SZEGŐ-TYPE ASYMPTOTICS FOR RAY SEQUENCES OF FROBENIUS-PADÉ APPROXIMANTS

ALEXANDER I. APTEKAREV, ALEXEY I. BOGOLUBSKY, AND MAXIM L. YATSELEV

Abstract. Let \( \hat{\sigma} \) be a Cauchy transform of a possibly complex-valued Borel measure \( \sigma \) and \( \{p_n\} \) be a system of orthonormal polynomials with respect to a measure \( \mu \), \( \text{supp}(\mu) \cap \text{supp}(\sigma) = \emptyset \). An \((m, n)\)th Frobenius-Padé approximant to \( \hat{\sigma} \) is a rational function \( P/Q \), \( \deg(P) \leq m, \deg(Q) \leq n \), such that the first \( m+n+1 \) Fourier coefficients of the linear form \( Q\hat{\sigma} - P \) vanish when the form is developed into a series with respect to the polynomials \( p_n \). We investigate the convergence of the Frobenius-Padé approximants to \( \hat{\sigma} \) along ray sequences \( \frac{n}{n+m+1} \to c > 0, n-1 \leq m \), when \( \mu \) and \( \sigma \) are supported on intervals on the real line and their Radon-Nikodym derivatives with respect to the arcsine distribution of the respective interval are holomorphic functions.

1 Introduction

Representation of functions by means of series with respect to the Chebyshev polynomials is a very convenient tool in numerical analysis (see, for example, [1]). Such a series converges in the interior of the largest ellipse into which the function has holomorphic continuation. However, if we need to compute the function beyond the boundary of the maximal ellipse of convergence of the series with respect to the Chebyshev polynomials (or any other orthonormal polynomial sequences) then one has to employ rational rather than polynomial approximation of the orthogonal polynomial expansion (see [2], [3]). The construction of these rational approximants is related to the notion of the generalized Padé table [4]. We call them Padé approximants of an orthogonal expansion. In this paper we focus on the Frobenius-Padé approximants, which are defined by means of a linear system with constant coefficients which are precisely the coefficients of the polynomial expansion of the approximated function (see (2), below). This type of approximants is the most popular in practice due to the ease of their numerical computation.

Let \( \mu \) be a possibly complex-valued Borel measure supported on an interval \( \Delta_\mu \subset \mathbb{R} \). Assume that \( \mu \) possesses full orthonormal system of polynomials that we denote by \( \{p_n\} \), i.e.,

\[
\int p_n(x)p_m(x)d\mu(x) = \delta_{n,m},
\]

where \( \delta_{n,m} \) is the usual Kronecker symbol. When \( \mu \) is a positive measure such a system always exists. For complex measures orthonormal polynomials of minimal degree uniquely exist as well, but it might happen that \( \deg(p_n) < n \), in which case \( p_n \) is orthogonal to itself and therefