2]$.

Theorem 2.3 indicates that $\mathbf{K}_{N,s}(z, w)$ has a non-zero exterior limit only when $c < \infty$.

Theorem 2.4. *Under the conditions of Theorem 2.3, assume in addition that $c < \infty$. Then it holds that*

$$\lim_{N \rightarrow \infty} \begin{cases} \epsilon \kappa_{N,s} \epsilon(x, y) &= F(x, y), \\ (|\Phi(z)|/|\Phi(z)|)^N \kappa_{N,s} \epsilon(z, y) &= -(\partial_x F)(z, y), \\ (|\Phi(w)|/|\Phi(w)|)^N \epsilon \kappa_{N,s}(x, w) &= -(\partial_y F)(x, w), \end{cases} \quad (2.7)$$

locally uniformly for $x, y \in \mathbb{R} \setminus [-2, 2]$ and $z, w \notin [-2, 2]$, where

$$F(x, y) := \frac{1}{\pi} \int_{\text{sgn}(x)\infty}^{\Phi(x)} \int_{\text{sgn}(y)\infty}^{\Phi(y)} \left[c + \frac{1}{uv - 1} \right] \frac{1}{|uv|^c} \frac{u - v}{uv - 1} \frac{du dv}{\sqrt{u^2 - 1}\sqrt{v^2 - 1}} + \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{c+1}{2})}{\Gamma(\frac{c}{2})} \left(\text{sgn}(x) \int_{\text{sgn}(y)\infty}^{\Phi(y)} - \text{sgn}(y) \int_{\text{sgn}(x)\infty}^{\Phi(x)} \right) \frac{1}{|u|^c} \frac{du}{\sqrt{u^2 - 1}}$$

for $x, y \in \mathbb{R} \setminus [-2, 2]$, where $\sqrt{u^2 - 1}$ is understood to be holomorphic in $\mathbb{C} \setminus [-1, 1]$.

Even though $F(x, y)$ is defined for real arguments only, its partial derivatives naturally extend to complex ones. Notice also that $\frac{1}{c}(\partial_{xy}^2 F)(z, w)$ is equal to the right-hand side of (2.6).

2.3 Scaling Limits in the Real Bulk

To find the scaling limits of $K_{N,s}(z, w)$ and $\kappa_{N,s}(z, w)$ on $(-2, 2)$, it is convenient to compute these limits for $\phi(z)$ separately. Notice that $\phi(y) = 1$ whenever $y \in (-2, 2)$.

Proposition 2.5. *Given $x \in (-2, 2)$, set $\omega^{-1}(x) := \sqrt{4 - x^2}$. Assuming (2.1), it holds that*

$$\lim_{N \rightarrow \infty} \phi \left(x + \frac{a}{N\omega(x)} \right) = e^{-|\operatorname{im}(a)|/\lambda},$$

locally uniformly for $x \in (-2, 2)$ and $a \in \mathbb{C}$ when $\lambda > 0$ and locally uniformly for $x \in (-2, 2)$ and $a \in \mathbb{C} \setminus \mathbb{R}$ when $\lambda = 0$ (the limit is zero in this case).

In the light of Proposition 2.5, let us write

$$K_{N,s}(z, w) = \phi(z)\phi(w)\widetilde{K}_{N,s}(z, w). \quad (2.8)$$

Then the following theorem holds.

Theorem 2.6. *Assuming (2.1), it holds that*

$$\lim_{N \rightarrow \infty} \frac{1}{sN\omega^2(x)} \widetilde{K}_{N,s} \left(x + \frac{a}{N\omega(x)}, x + \frac{b}{N\omega(x)} \right) = \frac{1}{\pi} \int_0^1 (1 - (\lambda t)^2) \cos((\bar{b} - a)t) dt \quad (2.9)$$

locally uniformly for $x \in (-2, 2)$ and $a, b \in \mathbb{C}$.

In the real case, analogously to (2.8), let us set

$$\kappa_{N,s}(z, w) = \phi(z)\phi(w)\widetilde{\kappa}_{N,s}(z, w) \quad \text{and} \quad \kappa_{N,s}\epsilon(z, y) = \phi(z)\widetilde{\kappa_{N,s}\epsilon}(z, y), \quad (2.10)$$

where $y \in (-2, 2)$ ($\phi(y) = 1$ in this case). Then the following theorem holds.

Theorem 2.7. *Let N be even. Assuming (2.1), it holds that*

$$\lim_{N \rightarrow \infty} \frac{1}{N^2\omega^2(x)} \widetilde{\kappa}_{N,s} \left(x + \frac{a}{N\omega(x)}, x + \frac{b}{N\omega(x)} \right) = \frac{1}{\pi} \int_0^1 t(1 - (\lambda t)^2) \sin((b - a)t) dt \quad (2.11)$$

locally uniformly for $x \in (-2, 2)$ and $a, b \in \mathbb{C}$. Furthermore, it holds that

$$\lim_{N \rightarrow \infty} \frac{1}{N\omega(x)} \widetilde{\kappa_{N,s}\epsilon} \left(x + \frac{a}{N\omega(x)}, x + \frac{b}{N\omega(x)} \right) = \frac{1}{\pi} \int_0^1 (1 - (\lambda t)^2) \cos((b - a)t) dt \quad (2.12)$$

locally uniformly for $x \in (-2, 2)$, $a \in \mathbb{C}$, and $b \in \mathbb{R}$. Finally, we have that

$$\lim_{N \rightarrow \infty} \epsilon \kappa_{N,s} \left(x + \frac{a}{N\omega(x)}, x + \frac{b}{N\omega(x)} \right) = \frac{1}{\pi} \int_0^1 \frac{1 - (\lambda t)^2}{t} \sin((b - a)t) dt \quad (2.13)$$

locally uniformly for $x \in (-2, 2)$ and $a, b \in \mathbb{R}$.

Notice that knowing limit (2.12) is sufficient for our purposes as $\epsilon \kappa_N(z, w) = -\kappa_N\epsilon(w, z)$ by (1.12) as the orto-kernel is antisymmetric. Observe also that

$$\lim_{N \rightarrow \infty} \frac{1}{N\omega(x)} \kappa_{N,s}\epsilon \left(x + \frac{a}{N\omega(x)}, x + \frac{b}{N\omega(x)} \right) = \frac{1}{\pi} \frac{\sin(b - a)}{b - a}$$

uniformly for $a, b \in \mathbb{R}$ when $\lambda = 0$ by (2.12) ($\kappa_{N,s}\epsilon(x, x)$ is exactly the function needed to compute the expected number of real zeros, see (2.18) further below).

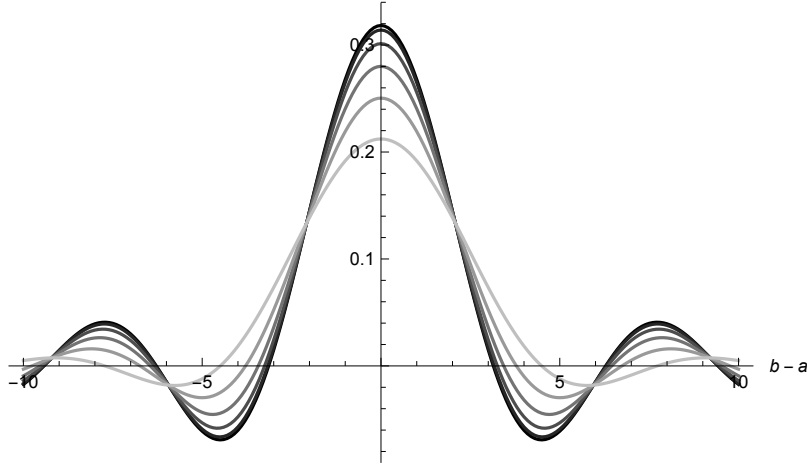


Figure 2: The scaled kernels (2.9) and (2.12) as a function $(b-a) \in \mathbb{R}$ for $\lambda \in [0, 1]$. The darkest curve is for $\lambda = 0$ and is equal to the classical sine kernel. Note that for the real ensemble, when $\lambda > 0$ this kernel does not tell us about the local density of real roots, but rather tells us about density of complex roots near the real axis. In this situation, as λ increases, the attraction of zeros to $[-2, 2]$ decreases and the zeros are more likely to drift into the complex plane. This phenomenon is captured by the decrease in amplitude of the kernel.

2.4 Scaling Limits at the Real Edge

Since $\phi(-z) = \phi(z)$, $K_{N,s}(-z, -w) = K_{N,s}(z, w)$, and $\kappa_{N,s}(-z, -w) = -\kappa_{N,s}(z, w)$, we report the scaling limits at 2 only. Again, we start with the function $\phi(z)$.

Proposition 2.8. *Assuming (2.1), it holds that*

$$\lim_{N \rightarrow \infty} \phi \left(2 - \frac{a^2}{N^2} \right) = e^{-|\operatorname{im}(a)|/\lambda}$$

uniformly on compact subsets of \mathbb{C} when $\lambda > 0$, and uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{R}$ when $\lambda = 0$ (the limit is zero).

In the case of random polynomials with complex coefficients the following theorem holds.

Theorem 2.9. *Let $\tilde{K}_{N,s}(z, w)$ be as (2.8). Assuming (2.1), it holds that*

$$\lim_{N \rightarrow \infty} \frac{1}{sN^3} \tilde{K}_{N,s} \left(2 - \frac{a^2}{N^2}, 2 - \frac{b^2}{N^2} \right) = \frac{1}{2\pi a \bar{b}} \int_0^1 (1 - (\lambda t)^2) \sin(at) \sin(\bar{b}t) dt \quad (2.14)$$

uniformly for a, b on compact subsets of \mathbb{C} .

Recall that the Bessel functions of the first kind are defined by

$$J_\nu(z) = \left(\frac{z}{2} \right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+\nu+1)} \left(\frac{z}{2} \right)^{2n},$$

where we take the principal branch of the ν 's power of $z/2$. Since $J_{1/2}(z) = 2 \sin(z)/\sqrt{2\pi z}$, the right-hand side of (2.14) specializes to

$$\frac{1}{2\sqrt{ab}} \frac{J_{1/2}(a)bJ'_{1/2}(b) - J_{1/2}(b)aJ'_{1/2}(a)}{2(a^2 - b^2)}$$

when $\lambda = 0$ and $a, b \in \mathbb{R}$, which is a classical Bessel kernel [references](#) up to the factor $1/2\sqrt{ab}$.

Theorem 2.10. *Let $\tilde{\kappa}_{N,s}(z, w)$ and $\widetilde{\kappa}_{N,s\epsilon}(z, y)$ be as in (2.10) and N be even. Assuming (2.1), it holds that*

$$\lim_{N \rightarrow \infty} \frac{1}{N^4} \tilde{\kappa}_{N,s} \left(2 - \frac{a^2}{N^2}, 2 - \frac{b^2}{N^2} \right) = \frac{1}{8ab} \int_0^1 t(1 - (\lambda t)^2) \mathbb{J}_{1,1}(at, bt) dt \quad (2.15)$$

uniformly for $a, b \in \mathbb{C}$, where $\mathbb{J}_{1,1}(u, v) := J_1(u)vJ_0(v) - J_1(v)uJ_0(u)$. Furthermore, we have that

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \widetilde{\kappa}_{N,s\epsilon} \left(2 - \frac{a^2}{N^2}, 2 - \frac{b^2}{N^2} \right) = \frac{1}{4a} \int_0^1 (1 - (\lambda t)^2) \mathbb{J}_{1,2}(at, bt) dt \quad (2.16)$$

uniformly for $a \in \mathbb{C}$ and $b \in \mathbb{R}$, where $\mathbb{J}_{1,2}(u, v) := J_1(u)vJ_1(v) + J_0(v)uJ_0(u)$. Finally, it holds that

$$\lim_{N \rightarrow \infty} \epsilon \kappa_{N,s\epsilon} \left(2 - \frac{a^2}{N^2}, 2 - \frac{b^2}{N^2} \right) = \frac{1}{2} \int_0^1 \frac{1 - (\lambda t)^2}{t} \mathbb{J}_{2,2}(at, bt) dt \quad (2.17)$$

uniformly for $a, b \in \mathbb{R}$, where $\mathbb{J}_{2,2}(u, v) := uJ_1(u)J_0(v) - vJ_1(v)J_0(u)$.

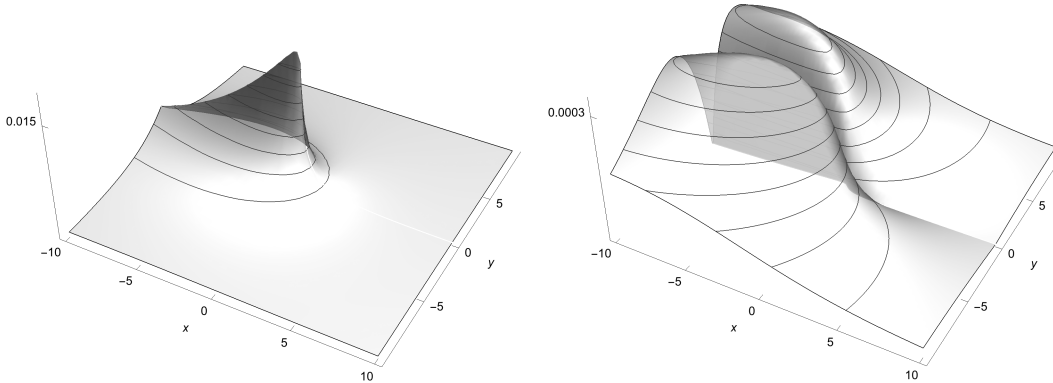


Figure 3: The scaled spatial density of complex roots near 2 for the complex ensemble on the left and the real ensemble on the right. Here we see a desire for roots to accumulate near 2 (the origin) with a sharper decrease in the density as we move along the positive x -axis (away from the bulk) than along the negative x -axis (into the bulk). The difference between the ensembles is starkest in the y direction, and we can see the competition between the attraction to 2 caused by the potential and the repulsion from the x -axis caused by the repulsion between complex conjugate pairs of roots. These images are produced from (2.14) and (2.16).

Equations (2.16) and (2.17) do not cover the cases $b \in i\mathbb{R}$ and $a, b \in i\mathbb{R}$, respectively. Such limits exist and we do derive formulas for them. Unfortunately, these formulas are much more cumbersome, which is the reason they are not presented here.

2.5 Expected Number of Real Zeros

The zeros of polynomials with real coefficients are either real or come in conjugate symmetric pairs. Hence, one of the interesting questions about such polynomials is the expected number

of real zeros. Given a closed set $A \subset \mathbb{R}$, denote by N_A the number of real roots belonging to A of a random degree N polynomial chosen from the real reciprocal Mahler ensemble. Then

$$E[N_A] = \int_A \text{Pf} \begin{bmatrix} 0 & \kappa_{N,s}\epsilon(x,x) \\ \epsilon\kappa_{N,s}(x,x) & 0 \end{bmatrix} d\mu_{\mathbb{R}}(x) = \int_A \kappa_{N,s}\epsilon(x,x) d\mu_{\mathbb{R}}(x) \quad (2.18)$$

by (1.8), (1.9), (1.10), and the anti-symmetry of $\kappa_{N,s}(z,w)$ and $\epsilon\kappa_{N,s}\epsilon(z,w)$. Moreover, the following theorem holds.

Theorem 2.11. *Let N_{in} be the number of real roots on $[-2, 2]$ of a random degree $N = 2J$ polynomial chosen from the real reciprocal Mahler ensemble. Then*

$$E[N_{\text{in}}] = N \left[1 - \frac{(N+1)(2N+1)}{6s^2} \right]. \quad (2.19)$$

Furthermore, let N_{out} be the number of real roots on $\mathbb{R} \setminus (-2, 2)$ of the said polynomial. Then

$$E[N_{\text{out}}] \sim -\log(1 - Ns^{-1}), \quad (2.20)$$

where $f(N, s) \sim g(N, s)$ if there exists $C > 1$ such that $C^{-1}g(N, s) \leq f(N, s) \leq Cg(N, s)$.

3 Proofs

3.1 Proof of Theorem 2.1

We start by representing polynomials from (2.4) as series in Chebyshëv polynomials.

Lemma 3.1. *Let π_{2n} and π_{2n+1} be given by (2.4). Then it holds that*

$$\begin{cases} \pi_{2n}(z) &= \frac{n + \frac{3}{4}}{2\pi} \sum_{i=0}^n (2i+1) \frac{\Gamma(n-i+\frac{1}{2}) \Gamma(n+i+\frac{3}{2})}{\Gamma(n-i+1) \Gamma(n+i+2)} U_{2i}(z), \\ \pi_{2n+1}(z) &= -\frac{1}{2\pi} \sum_{i=0}^n (2i+2) \left(1 - \frac{(2i+2)^2}{s^2} \right) \frac{\Gamma(n-i-\frac{1}{2}) \Gamma(n+i+\frac{3}{2})}{\Gamma(n-i+1) \Gamma(n+i+3)} U_{2i+1}(z), \end{cases} \quad (3.1)$$

where U_n is the degree n monic Chebyshëv polynomials of the second kind for $[-2, 2]$, see (2.3).

Proof. Observe that $C_k^{(1)}(\frac{z}{2}) = U_k(z)$. Hence, it follows from [15, Eq. (4.10.27)] that

$$C_{2n+\delta}^{(\alpha)}\left(\frac{z}{2}\right) = \frac{1}{\Gamma(\alpha)\Gamma(\alpha-1)} \sum_{i=0}^n (2i+1+\delta) \frac{\Gamma(n-i-1+\alpha) \Gamma(n+\delta+i+\alpha)}{\Gamma(n-i+1) \Gamma(n+\delta+i+2)} U_{2i+\delta}(z).$$

The first relation in (3.1) can be obtained now by setting $\alpha = \frac{3}{2}$ and $\delta = 0$ in the above formula. The second one also follows from the above representation combined with the identity

$$\begin{aligned} -\frac{1}{2\pi} \left(1 - \frac{(2n+1)^2}{s^2} \right) \frac{\Gamma(n-i-\frac{1}{2}) \Gamma(n+i+\frac{3}{2})}{\Gamma(n-i+1) \Gamma(n+i+3)} - \frac{2}{\pi s^2} \frac{\Gamma(n-i+\frac{1}{2}) \Gamma(n+i+\frac{5}{2})}{\Gamma(n-i+1) \Gamma(n+i+3)} &= \\ &= -\frac{1}{2\pi} \left(1 - \frac{(2i+2)^2}{s^2} \right) \frac{\Gamma(n-i-\frac{1}{2}) \Gamma(n+i+\frac{3}{2})}{\Gamma(n-i+1) \Gamma(n+i+3)}. \quad \square \end{aligned}$$

Next, we compute the skew-moments of the Chebyshëv polynomials of the second kind.

Lemma 3.2. *Let $\langle \cdot | \cdot \rangle$ be skew-inner product (1.13). Then it holds that*

$$\langle U_{m-1} | U_{n-1} \rangle = \begin{cases} 0, & m+n \text{ is even,} \\ \frac{n}{m} \frac{16s^2}{(n^2 - m^2)(s^2 - n^2)}, & m \text{ is odd, } n \text{ is even,} \\ -\langle U_{n-1} | U_{m-1} \rangle, & m \text{ is even, } n \text{ is odd.} \end{cases}$$

Proof. If f and g satisfy $f(\bar{z}) = \overline{f(z)}$ and $g(\bar{z}) = \overline{g(z)}$, then it holds that

$$\langle f, g \rangle = \operatorname{re} \left(4i \int_{\mathbb{C}_+} f(z) \overline{g(z)} \phi^2(z) d\mu_{\mathbb{C}}(z) \right) + \iint f(x) \phi(x) g(y) \phi(y) \operatorname{sgn}(y-x) d\mu_{\mathbb{R}}(x) d\mu_{\mathbb{R}}(y).$$

In what follows, we denote the first summand above by $\langle f, g \rangle_{\mathbb{C}_+}$ and the second one by $\langle f, g \rangle_{\mathbb{R}}$. An elementary change of variables and (2.3) yield

$$\begin{aligned} \langle U_{m-1} | U_{n-1} \rangle_{\mathbb{C}_+} &= 4 \frac{1 - (-1)^{m-n}}{m-n} \left(\frac{1}{2s+m+n} - \frac{1}{2s-m-n} \right) + \\ &\quad + 4 \frac{1 - (-1)^{m-n}}{m+n} \left(\frac{1}{2s-m+n} - \frac{1}{2s+m-n} \right). \end{aligned} \quad (3.2)$$

Expression (3.2) is zero when m and n have the same parity. Notice also that if f and g are either both even or both odd, it holds that $\langle f, g \rangle_{\mathbb{R}} = -\langle f, g \rangle_{\mathbb{R}} = 0$. Hence, $\langle U_{m-1} | U_{n-1} \rangle = 0$ when m and n have the same parity. Moreover, since $\langle U_{m-1} | U_{n-1} \rangle = -\langle U_{n-1} | U_{m-1} \rangle$, we only need to consider the case where m is odd and n is even. In this situation, we get

$$\begin{aligned} \langle U_{m-1} | U_{n-1} \rangle_{\mathbb{R}} &= 2 \int_0^\infty \int_0^\infty U_{m-1}(x) \phi(x) U_{n-1}(y) \phi(y) (\operatorname{sgn}(y-x) + 1) d\mu_{\mathbb{R}}(x) d\mu_{\mathbb{R}}(y) \\ &= 4 \int_0^\infty \int_0^y U_{m-1}(x) \phi(x) U_{n-1}(y) \phi(y) d\mu_{\mathbb{R}}(x) d\mu_{\mathbb{R}}(y). \end{aligned}$$

Moreover, since $\phi(x) \equiv 1$ on $(0, 2)$ and $\phi(x) = \Phi^{-s}(x)$ for $x > 2$, it holds that

$$\int_0^y U_{m-1}(x) \phi(x) d\mu_{\mathbb{R}}(x) = \begin{cases} T_m(y)/m, & y \in [0, 2], \\ \frac{2s^2}{m(s^2 - m^2)} + \frac{\Phi^{-s-m}(y)}{s+m} - \frac{\Phi^{-s+m}(y)}{s-m}, & y \in (2, \infty), \end{cases} \quad (3.3)$$

where, $T_m(z) = \Phi^m(z) + \Phi^{-m}(z)$, $m \geq 1$, is the degree m monic Chebyshëv polynomial of the first kind for $[-2, 2]$. Therefore, $\langle U_{m-1} | U_{n-1} \rangle_{\mathbb{R}}$ is equal to the sum of

$$\frac{4}{m} \int_0^2 T_m(y) U_{n-1}(y) d\mu_{\mathbb{R}}(y) = \frac{16n}{m(n^2 - m^2)} \quad (3.4)$$

and of

$$\begin{aligned} 4 \int_2^\infty U_{n-1}(y) \left(\frac{2s^2 \Phi^{-s}(y)}{m(s^2 - m^2)} + \frac{\Phi^{-2s-m}(y)}{s+m} - \frac{\Phi^{-2s+m}(y)}{s-m} \right) d\mu_{\mathbb{R}}(y) &= \frac{n}{m} \frac{16s^2}{(s^2 - n^2)(s^2 - m^2)} + \\ + \frac{4}{s+m} \left(\frac{1}{2s+m-n} - \frac{1}{2s+m+n} \right) - \frac{4}{s-m} \left(\frac{1}{2s-m-n} - \frac{1}{2s-m+n} \right). \end{aligned} \quad (3.5)$$

By grouping corresponding $2s$ terms in (3.2) and (3.5), simplifying the resulting expression, and then adding (3.4), we get

$$\frac{16mn}{(n^2 - m^2)(s^2 - m^2)} + \frac{16n}{m(n^2 - m^2)} = \frac{n}{m} \frac{16s^2}{(n^2 - m^2)(s^2 - m^2)}.$$

The claim now follows by adding the first term on the right-hand side of (3.5) to the above expression. \square

The next lemma is a technical result that we need to continue with the proof of Theorem 2.1.

Lemma 3.3. *Put*

$$\Gamma_{2n,i} := \frac{\Gamma(n-i+\frac{1}{2})\Gamma(n+i+\frac{3}{2})}{\Gamma(n-i+1)\Gamma(n+i+2)} \quad \text{and} \quad \Gamma_{2n+1,i} := \frac{\Gamma(n-i-\frac{1}{2})\Gamma(n+i+\frac{3}{2})}{\Gamma(n-i+1)\Gamma(n+i+3)}. \quad (3.6)$$

Given $a \in \mathbb{C}$, it holds that

$$\sum_{i=0}^n \frac{4\Gamma_{2n,i}}{(2a)^2 - (2i+1)^2} = \frac{\pi}{2a} \frac{\Gamma(n+a+1)\Gamma(a-n-\frac{1}{2})}{\Gamma(n+a+\frac{3}{2})\Gamma(a-n)} \quad (3.7)$$

and that

$$\sum_{i=0}^n \frac{(2i+2)^2\Gamma_{2n+1,i}}{(2i+2)^2 - (2a+1)^2} = \frac{(2a+1)\pi}{4} \frac{\Gamma(n+a+1)\Gamma(a-n-\frac{1}{2})}{\Gamma(n+a+\frac{5}{2})\Gamma(a-n+1)}. \quad (3.8)$$

Moreover,

$$\sum_{i=0}^n \Gamma_{2n,i} = \frac{\pi}{2} \quad \text{and} \quad \sum_{i=0}^n (2i+2)^2\Gamma_{2n+1,i} = -2\pi. \quad (3.9)$$

Proof. We use the Pochhammer symbol, defined by

$$x_{(\ell)} := \frac{\Gamma(x+\ell)}{\Gamma(x)} = x(x+1)\cdots(x+\ell-1),$$

and the elementary transformation, $(1-x)_\ell = (-1)^\ell \Gamma(x)/\Gamma(x-\ell)$ to write the right hand side of (3.7) as

$$\frac{\pi}{2a} \frac{\Gamma(n+a+1)\Gamma(a-n-\frac{1}{2})}{\Gamma(a-n)\Gamma(n+a+\frac{3}{2})} = \frac{\pi}{2} \frac{\Gamma(n+a+1)}{\Gamma(1+a)} \frac{\Gamma(a)}{\Gamma(a-n)} \frac{\Gamma(a+\frac{1}{2})}{\Gamma(n+1+a+\frac{1}{2})} \frac{\Gamma(a+\frac{1}{2}-(n+1))}{\Gamma(a+\frac{1}{2})}.$$

It is obvious that the above expression is a rational function in a and we can write it as

$$-\frac{\pi}{2} \frac{(1+a)_{(n)}(1-a)_{(n)}}{(\frac{1}{2}+a)_{(n+1)}(\frac{1}{2}-a)_{(n+1)}}. \quad (3.10)$$

The rest of the proof of (3.7) is partial fractions decomposition of this rational function. Its poles are simple and are located at half integers with the residue of the pole at $a = m + 1/2$

being given by

$$\begin{aligned}
& \frac{\pi}{2} \frac{(1+a)_{(n)}(1-a)_{(n)}}{\left(\frac{1}{2}+a\right)_{(n+1)}} \left\{ \prod_{\ell=0}^{m-1} \frac{1}{2} - a + \ell \right\}^{-1} \left\{ \prod_{\ell=m+1}^n \frac{1}{2} - a + \ell \right\}^{-1} \Big|_{a=m+\frac{1}{2}} \\
&= \frac{\pi}{2} \frac{(m+\frac{3}{2})_{(n)}(-m+\frac{1}{2})_{(n)}}{(m+1)_{(n+1)}} \frac{(-1)^m}{m!(n-m)!} \\
&= \frac{\pi}{2} \frac{\Gamma(m+n+\frac{3}{2})}{\Gamma(m+\frac{3}{2})} \frac{\Gamma(n-m+\frac{1}{2})}{\Gamma(-m+\frac{1}{2})} \frac{\Gamma(m+1)}{\Gamma(m+n+2)} \frac{(-1)^m}{m!(n-m)!} \\
&= \frac{\pi}{2m+1} \frac{\Gamma(m+n+\frac{3}{2})}{\Gamma(m+\frac{1}{2})} \frac{\Gamma(n-m+\frac{1}{2})}{\Gamma(-m+\frac{1}{2})} \frac{\Gamma(m+1)}{\Gamma(m+n+2)} \frac{(-1)^m}{m!(n-m)!} \\
&= \frac{1}{2m+1} \frac{\Gamma(m+n+\frac{3}{2})}{\Gamma(m+n+2)} \frac{\Gamma(n-m+\frac{1}{2})}{\Gamma(n-m+1)},
\end{aligned}$$

where we used the formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ to write

$$\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-m\right) = \frac{\pi}{\sin(\pi(m+\frac{1}{2}))} = (-1)^m \pi.$$

Since (3.10) is even in a , the residue of the pole at $a = -m - \frac{1}{2}$ must differ from that of $a = -m - \frac{1}{2}$ by a minus sign. We have identified all the poles and their residues, and hence there exists a polynomial f such that

$$\begin{aligned}
& \frac{\pi}{2} \frac{(1+a)_{(n)}(1-a)_{(n)}}{\left(\frac{1}{2}+a\right)_{(n+1)}\left(\frac{1}{2}-a\right)_{(n+1)}} \\
&= f(a) + \sum_{m=0}^n \frac{1}{2m+1} \frac{\Gamma(m+n+\frac{3}{2})}{\Gamma(m+n+2)} \frac{\Gamma(n-m+\frac{1}{2})}{\Gamma(n-m+1)} \left[\frac{1}{a-(m+\frac{1}{2})} - \frac{1}{a+(m+\frac{1}{2})} \right] \\
&= f(a) + \sum_{m=0}^n \frac{\Gamma(m+n+\frac{3}{2})}{\Gamma(m+n+2)} \frac{\Gamma(n-m+\frac{1}{2})}{\Gamma(n-m+1)} \left[\frac{4}{(2a)^2 - (2m+1)^2} \right].
\end{aligned}$$

It remains to show that $f(a)$ is identically 0. But this is trivial, since, multiplying both sides by a^2 and taking the limit as $a \rightarrow \infty$ we expect the left-hand side to go to a non-zero constant. Equality can only be preserved under this procedure when $f(a) = 0$.

The proof of (3.8) is essentially identical, and we only record the most salient maneuvers.

$$\frac{(2a+1)\pi}{4} \frac{\Gamma(n+a+1)\Gamma(a-n-\frac{1}{2})}{\Gamma(a-n+1)\Gamma(n+a+\frac{5}{2})} = -\frac{\pi}{2} \frac{(1+a)_{(n)}(-a)_{(n)}}{\left(\frac{3}{2}+a\right)_{(n+1)}\left(\frac{1}{2}-a\right)_{(n+1)}}. \quad (3.11)$$

This is a rational function with poles at $a = -\frac{3}{2}, -\frac{5}{2}, \dots, -\frac{3}{2} - n$ and $a = \frac{1}{2}, \frac{3}{2}, \dots, \frac{1}{2} + n$. Since this rational function is invariant under the substitution $a \mapsto -1 - a$, the residues at $-(2m+1)/2$ and $(2m-1)/2$ are equal up to sign, the former being given by

$$\frac{m}{2} \frac{\Gamma(n+1-\frac{2m+1}{2})}{\Gamma(n+m+2)} \frac{\Gamma(n+\frac{2m+1}{2})}{\Gamma(n-m+2)}.$$

The proof of (3.8) follows by using these facts to write (3.11) in partial fractions form and simplifying.

Finally, the first sum in (3.9) is equal to

$$\frac{\pi}{2} \lim_{a \rightarrow \infty} a \frac{\Gamma(n+a+1)\Gamma(a-n-\frac{1}{2})}{\Gamma(n+a+\frac{3}{2})\Gamma(a-n)} = \frac{\pi}{2}$$

and the second one can be computed analogously. \square

Now, we are ready to prove Theorem 2.1.

Lemma 3.4. *Under the conditions of Theorem 2.1, (2.4) holds.*

Proof. Let π_n be given by (2.4). It follows immediately from Lemmas 3.1 and 3.2 that $\langle \pi_{2n} | \pi_{2m} \rangle = \langle \pi_{2n+1} | \pi_{2m+1} \rangle = 0$. Thus, to show that π_n 's are indeed skew-orthogonal, it suffices to prove that

$$\langle \pi_{2n} | U_{2j+1} \rangle = \langle U_{2j} | \pi_{2n+1} \rangle = 0, \quad j \in \{0, \dots, n-1\}. \quad (3.12)$$

Using (3.6), we deduce from Lemmas 3.1 and 3.2 applied with $n = 2j+2$ and $m = 2i+1$ that

$$\langle \pi_{2n} | U_{2j+1} \rangle = \frac{4n+3}{2\pi} \frac{(2j+2)s^2}{s^2 - (2j+2)^2} \sum_{i=0}^n \frac{4\Gamma_{2n,i}}{(2j+2)^2 - (2i+1)^2}.$$

Hence, the desired claim is a consequence of (3.7) applied with $a = j+1$ as $\frac{1}{\Gamma(j+1-n)} = 0$. Furthermore, it follows again from Lemmas 3.1 and 3.2 applied with $n = 2i+2$ and $m = 2j+1$ that

$$\langle U_{2j} | \pi_{2n+1} \rangle = -\frac{8}{\pi} \frac{1}{2j+1} \sum_{i=0}^n \frac{(2i+2)^2 \Gamma_{2n+1,i}}{(2i+2)^2 - (2j+1)^2}.$$

Equation (3.8) applied with $a = j$ yields that the above skew-inner product is 0. It remains to verify that $\langle \pi_{2n} | \pi_{2n+1} \rangle = 1$. It follows from (3.12) that

$$\begin{aligned} \langle \pi_{2n} | \pi_{2n+1} \rangle &= \frac{1}{\sqrt{\pi}} \left(1 - \frac{(2n+2)^2}{s^2} \right) \frac{\Gamma(2n+\frac{3}{2})}{\Gamma(2n+2)} \langle \pi_{2n} | U_{2n+1} \rangle \\ &= \frac{2n+2}{\pi^{3/2}} \frac{\Gamma(2n+\frac{5}{2})}{\Gamma(2n+2)} \sum_{i=0}^n \frac{4\Gamma_{2n,i}}{(2n+2)^2 - (2i+1)^2}. \end{aligned}$$

Upon applying (3.7) with $a = n+1$, the desired result follows. \square

3.2 Proof of Theorem 2.2

Set $u := \Phi(z)\overline{\Phi(w)}$. Then $|\Phi(z)\Phi(w)|^s K_{N,s}(z,w)/(\Phi'(z)\overline{\Phi'(w)})^N$ is equal to

$$\sum_{n=0}^{N-1} \frac{s^2 - (n+1)^2}{2\pi s} u^n \left(1 + \mathcal{O}(\rho^{-2(n+1)}) \right)$$

uniformly for $|\Phi(z)|, |\Phi(w)| \geq \rho > 1$ by (2.2) and (2.3). It can be readily verified that

$$\begin{aligned} \sum_{n=0}^{N-1} \frac{s^2 - (n+1)^2}{2\pi s(s-N)} u^{n-N} &= \sum_{m=1}^N \frac{s^2 - N^2 - (2N-1)(1-m) - (2-m)(1-m)}{2\pi s(s-N)} u^{-m} \\ &= \frac{1 + Ns^{-1}}{2\pi} \sum_{m=1}^N u^{-m} - \frac{2Ns^{-1} - s^{-1}}{2\pi(s-N)} \left(\sum_{m=1}^N u^{-m+1} \right)' - \frac{1}{2\pi s(s-N)} \left(\sum_{m=1}^N u^{-m+2} \right)'' , \end{aligned}$$

which converges to

$$\frac{1+\lambda}{2\pi} \frac{1}{u-1} + \frac{\lambda}{\pi} \frac{c^{-1}}{(u-1)^2}$$

uniformly for $|\Phi(z)|, |\Phi(w)| \geq \rho > 1$. This proves (2.5) since $\lambda = 1$ when $c^{-1} > 0$ and clearly

$$\lim_{N \rightarrow \infty} \sum_{n=0}^{N-1} \frac{s^2 - (n+1)^2}{2\pi s(s-N)} u^{n-N} \mathcal{O}(\rho^{-2(n+1)}) = 0$$

uniformly for $|\Phi(z)|, |\Phi(w)| \geq \rho$.

3.3 Proof of Theorem 2.3

It is known, see [15, Theorem 8.21.7] or [1, Theorem 2.11], that

$$C_n^{(\alpha)} \left(\frac{z}{2} \right) = \frac{(n+1)^{\alpha-1}}{\Gamma(\alpha)} \left(1 + \mathcal{O} \left(\frac{1}{n+1} \right) \right) (\Phi'(z))^\alpha \Phi^n(z) \quad (3.13)$$

locally uniformly in $\overline{\mathbb{C}} \setminus [-2, 2]$, where $(\Phi'(z))^\alpha$ is the principal branch. Set, for brevity,

$$S_N(z, w) := \frac{2}{s-N} \frac{\sum_{j=0}^{J-1} \pi_{2j}(z) \pi_{2j+1}(w)}{\Phi'(z) \Phi'(w) (\Phi(z) \Phi(w))^N}.$$

Further, put $u = \Phi(z) \Phi(w)$. Then it follows from (2.4) and (3.13) that

$$\begin{aligned} S_N(z, w) &= \frac{\Phi(w)}{4\pi} \frac{\sqrt{\Phi'(z)}}{\sqrt{\Phi'(w)}} \sum_{j=0}^{J-1} \left(1 + \mathcal{O} \left(\frac{1}{2j+1} \right) \right) \frac{4j+3}{s} \frac{s^2 - (2j+1)^2}{s(s-N)} u^{2j-N} \\ &\quad - \sqrt{\Phi'(z)} \sqrt{\Phi'(w)} \frac{\Phi(w)}{2\pi} \sum_{j=0}^{J-1} \frac{(4j+3)(2j+1)}{s^2(s-N)} \left(1 + \mathcal{O} \left(\frac{1}{2j+1} \right) \right) u^{2j-N}. \end{aligned}$$

Upon replacing j with $J-j$, the first sum above can be rewritten as

$$\sum_{j=1}^J \frac{2N-4j+3}{s} \frac{s+N-2j+1}{s} \frac{s-N+2j-1}{s-N} \left(1 + \mathcal{O} \left(\frac{1}{N-2j+1} \right) \right) u^{-2j}.$$

It is easy to see that the coefficient next to u^{-2j} is positive and bounded by C_j for some absolute constant C . Hence, these sums form a normal family in $|u| > 1$ and to find their limit as $N \rightarrow \infty$ is enough to take the limits of the individual coefficients. Clearly, the second sum can be treated similarly. Therefore,

$$\lim_{N \rightarrow \infty} S_N(z, w) = \frac{\lambda(1+\lambda)}{2\pi} \frac{\sqrt{\Phi'(z)}}{\sqrt{\Phi'(w)}} \frac{\Phi(w)}{u^2-1} \left(1 + \frac{1}{c} \frac{u^2+1}{u^2-1} \right) - \sqrt{\Phi'(z)} \sqrt{\Phi'(w)} \frac{\lambda^2}{\pi c} \frac{\Phi(w)}{u^2-1},$$

where the limit holds locally uniformly in $|u| > 1$. Notice that $\Phi' = \Phi^2/(\Phi^2-1)$. Hence, when $c = \infty$, we get that

$$\lim_{N \rightarrow \infty} (S_N(z, w) - S_N(w, z)) = \frac{\lambda(1+\lambda)}{2\pi} \frac{1}{\sqrt{\Phi^2(z)-1} \sqrt{\Phi^2(w)-1}} \frac{\Phi(w) - \Phi(z)}{\Phi(z) \Phi(w) - 1}.$$

On the other hand, when $c < \infty$, it holds that $\lambda = 1$ and therefore the above limit is equal to

$$\frac{1}{\pi} \frac{1}{\sqrt{\Phi^2(z)-1} \sqrt{\Phi^2(w)-1}} \frac{\Phi(w) - \Phi(z)}{\Phi(z) \Phi(w) - 1} \left(1 + \frac{c^{-1}}{\Phi(z) \Phi(w) - 1} \right).$$

The last two formulas and the definition of $S_N(z, w)$ clearly yield (2.6).

3.4 Proof of Theorem 2.4

Lemma 3.5. *It holds that*

$$\begin{cases} \epsilon(\phi\pi_{2n+1})(x) &= -\int_{\xi\infty}^x \pi_{2n+1}(u)\phi(u)du, \\ \epsilon(\phi\pi_{2n})(x) &= -\int_{\xi\infty}^x \pi_{2n}(u)\phi(u)du - \frac{\xi}{2}\Gamma_n(s), \end{cases}$$

where $\xi = \pm 1$ and $\Gamma_n(s) := \int_{-\infty}^{\infty} \phi(x)\pi_{2n}(x)dx = \frac{s}{2} \left(n + \frac{3}{4}\right) \frac{\Gamma(\frac{s}{2} + n + 1)\Gamma(\frac{s}{2} - n - \frac{1}{2})}{\Gamma(\frac{s}{2} + n + \frac{3}{2})\Gamma(\frac{s}{2} - n)}$.

Proof. The second equality in the definition of $\Gamma_n(s)$ follows from (3.1), (3.3), and (3.7) applied with $a = s/2$. Moreover, $\int_{-\infty}^{\infty} \phi(x)\pi_{2n+1}(x)dx = 0$ since the integrand is an odd function. The claim then is a simple consequence of (1.11). \square

Lemma 3.6. *Let c be given by (2.1). Assuming $c < \infty$, it holds that*

$$\lim_{N \rightarrow \infty} \Phi^{-N}(z) \sum_{j=0}^{J-1} \Gamma_j(s)\pi_{2j+1}(z) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{c+1}{2})}{\Gamma(\frac{c}{2})} \frac{\Phi'(z)}{\sqrt{\Phi^2(z) - 1}}$$

locally uniformly for $z \notin [-2, 2]$, where $N = 2J$. When $c = \infty$, the limit above is convergent if it is additionally normalized by $\sqrt{s - N}$.

Proof. By changing the index of summation to $J - 1 - j$ and using (2.4) and (3.13), we see that we need to compute the limit of

$$\begin{aligned} & \frac{\sqrt{\Phi'(z)}}{\sqrt{\pi}} \sum_{j=0}^{J-1} \left(1 + \mathcal{O}\left(\frac{1}{J-j}\right)\right) \frac{\sqrt{N-2j}}{s} \frac{\Gamma(\frac{s}{2} + J - j)\Gamma(\frac{s}{2} - J + j + \frac{3}{2})}{\Gamma(\frac{s}{2} + J - j - \frac{1}{2})\Gamma(\frac{s}{2} - J + j + 1)} \Phi^{-2j-1}(z) - \\ & - \frac{(\Phi'(z))^{3/2}}{2\sqrt{\pi}} \sum_{j=0}^{J-1} \left(1 + \mathcal{O}\left(\frac{1}{J-j}\right)\right) \frac{(N-2j)^{3/2}}{s} \frac{\Gamma(\frac{s}{2} + J - j)\Gamma(\frac{s}{2} - J + j + \frac{1}{2})}{\Gamma(\frac{s}{2} + J - j + \frac{1}{2})\Gamma(\frac{s}{2} - J + j + 1)} \Phi^{-2j-1}(z). \end{aligned}$$

Straightforward estimates show that the above sums form normal families and therefore the limiting function can be obtained by simply evaluating the limits of the coefficients. That is, the above expression converges to

$$\frac{\sqrt{\Phi'(z)}}{\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{\Gamma(\frac{c+3}{2} + j)}{\Gamma(\frac{c}{2} + 1 + j)} \Phi^{-2j-1}(z) - \frac{(\Phi'(z))^{3/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{\Gamma(\frac{c+1}{2} + j)}{\Gamma(\frac{c}{2} + 1 + j)} \Phi^{-2j-1}(z)$$

locally uniformly for $z \notin [-2, 2]$. Since $\Phi' = \Phi^2/(\Phi^2 - 1) = \sum_{j=0}^{\infty} \Phi^{-2j}$, we get that

$$\begin{aligned} \frac{\Phi'(z)}{2} \sum_{j=0}^{\infty} \frac{\Gamma(\frac{c+1}{2} + j)}{\Gamma(\frac{c}{2} + 1 + j)} \Phi^{-2j-1}(z) &= \sum_{j=0}^{\infty} \left(\frac{1}{2} \sum_{k=0}^j \frac{\Gamma(\frac{c+1}{2} + k)}{\Gamma(\frac{c}{2} + 1 + k)} \right) \Phi^{-2j-1}(z) = \\ &= \sum_{j=0}^{\infty} \left(\frac{\Gamma(\frac{c+3}{2} + j)}{\Gamma(\frac{c}{2} + 1 + j)} - \frac{\Gamma(\frac{c+1}{2})}{\Gamma(\frac{c}{2})} \right) \Phi^{-2j-1}(z), \end{aligned}$$

where the last equality can be shown by induction. Using the identity $\Phi' = \Phi^2/(\Phi^2 - 1)$ once more, we arrive at the first claim of the lemma. The second claim can be shown analogously. \square

Lemma 3.7. *Under the conditions of Theorem 2.4, (2.7) holds.*

Proof. For $x, y \in \mathbb{R}$, it holds by (1.12) and Lemma 3.5 that

$$\epsilon \kappa_{N,s} \epsilon(x, y) = \int_{\xi_x \infty}^x \int_{\xi_y \infty}^y \kappa_{N,s}(u, v) du dv + \left(\xi_x \int_{\xi_y \infty}^y - \xi_y \int_{\xi_x \infty}^x \right) \sum_{j=0}^{J-1} \Gamma_j(s) \pi_{2j+1}(u) \phi(u) du,$$

where $\xi_u = \text{sgn}(u)$, $u \in \mathbb{R} \setminus \{0\}$. The first limit in (2.7) now follows from (2.6), Lemma 3.6, the change of variables $\Phi(u) \mapsto u$ and $\Phi(v) \mapsto v$, and upon observing that $(|\Phi(u)|/\Phi(u))^N = 1$ for $u \in \mathbb{R} \setminus [-2, 2]$ as $\Phi(u)$ is real for such u and N is even. The other two limits can be obtained similarly. \square

3.5 Proof of Proposition 2.5

Set $x_{N,a} := x + a/N\omega(x)$. For $\pm \text{im}(a) \geq 0$, we have that

$$(x_{N,a}^2 - 4)^{1/2} = (x^2 - 4)_{\pm}^{1/2} + \frac{a}{N\omega(x)} \frac{2 + a/N\omega(x)}{(x_{N,a}^2 - 4)^{1/2} + (x^2 - 4)_{\pm}^{1/2}} = (x^2 - 4)_{\pm}^{1/2} + \frac{a + \mathcal{O}(N^{-1})}{N\omega(x)}, \quad (3.14)$$

locally uniformly for $x \in (-2, 2)$ and $a \in \mathbb{C}$. Hence, since $(4 - x^2)^{1/2} = \mp i(x^2 - 4)_{\pm}^{1/2}$, we have that

$$\Phi(x_{N,a}) = \Phi_{\pm}(x) + \frac{a + \mathcal{O}(N^{-1})}{N\omega(x)} = \Phi_{\pm}(x) \left(1 \mp \frac{ia + \mathcal{O}(N^{-1})}{N} \right), \quad \pm \text{im}(a) > 0, \quad (3.15)$$

locally uniformly for $x \in (-2, 2)$ and $a \in \mathbb{C}$. Since $2\Phi^{-1}(z) = z - \sqrt{z^2 - 4}$ and $(x^2 - 4)_{+}^{1/2} + (x^2 - 4)_{-}^{1/2} \equiv 0$ on $(-2, 2)$, it holds that

$$\Phi^{-1}(x_{N,a}) = \Phi_{\mp}(x) \left(1 \pm \frac{ia + \mathcal{O}(N^{-1})}{N} \right), \quad \pm \text{im}(a) > 0, \quad (3.16)$$

from which the claim of the proposition easily follows.

3.6 Proof of Theorem 2.6

Lemma 3.8. *Let $p(\cdot)$ be a monic polynomial of degree d and α be a real constant. Then*

$$\lim_{N \rightarrow \infty} \sum_{n=0}^{N-1} \frac{p(n)}{N^{d+1}} \left(1 \pm \frac{\eta}{N} \right)^{n+\alpha} = \int_0^1 t^d e^{\pm \eta t} dt,$$

where we take the principal α -root.

Proof. It holds that

$$\begin{aligned} \sum_{n=0}^{N-1} \frac{p(n)}{N^{d+1}} \left(1 \pm \frac{\eta}{N} \right)^{n+\alpha} &= \left(1 \pm \frac{\eta}{N} \right)^{\alpha} \sum_{k=0}^d p_k \sum_{n=0}^{N-1} \frac{(n+k) \cdots (n+1)}{N^{d+1}} \left(1 \pm \frac{\eta}{N} \right)^n \\ &= \left(1 \pm \frac{\eta}{N} \right)^{\alpha} \sum_{k=0}^d \frac{(\pm 1)^k p_k}{N^{d-k}} \left(\frac{(1 \pm \frac{\eta}{N})^{N+k} - (1 \pm \frac{\eta}{N})^k}{\pm \eta} \right)^{(k)}_{\eta} \end{aligned}$$

for some constants p_k , which depend on the polynomial $p(\cdot)$. Since $p_d = 1$, the desired limit is equal to

$$(-1)^d \left(\frac{e^{\pm\eta} - 1}{\pm\eta} \right)_\eta^{(d)} = (-1)^d \left(\int_0^1 e^{\pm\eta t} dt \right)_\eta^{(d)} = \int_0^1 t^d e^{\pm\eta t} dt. \quad \square$$

Lemma 3.9. *Assuming (2.1), limit (2.9) holds.*

Proof. First we shall consider the case $\text{im}(a)\text{im}(b) \neq 0$. Clearly, it follows from (3.14) that

$$\lim_{N \rightarrow \infty} (x_{N,a}^2 - 4)^{-1/2} \overline{(x_{N,b}^2 - 4)^{-1/2}} = \pm \omega^2(x), \quad \pm \text{im}(a)\text{im}(b) > 0, \quad (3.17)$$

where, as before, $x_{N,a} = x + a/N\omega(x)$. Hence, we see from (2.2) and (2.3) that we need to compute the limit of

$$\begin{aligned} \sum_{n=0}^{N-1} \frac{s^2 - (n+1)^2}{2\pi s^2 N} & \left((\Phi(x_{N,a}) \overline{\Phi(x_{N,b})})^{n+1} + (\Phi^{-1}(x_{N,a}) \overline{\Phi^{-1}(x_{N,b})})^{n+1} \right. \\ & \left. - (\Phi^{-1}(x_{N,a}) \overline{\Phi(x_{N,b})})^{n+1} - (\Phi(x_{N,a}) \overline{\Phi^{-1}(x_{N,b})})^{n+1} \right). \end{aligned} \quad (3.18)$$

Set $\tau := i(a - \bar{b})$. It follows from (3.15) and (3.16) that (3.18) can be rewritten as

$$\begin{aligned} \pm \sum_{n=0}^{N-1} \frac{s^2 - (n+1)^2}{2\pi s^2 N} & \left(\left(1 + \frac{\tau + o(1)}{N} \right)^{n+1} + \left(1 - \frac{\tau + o(1)}{N} \right)^{n+1} \right. \\ & \left. - (\Phi_-^2(x) + o(1))^{n+1} - (\Phi_+^2(x) + o(1))^{n+1} \right) \end{aligned} \quad (3.19)$$

for $\pm \text{im}(a)\text{im}(b) > 0$, where $o(1)$ holds locally uniformly on $(-2, 2) \times \mathbb{C}^2$. Therefore, it follows from Lemma 3.8 that

$$\begin{aligned} \lim_{N \rightarrow \infty} \sum_{n=0}^{N-1} \frac{s^2 - (n+1)^2}{2\pi s^2 N} & \left(\left(1 + \frac{\tau + o(1)}{N} \right)^{n+1} + \left(1 - \frac{\tau + o(1)}{N} \right)^{n+1} \right) \\ & = \frac{1}{\pi} \int_0^1 (1 - (\lambda t)^2) \frac{e^{\tau t} + e^{-\tau t}}{2} dt = \frac{1}{\pi} \int_0^1 (1 - (\lambda t)^2) \cos(-i\tau t) dt. \end{aligned} \quad (3.20)$$

As $\Phi_\pm^2(x) \neq 1$ and $|\Phi_\pm(x)| = 1$, it clearly holds that

$$\lim_{N \rightarrow \infty} \sum_{n=0}^{N-1} \frac{s^2 - (n+1)^2}{2\pi s^2 N} (\Phi_\pm^2(x) + o(1))^{n+1} = 0.$$

The claim now follows from the last two limits, (3.17), and (3.19). When either $\text{im}(a) = 0$ or $\text{im}(b) = 0$, (2.6) can be deduced similarly. \square

3.7 Proof of Theorem 2.7

Lemma 3.10. *Let $j_1, j_2 \in \mathbb{Z}$, and $p(\cdot)$ be a monic polynomial of degree d . Then*

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N^{d+1} \omega^2(x)} \sum_{j=0}^{J-1} p(2j) C_{2j+j_1}^{(3/2)} \left(\frac{x_{N,u}}{2} \right) C_{2j+j_2}^{(1/2)} \left(\frac{x_{N,v}}{2} \right) & = \\ & = \mp \frac{i}{\pi} \int_0^1 t^d \left(\Phi_\pm^{j_1-j_2+1}(x) e^{\mp \tau t} - \Phi_\mp^{j_1-j_2+1}(x) e^{\pm \tau t} \right) dt \end{aligned}$$

for $\pm \text{im}(u) \geq 0$, where $\tau := i(u - v)$, $x_{N,u} = x + u/N\omega(x)$, and $N = 2J$.

Proof. Using the asymptotic formulas for $C_n^{(\alpha)}(x)$ in a neighborhood of a point $x \in (-1, 1)$, see [15, Theorem 8.21.8] or [1, Theorem 2.11], we get that

$$C_n^{(\alpha)}\left(\frac{x_{N,u}}{2}\right) = \frac{(n+1)^{\alpha-1}}{\Gamma(\alpha)} \left(\frac{\Phi^{n+\alpha}(x_{N,u}) + e^{\pm\alpha\pi i} \Phi^{-n-\alpha}(x_{N,u})}{(x_{N,u}^2 - 4)^{\alpha/2}} + \mathcal{O}\left(\frac{1}{n+1}\right) \right) \quad (3.21)$$

for $\pm \operatorname{im}(u) > 0$ and locally uniformly for $x \in (-2, 2)$ and $u \in \mathbb{C}$. Observe that

$$\lim_{N \rightarrow \infty} (x_{N,u}^2 - 4)^{-3/4} (x_{N,v}^2 - 4)^{-1/4} = \omega^2(x) \begin{cases} -1, & \operatorname{im}(u) \operatorname{im}(v) > 0, \\ \mp i, & \operatorname{im}(u) \operatorname{im}(v) < 0, \end{cases} \quad (3.22)$$

when $\pm \operatorname{im}(u) > 0$. Thus, to compute the desired limit it is enough to compute the limit of

$$\begin{aligned} \frac{1}{\pi} \sum_{j=0}^{J-1} \frac{2p(2j)}{N^{d+1}} \left(\Phi^{2j+j_1+3/2}(x_{N,u}) - \operatorname{sgn}(\operatorname{im}(u)) i \Phi^{-2j-j_1-3/2}(x_{N,u}) \right) \times \\ \times \left(\Phi^{2j+j_2+1/2}(x_{N,v}) + \operatorname{sgn}(\operatorname{im}(v)) i \Phi^{-2j-j_2-1/2}(x_{N,v}) \right). \end{aligned} \quad (3.23)$$

Assume that $\operatorname{im}(u) \operatorname{im}(v) > 0$. Then it follows from (3.15) and (3.16) that the dominant term in (3.23) is

$$\pm \frac{i}{\pi} \sum_{j=0}^{J-1} \frac{2p(2j)}{N^{d+1}} \left(\frac{\Phi^{2j+j_1+3/2}(x_{N,u})}{\Phi^{2j+j_2+1/2}(x_{N,v})} - \frac{\Phi^{2j+j_2+1/2}(x_{N,v})}{\Phi^{2j+j_1+3/2}(x_{N,u})} \right) \quad (3.24)$$

for $\pm \operatorname{im}(u) > 0$. Using (3.15) and (3.16) once more we can rewrite (3.24) as

$$\pm \frac{i}{\pi} \sum_{j=0}^{J-1} \frac{\tilde{p}(j)}{J^{d+1}} \left(\frac{\left(1 \mp \frac{\tau+o(1)}{J}\right)^{j+\frac{2j_1+3}{4}}}{\Phi^{j_2-j_1-1}(x_{N,v})} - \frac{\left(1 \pm \frac{\tau+o(1)}{J}\right)^{j+\frac{2j_2+1}{4}}}{\Phi^{j_1-j_2+1}(x_{N,u})} \right) \quad (3.25)$$

for $\pm \operatorname{im}(u) > 0$, where $\tilde{p}(\cdot)$ is a monic polynomial of degree d and $o(1)$ holds locally uniformly in $(-2, 2) \times \mathbb{C}^2$. Since $\Phi_+(x)\Phi_-(x) \equiv 1$ for $x \in (-2, 2)$, the claim of the lemma follows from (3.22) and Lemma 3.8.

The case $\operatorname{im}(u) \operatorname{im}(v) < 0$ is absolutely analogous. When $\operatorname{im}(u) = 0$ (resp. $\operatorname{im}(v) = 0$) we can replace $\Phi(x_{N,u})$ (resp. $\Phi(x_{N,v})$) with $\Phi_+(x_{N,u})$ (resp. $\Phi_+(x_{N,v})$) and carry the computations as before (notice that replacing \pm with \mp and \mp with \pm on the right-hand side of the limit in the statement of the lemma does not change the said limit). \square

Lemma 3.11. *Under the conditions of Theorem 2.7, limit (2.11) holds.*

Proof. It follows from Theorem 2.1 that we need to compute the limit of

$$\frac{1}{N^2 \omega^2(x)} \sum_{n=0}^{J-1} \frac{4n+3}{8} \left(1 - \frac{(2n+1)^2}{s^2} \right) C_{2n}^{(3/2)}\left(\frac{x_{N,a}}{2}\right) C_{2n+1}^{(1/2)}\left(\frac{x_{N,b}}{2}\right),$$

which is equal to

$$\frac{\mp i}{4\pi} \int_0^1 t(1 - (\lambda t)^2) (e^{\mp \tau t} - e^{\pm \tau t}) dt = \frac{1}{2\pi} \int_0^1 t(1 - (\lambda t)^2) \sin((b-a)t) dt$$

by Lemma 3.10. We get (2.7) by interchanging the roles of a and b in the above formula and taking the difference. \square

Lemma 3.12. *It holds that*

$$\begin{cases} \epsilon(\phi\pi_{2n+1})(x) &= -\frac{2}{4n+3} \left[\left(1 - \frac{(2n+2)^2}{s^2}\right) C_{2n+2}^{(1/2)}\left(\frac{x}{2}\right) - \left(1 - \frac{(2n+1)^2}{s^2}\right) C_{2n}^{(1/2)}\left(\frac{x}{2}\right) \right] + \Delta_n(s), \\ \epsilon(\phi\pi_{2n})(x) &= -\frac{4n+3}{8} C_{2n+1}^{(1/2)}\left(\frac{x}{2}\right), \end{cases}$$

for $x \in (-2, 2)$, where $\Delta_n(s) = \frac{(-1)^{n+1} \Gamma(n+\frac{1}{2})}{\sqrt{\pi} \Gamma(n+2)} \left(1 + \frac{(-1)^{n+1} \Gamma(n+\frac{1}{2})}{2\sqrt{\pi} \Gamma(n+2)} - \frac{(2n+1)(2n+2)}{s^2}\right)$.

Proof. Observe first that

$$\begin{cases} \epsilon(\phi\pi_{2n+1})(x) &= -\int_0^x \pi_{2n+1}(u)\phi(u)du + \int_0^\infty \pi_{2n+1}(u)\phi(u)du, \\ \epsilon(\phi\pi_{2n})(x) &= -\int_0^x \pi_{2n}(u)\phi(u)du. \end{cases}$$

Indeed, by the very definition (1.11), we have that $(\epsilon f)(x) = -\int_0^x f(t)dt - \frac{1}{2} \int_{-\infty}^0 f(t)dt + \frac{1}{2} \int_0^\infty f(t)dt$. Hence, for even functions f it holds that $(\epsilon f)(x) = -\int_0^x f(t)dt$ and for odd functions f it holds that $(\epsilon f)(x) = -\int_0^x f(t)dt + \int_0^\infty f(t)dt$, from which the claim easily follows. Using [5, Eq. (18.9.19)] and [5, Table 18.6.1], one can also get that

$$\begin{cases} \int_0^x C_{2n}^{(3/2)}\left(\frac{u}{2}\right) du &= 2C_{2n+1}^{(1/2)}\left(\frac{x}{2}\right) \\ \int_0^x C_{2n+1}^{(3/2)}\left(\frac{u}{2}\right) du &= 2 \left[C_{2n+2}^{(1/2)}\left(\frac{x}{2}\right) - \frac{(-1)^{n+1} \Gamma(n+\frac{3}{2})}{\sqrt{\pi} \Gamma(n+2)} \right]. \end{cases}$$

Since $\phi(u) = 1$ for $u \in (-2, 2)$, the expression for $\epsilon(\phi\pi_{2n})(x)$ now follows from Theorem 2.1. It further follows from [5, Eq. (18.9.7)] that

$$\pi_{2n+1}(u) = \frac{1}{4n+3} \left[\left(1 - \frac{(2n+2)^2}{s^2}\right) C_{2n+1}^{(3/2)}\left(\frac{u}{2}\right) - \left(1 - \frac{(2n+1)^2}{s^2}\right) C_{2n-1}^{(3/2)}\left(\frac{u}{2}\right) \right]. \quad (3.26)$$

Hence, we only need to compute the constant term. Clearly,

$$\begin{aligned} \left(1 - \frac{(2n+2)^2}{s^2}\right) \frac{(-1)^{n+1} \Gamma(n+\frac{3}{2})}{\sqrt{\pi} \Gamma(n+2)} - \left(1 - \frac{(2n+1)^2}{s^2}\right) \frac{(-1)^n \Gamma(n+\frac{1}{2})}{\sqrt{\pi} \Gamma(n+1)} &= \\ \frac{4n+3}{2} \frac{(-1)^{n+1} \Gamma(n+\frac{1}{2})}{\sqrt{\pi} \Gamma(n+2)} \left(1 - \frac{(2n+1)(2n+2)}{s^2}\right). \end{aligned}$$

Thus, it remains to compute $\int_0^\infty \pi_{2n+1}(u)\phi(u)du$. It follows from (3.1) and (3.3) that

$$\int_0^\infty \pi_{2n+1}(u)\phi(u)du = -\frac{1}{\pi} \sum_{i=0}^n \frac{\Gamma(n-i-\frac{1}{2}) \Gamma(n+i+\frac{3}{2})}{\Gamma(n-i+1) \Gamma(n+i+3)} = \frac{1}{2\pi} \left(\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+2)} \right)^2,$$

where the second equality follows from (3.8) after taking the limit as $a \rightarrow -1/2$ and observing that

$$\frac{\Gamma(a-n-\frac{1}{2})}{\Gamma(a-n+1)} = \frac{\Gamma(n-a)}{\Gamma(n-a+\frac{3}{2})} \frac{\sin(\pi(n-a-1))}{\sin(\pi(n+a+\frac{1}{2}))}. \quad \square$$

Lemma 3.13. *Assuming (2.1), it holds that*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{J-1} \Delta_j(s) \pi_{2j}(x_{N,a}) = 0$$

locally uniformly for $x \in (-2, 2)$ and $a \in \mathbb{C}$, where $x_{N,a} = x + a/N\omega(x)$.

Proof. Observe that it is enough to prove the claim with $\Delta_j(s)$ replaced by $(-1)^{j+1}\Gamma(j+1/2)/\Gamma(j+2)$ as these constants have the same leading order behavior. Hence, we need to analyze the sum

$$\sum_{j=0}^{J-1} \frac{(-1)^{j+1} + o(1)}{\sqrt{2j+1}} C_{2j}^{(3/2)} \left(\frac{x_{N,a}}{2} \right) = \frac{2}{\sqrt{\pi}} \sum_{j=0}^{J-1} \frac{(-1)^{j+1} + o(1)}{(x_{N,a}^2 - 4)^{3/4}} \left(\Phi^{2j+3/2}(x_{N,a}) \mp i\Phi^{-2j-3/2}(x_{N,a}) \right)$$

for $\pm \operatorname{im}(a) \geq 0$ by Theorem 2.1 and (3.21). Since $\Phi_{\pm}(x_{N,a}) \rightarrow \Phi_{\pm}(x) \neq 1$ for $\pm \operatorname{im}(a) \geq 0$, the geometric sum formula and straightforward estimates show that the above sum converges to zero when normalized by $1/N$. \square

Lemma 3.14. *Given (2.1), limits (2.12) and (2.13) hold.*

Proof. Set, as usual, $x_{N,a} = x + a/N\omega(x)$. Put $K_{n,i} := 1 - \frac{(2n+i)^2}{s^2}$. Then

$$\begin{aligned} \widetilde{\kappa}_{N,s}\epsilon(x_{N,a}, x_{N,b}) &= \frac{1}{4} \sum_{j=0}^{J-1} C_{2j+1}^{(1/2)} \left(\frac{x_{N,b}}{2} \right) \left[K_{j,2} C_{2j+1}^{(3/2)} \left(\frac{x_{N,a}}{2} \right) - K_{j,1} C_{2j-1}^{(3/2)} \left(\frac{x_{N,a}}{2} \right) \right] - \\ &\quad - \frac{1}{4} \sum_{j=0}^{J-1} C_{2j}^{(3/2)} \left(\frac{x_{N,a}}{2} \right) \left[K_{j,2} C_{2j+2}^{(1/2)} \left(\frac{x_{N,b}}{2} \right) - K_{j,1} C_{2j}^{(1/2)} \left(\frac{x_{N,b}}{2} \right) \right] + 2 \sum_{j=0}^{J-1} \Delta_j(s) \pi_{2j}(x_{N,a}) \end{aligned}$$

by Theorem 2.1, (1.11), Lemma 3.12, and (3.26). Denote the three sums above by $S_{N,1}$, $S_{N,2}$, and $S_{N,3}$. Then it follows from Lemma 3.10 that

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{S_{N,1}}{N\omega(x)} &= \mp \frac{i\omega(x)}{4\pi} \int_0^1 (1 - (\lambda t)^2) (\Phi_{\pm}(x) - \Phi_{\mp}(x)) (e^{\mp i(b-a)t} + e^{\pm i(b-a)t}) dt \\ &= \frac{1}{2\pi} \int_0^1 (1 - (\lambda t)^2) \cos((b-a)t) dt, \end{aligned}$$

where one needs to observe that $\Phi_{\pm}(x)\Phi_{\mp}(x) \equiv 1$ and $\Phi_{\pm}(x) - \Phi_{\mp}(x) = \pm i/\omega(x)$. Similarly, one can check that $S_{N,2}$ has the same limit and therefore (2.12) follows from Lemma 3.13.

Since $\epsilon\kappa_{N,s}\epsilon(x, y)$ is anti-symmetric (in particular, zero on the diagonal) and $(\epsilon f)'(x) = -f(x)$ for x real, we have that $\epsilon\kappa_{N,s}\epsilon(x, y) = \int_x^y \kappa_{N,s}\epsilon(u, y) du$ for $x, y \in (-2, 2)$. Hence, it holds that

$$\epsilon\kappa_{N,s}\epsilon(x_{N,a}, x_{N,b}) = \int_a^b \frac{1}{N\omega(x)} \kappa_{N,s}\epsilon(x_{N,u}, x_{N,b}) du$$

and therefore (2.13) easily follows from (2.12). \square

3.8 Proof of Proposition 2.8

Let $y_{N,a} := 2 - (a/N)^2$. Since we take the principal branch of the square root, it holds that

$$\sqrt{y_{N,a}^2 - 4} = \frac{2}{N} \sqrt{-a^2(1 - (a/2N)^2)} = \mp \frac{2ia}{N} (1 + \mathcal{O}(N^{-2})), \quad \pm \operatorname{im}(a) \geq 0, \quad (3.27)$$

locally uniformly in \mathbb{C} . Hence,

$$\Phi(y_{N,a}) = 1 \mp \frac{ia}{N} + \mathcal{O}(N^{-2}) \quad \text{and} \quad \Phi^{-1}(y_{N,a}) = 1 \pm \frac{ia}{N} + \mathcal{O}(N^{-2}) \quad (3.28)$$

for $\pm \operatorname{im}(a) \geq 0$ uniformly on compact subsets of \mathbb{C} , from which the desired claim easily follows.

3.9 Proof of Theorem 2.9

Assume first that $\text{im}(a)\text{im}(b) \neq 0$. It follows from (3.27) that

$$\lim_{N \rightarrow \infty} N^{-2} (y_{N,a}^2 - 4)^{-1/2} \overline{(y_{N,b}^2 - 4)^{-1/2}} = \frac{\pm 1}{4ab}, \quad \pm \text{im}(a)\text{im}(b) > 0, \quad (3.29)$$

where $y_{N,a} = 2 - (a/N)^2$. Hence, we need to compute the limit of (3.18) with $x_{N,a}$ and $x_{N,b}$ replaced by $y_{N,a}$ and $y_{N,b}$. To this end observe that (3.28) yields that

$$\begin{cases} (\Phi(y_{N,a}) \overline{\Phi(y_{N,b})})^{n+1} + (\Phi^{-1}(y_{N,a}) \overline{\Phi^{-1}(y_{N,b})})^{n+1} = \left(1 + \frac{\tau_1 + o(1)}{N}\right)^{n+1} + \left(1 - \frac{\tau_1 + o(1)}{N}\right)^{n+1} \\ (\Phi^{-1}(y_{N,a}) \overline{\Phi(y_{N,b})})^{n+1} + (\Phi(y_{N,a}) \overline{\Phi^{-1}(y_{N,b})})^{n+1} = \left(1 + \frac{\tau_2 + o(1)}{N}\right)^{n+1} + \left(1 - \frac{\tau_2 + o(1)}{N}\right)^{n+1} \end{cases}$$

$\tau_1 := i(a \mp \bar{b})$, $\tau_2 := i(a \pm \bar{b})$ when $\pm \text{im}(a)\text{im}(b) > 0$, where $o(1)$ holds locally uniformly on \mathbb{C}^2 . Hence, we get by (3.20) that the desired limit is equal to

$$\frac{1}{\pi} \int_0^1 (1 - (\lambda t)^2) [\cos(-i\tau_1 t) - \cos(-i\tau_2 t)] dt = \pm \frac{2}{\pi} \int_0^1 (1 - (\lambda t)^2) \sin(at) \sin(\bar{b}t) dt, \quad (3.30)$$

for $\pm \text{im}(a)\text{im}(b) > 0$. Combining (3.29) and (3.30), we get (2.14). When either $\text{im}(a) = 0$ or $\text{im}(b) = 0$, (2.14) can be deduced similarly.

3.10 Proof of Theorem 2.10

To prove Theorem 2.10, it will be convenient to set $\tilde{J}_\nu(z) := (2/z)^\nu J_\nu(z)$, which is always an entire function. It follows from [5, Eq. (10.9.4)] that

$$\tilde{J}_\nu(z) = \frac{1}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_{-1}^1 e^{izt} (1 - t^2)^{\nu-1/2} dt. \quad (3.31)$$

Lemma 3.15. *Let $2(\alpha_1 + \alpha_2) \in \mathbb{N}$, $j_1, j_2 \in \mathbb{N}$ be fixed, and $p(\cdot)$ be a monic polynomial of degree d . Then*

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N^{d+2\alpha_1+2\alpha_2+1}} K_N^{\alpha_1, \alpha_2, d} \left(1 - \frac{a^2}{2N^2}, 1 - \frac{b^2}{2N^2}\right) &= \\ &= \frac{1}{2} \frac{\Gamma(\alpha_1 + 1)}{\Gamma(2\alpha_1 + 1)} \frac{\Gamma(\alpha_2 + 1)}{\Gamma(2\alpha_2 + 1)} \int_0^1 t^{d+2(\alpha_1+\alpha_2)} \tilde{J}_{\alpha_1}(at) \tilde{J}_{\alpha_2}(bt) dt \end{aligned} \quad (3.32)$$

uniformly for a, b on compact subsets of \mathbb{C} , where $N = 2J$ and

$$K_N^{\alpha_1, \alpha_2, d}(z, w) := \sum_{j=0}^{J-1} p(2j) C_{2j+j_1}^{(\alpha_1+\frac{1}{2})}(z) C_{2j+j_2}^{(\alpha_2+\frac{1}{2})}(w).$$

Proof. It follows from [5, Eq. (18.10.4) and Table 18.6.1] that

$$C_n^{(\alpha+\frac{1}{2})}(z) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})\Gamma(2\alpha+1)} \frac{\Gamma(n+2\alpha+1)}{\Gamma(n+1)} \int_{-1}^1 (z + v\sqrt{z^2-1})^n d\mu_\alpha(v), \quad (3.33)$$

for any determination of the square root, where $d\mu_\alpha(v) := (1-v^2)^{\alpha-\frac{1}{2}} dv$. If $z = 1 - a^2/2N^2$ and $w = 1 - b^2/2N^2$, then

$$(z + v\sqrt{z^2-1})^2 (w + u\sqrt{w^2-1})^2 = 1 + i \frac{av + bu + o(1)}{J},$$

where $o(1)$ is uniform with respect to a, b on compact subsets of \mathbb{C} and $t, u \in [-1, 1]$, and we do not keep track of the determination of the square roots as it is not important in (3.33), (observe also that the substitutions $a \mapsto -a$ and $b \mapsto -b$ do not change either side of (3.32)). As

$$\frac{1}{N^{2(\alpha_1+\alpha_2)}} \frac{\Gamma(2j+j_1+2\alpha_1+1)}{\Gamma(2j+j_1+1)} \frac{\Gamma(2j+j_2+2\alpha_2+1)}{\Gamma(2j+j_2+1)} = (1+o(1)) \frac{(j+1)^{2(\alpha_1+\alpha_2)}}{J^{2(\alpha_1+\alpha_2)}}$$

uniformly for all $j \in \{0, \dots, J-1\}$, the limit of the left-hand side of (3.32) is the same as the limit of

$$\frac{C}{\pi\Gamma(\alpha_1+\frac{1}{2})\Gamma(\alpha_2+\frac{1}{2})} \int_{-1}^1 \int_{-1}^1 \sum_{j=0}^{J-1} \frac{q(j)}{J^{d+2(\alpha_1+\alpha_2)+1}} \left(1 + i \frac{av+bu}{J}\right)^j d\mu_{\alpha_1}(v) d\mu_{\alpha_2}(u), \quad (3.34)$$

where $C = \frac{1}{2} \frac{\Gamma(\alpha_1+1)}{\Gamma(2\alpha_1+1)} \frac{\Gamma(\alpha_2+1)}{\Gamma(2\alpha_2+1)}$ and $q(\cdot)$ is a monic polynomial of degree $d+2(\alpha_1+\alpha_2)$. Then it follows from Lemma 3.8 that the limit of (3.34) is equal to

$$\frac{C}{\pi\Gamma(\alpha_1+\frac{1}{2})\Gamma(\alpha_2+\frac{1}{2})} \int_{-1}^1 \int_{-1}^1 \int_0^1 t^{d+2(\alpha_1+\alpha_2)} e^{i(av+bu)t} dt d\mu_{\alpha_1}(v) d\mu_{\alpha_2}(u).$$

The claim of the lemma now follows from (3.31). \square

Lemma 3.16. *Under the conditions of Theorem 2.10, (2.15) holds.*

Proof. It follows from Theorems 1.2 and 2.1 that we need to evaluate the limit of

$$\begin{aligned} \frac{1}{4N^4} \sum_{j=0}^{J-1} \left(1 - \frac{(2j+1)^2}{s^2}\right) \left(2j + \frac{3}{2}\right) C_{2j}^{(3/2)} \left(1 - \frac{a^2}{2N^2}\right) C_{2j+1}^{(1/2)} \left(1 - \frac{b^2}{2N^2}\right) - \\ - \frac{1}{4s^2N^4} \sum_{j=0}^{J-1} \left(2j + \frac{3}{2}\right) C_{2j}^{(3/2)} \left(1 - \frac{a^2}{2N^2}\right) C_{2j+1}^{(3/2)} \left(1 - \frac{b^2}{2N^2}\right), \end{aligned}$$

which is equal to

$$\frac{1}{16} \int_0^1 t^3 (1 - (\lambda t)^2) \tilde{J}_1(at) \tilde{J}_0(bt) dt - \frac{\lambda^2}{32} \int_0^1 t^5 \tilde{J}_1(at) \tilde{J}_1(bt) dt$$

by (3.32). By swapping the the roles of a and b , we get that the limit of the left-hand side of (2.15) is equal to

$$\frac{1}{16} \int_0^1 t^3 (1 - (\lambda t)^2) (\tilde{J}_1(at) \tilde{J}_0(bt) - (\tilde{J}_0(at) \tilde{J}_1(bt))) dt.$$

The desired result now follows from the identities $J_1(z) = (z/2)\tilde{J}_1(z)$ and $J_0(z) = \tilde{J}_0(z)$. \square

Lemma 3.17. *It holds that*

$$\begin{cases} \epsilon(\phi\pi_{2n+1})(x) &= -\int_2^x \pi_{2n+1}(u)\phi(u)du + \frac{2}{s^2}, \\ \epsilon(\phi\pi_{2n})(x) &= -\int_2^x \pi_{2n}(u)\phi(u)du - \frac{4n+3}{8}. \end{cases}$$

Proof. The first relation in the proof of Lemma 3.12 gives us

$$\begin{cases} \epsilon(\phi\pi_{2n+1})(x) &= -\int_2^x \pi_{2n+1}(u)\phi(u)du + \int_2^\infty \pi_{2n+1}(u)\phi(u)du, \\ \epsilon(\phi\pi_{2n})(x) &= -\int_2^x \pi_{2n}(u)\phi(u)du - \int_0^2 \pi_{2n}(u)\phi(u)du. \end{cases}$$

The claim of the lemma now follows from (3.1), (3.3), and (3.9). \square

Lemma 3.18. *Under conditions of Theorem 2.10, it holds locally uniformly for $u \in \mathbb{C}$ that*

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{j=0}^{J-1} \left(\frac{4n+3}{4} \pi_{2j+1} + \frac{4}{s^2} \pi_{2j} \right) \left(2 - \frac{u^2}{N^2} \right) = \frac{1}{4} \int_0^1 t(1 - (\lambda t)^2) J_0(ut) dt.$$

Proof. Repeating the proof of Lemma 3.15, we can show the following. Let $2\alpha \in \mathbb{N}$, $m \in \mathbb{N}$ be fixed, and $p(\cdot)$ be a monic polynomial of degree d . Then

$$\lim_{N \rightarrow \infty} \frac{1}{N^{d+2\alpha+1}} \sum_{j=0}^{J-1} p(2j) C_{2j+m}^{(\alpha+\frac{1}{2})} \left(1 - \frac{u^2}{2N^2} \right) = \frac{1}{2} \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} \int_0^1 t^{d+2\alpha} \tilde{J}_\alpha(ut) dt \quad (3.35)$$

locally uniformly for $u \in \mathbb{C}$. Now, it follow from Theorem 2.1 that $\frac{4n+3}{4} \pi_{2n+1}(z) + \frac{4}{s^2} \pi_{2n}(z)$ is equal to

$$\frac{2j+3/2}{2s^2} \left[C_{2j}^{(3/2)} \left(\frac{z}{2} \right) + (s^2 - (2n+1)^2) C_{2j+1}^{(1/2)} \left(\frac{z}{2} \right) - C_{2j+1}^{(3/2)} \left(\frac{z}{2} \right) \right].$$

Clearly, the claim of the lemma is a consequence of (3.35) as $\tilde{J}_0(z) = J_0(z)$. \square

Lemma 3.19. *Under the conditions of Theorem 2.10, limits (2.16) and (2.17) hold.*

Proof. It follows from (1.11) and Lemma 3.17 that

$$\kappa_{N,s}\epsilon(z, y) = -\int_2^y \kappa_{N,s}(z, u) du + \phi(z) \sum_{j=0}^{J-1} \left(\frac{4n+3}{4} \pi_{2j+1} + \frac{4}{s^2} \pi_{2j} \right) (z).$$

for $y \in \mathbb{R}$. Hence, Proposition 2.8, (2.15), Lemma 3.18, and the change of variable $u \mapsto 2 - \frac{u^2}{N^2}$ imply that

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \widetilde{\kappa_{N,s}\epsilon}(y_{N,a}, y_{N,b}) = \frac{1}{4} \int_0^1 t(1 - (\lambda t)^2) \left[\frac{1}{a} \int_0^b e^{-|\operatorname{im}(u)|/\lambda} \mathbb{J}_{1,1}(at, ut) du + J_0(at) \right] dt$$

uniformly for $a \in \mathbb{C}$ and $b^2 \in \mathbb{R}$, where $y_{N,a} = 2 - a^2/N^2$. When $b \in \mathbb{R}$, the expression in square parenthesis becomes

$$\frac{1}{a} J_1(at) \int_0^b ut J_0(ut) du - t J_0(at) \int_0^b J_1(ut) du + J_0(at) = \frac{b}{a} J_1(at) J_1(bt) + J_0(at) J_0(bt)$$

as $J_0'(z) = -J_1(z)$ and $(zJ_1(z))' = zJ_0(z)$, from which (2.16) is immediate. Similarly, we get from Lemma 3.17 that

$$\epsilon\kappa_{N,s}\epsilon(x, y) = \int_2^x \int_2^y \kappa_{N,s}(u, v) dv du + \left(\int_0^y - \int_0^x \right) \phi(u) \sum_{n=0}^{J-1} \left(\frac{4n+3}{4} \pi_{2n+1} + \frac{4}{s^2} \pi_{2n} \right) (u) du.$$

The same change of variables allows us to show that the limit of $\epsilon\kappa_{N,s}\epsilon(y_{N,a}, y_{N,b})$ is equal to

$$\frac{1}{2} \int_0^1 t(1-(\lambda t)^2) \left[\int_0^a \int_0^b e^{-\frac{|\operatorname{im}(u)|+|\operatorname{im}(v)|}{\lambda}} \mathbb{J}_{1,1}(ut, vt) dv du - \left(\int_0^b - \int_0^a \right) e^{-\frac{|\operatorname{im}(u)|}{\lambda}} u J_0(ut) du \right] dt.$$

When a, b are real and therefore the integrals are evaluated along the real axis, we get that

$$\int_0^a \int_0^b \mathbb{J}_{1,1}(ut, vt) dv du + \left(\int_0^b - \int_0^a \right) u J_0(ut) du = \frac{aJ_1(at)J_0(bt) - bJ_1(bt)J_0(at)}{t},$$

which proves (2.17). \square

3.11 Proof of Theorem 2.11

Lemma 3.20. *Equation (2.19) holds.*

Proof. It follows from (2.18) and Theorem 1.2 that

$$E[N_{\text{in}}] = \sum_{n=0}^{J-1} \int_{-2}^2 2[(\pi_{2n}\phi)(x)\epsilon(\pi_{2n+1}\phi)(x) - (\pi_{2n+1}\phi)(x)\epsilon(\pi_{2n}\phi)(x)] dx.$$

Since $(\epsilon f)'(x) = -f(x)$ for x real, we can rewrite the above equality as

$$E[N_{\text{in}}] = -2 \sum_{n=0}^{J-1} \epsilon(\pi_{2n}\phi)(x)\epsilon(\pi_{2n+1}\phi)(x)|_{-2}^2 - 4 \sum_{n=0}^{J-1} \int_{-2}^2 (\pi_{2n+1}\phi)(x)\epsilon(\pi_{2n}\phi)(x) dx.$$

Since $\epsilon(\pi_{2n}\phi)(-2) = -\epsilon(\pi_{2n}\phi)(2)$ and $\epsilon(\pi_{2n+1}\phi)(-2) = \epsilon(\pi_{2n+1}\phi)(2)$ by Lemma 3.12, it follows from Lemma 3.17 that the first sum above is equal to $J(2J+1)/s^2$. Set

$$I_n := -4 \int_{-2}^2 (\pi_{2n+1}\phi)(x)\epsilon(\pi_{2n}\phi)(x) dx = 8 \int_0^2 \pi_{2n+1}(x) \int_0^x \pi_{2n}(u) du dx,$$

where the second equality holds by Lemma 3.12 and since $\phi(x) \equiv 1$ on $[-2, 2]$ (notice that the integrand in the first integral is an even function by Lemma 3.12). Thus, it follows from Lemma 3.1 and (3.3) that

$$I_n = \frac{4n+3}{\pi} \sum_{i=0}^n \frac{\Gamma(n-i+\frac{1}{2})}{\Gamma(n-i+1)} \frac{\Gamma(n+i+\frac{3}{2})}{\Gamma(n+i+2)} \int_0^2 \pi_{2n+1}(x) T_{2i+1}(x) dx.$$

Using notation (3.6) and Lemma 3.1 once more, this time with (3.4), we obtain that

$$I_n = -\frac{4n+3}{2\pi^2} \sum_{j=0}^n \left(\sum_{i=0}^n \frac{4\Gamma_{2n,i}}{(2j+2)^2 - (2i+1)^2} \right) (2j+2)^2 \left(1 - \frac{(2j+2)^2}{s^2} \right) \Gamma_{2n+1,j}.$$

Further, we deduce from (3.7) applied with $a = j+1$ that the inner sum is zero for all $j < n$ and for $j = n$ it is equal to $\pi^{3/2}\Gamma(2n+2)/(2n+2)\Gamma(2n+5/2)$. Hence, $I_n = 2(1-(2n+2)^2/s^2)$ and therefore

$$E[N_{\text{in}}] = \frac{N(N+1)}{2s^2} + \sum_{n=0}^{J-1} I_n = \frac{N(N+1)}{2s^2} + N - \frac{N(N+1)(N+2)}{3s^2}$$

from which the claim of the lemma follows. \square

Lemma 3.21. *Equation (2.20) holds.*

Proof. As in Lemma 3.20, we have that

$$E[N_{\text{out}}] = \sum_{n=0}^{J-1} \int_2^{\infty} 4[(\pi_{2n}\phi)(x)\epsilon(\pi_{2n+1}\phi)(x) - (\pi_{2n+1}\phi)(x)\epsilon(\pi_{2n}\phi)(x)] dx$$

where we used Lemma 3.12 to deduce that the integrand is an even function. It follows from the integration by parts, Lemma 3.5, and Lemma 3.17 that

$$E[N_{\text{out}}] = -\frac{N(N+1)}{2s^2} + \sum_{n=0}^{J-1} 8 \int_2^{\infty} \int_0^x (\pi_{2n+1}\phi)(x)(\pi_{2n}\phi)(u) du dx.$$

We get from (3.1), (2.3), (3.8), and (3.9) that

$$\begin{aligned} \frac{1}{s+m} \int_2^{\infty} \frac{\pi_{2n+1}(x)}{\Phi^{2s+m}(x)} dx &= -\frac{1}{\pi} \sum_{i=0}^n \frac{(2i+2)^2}{s^2(s+m)} \frac{s^2 - (2i+2)^2}{(2s+m)^2 - (2i+2)^2} \Gamma_{2n+1,i} \\ &= \frac{2}{s^2(s+m)} + \frac{s^2 - (2s+m)^2}{\pi s^2(s+m)} \sum_{i=0}^n \frac{(2i+2)^2 \Gamma_{2n+1,i}}{(2i+2)^2 - (2s+m)^2} \\ &= \frac{2}{s^2(s+m)} - \frac{(3s+m)(2s+m)}{4s^2} \frac{\Gamma(s+n+\frac{m+1}{2})}{\Gamma(s+n+\frac{m+4}{2})} \frac{\Gamma(s-n+\frac{m-2}{2})}{\Gamma(s-n+\frac{m+1}{2})}. \end{aligned}$$

Recall that $\int_2^{\infty} \pi_{2n+1}\phi = 2/s^2$, see Lemma 3.17. Denoting by $\Gamma_{\frac{m+1}{2}}^{s,n}$ the last summand in the equation above, we deduce from (3.3) that

$$8 \int_2^{\infty} (\pi_{2n+1}\phi)(x) \int_0^x (U_{2i}\phi)(u) du dx = \frac{32}{(2i+1)s^2} + 8(\Gamma_{-i}^{s,n} - \Gamma_{i+1}^{s,n}).$$

Then it follows from Lemma 3.1 and (3.9) that

$$E[N_{\text{out}}] = \frac{N(N+1)}{2s^2} + \frac{1}{\pi} \sum_{n=0}^{J-1} (4n+3) \sum_{i=0}^n (2i+1) \Gamma_{2n,i} (\Gamma_{-i}^{s,n} - \Gamma_{i+1}^{s,n}).$$

By the Gautschi's inequality for the ratio of Gamma functions [5, Eq. (5.6.4)] and straightforward estimates using the fact that $n \leq J-1 < s/2$, it holds that $\Gamma_{i+1}^{s,n} \lesssim 1/s^3$, where \lesssim means \leq with an absolute constant. Therefore, (3.9) and simple upper bounds yield that

$$E[N_{\text{out}}] = \mathcal{O}(1) + \frac{1}{\pi} \sum_{n=0}^{J-1} (4n+3) \sum_{i=0}^{n-1} (2i+1) \Gamma_{2n,i} \Gamma_{-i}^{s,n}, \quad (3.36)$$

where similar estimates allowed us to change the index of summation from n to $n-1$ in the inner sum. Using Gautschi's inequality and straightforward estimates once more, one can verify that $\Gamma_{-i}^{s,n} \sim (s(s-n-i))^{-3/2}$ and that $\Gamma_{2n,i} \sim (n(n-i))^{-1/2}$ when $0 \leq i \leq n-1$. Observe further that

$$\sum_{i=1}^n \frac{1}{\sqrt{i}(s-2n+i)^{3/2}} \sim \int_0^n \frac{dx}{\sqrt{x}(s-2n+x)^{3/2}} = \frac{2}{s-2n} \sqrt{\frac{x}{s-2n+x}} \Big|_0^n \sim \frac{1}{s-2n} \sqrt{\frac{n}{s}}.$$

Thus, the inner sum in (3.36) then can be estimated as

$$\frac{1}{s^{3/2}} \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{2(n-i)+1}{\sqrt{i}(s-2n+i)^{3/2}} \sim \frac{n}{s^2} \frac{1}{s-2n}.$$

After plugging the above expression into (3.36), (2.20) easily follows. \square

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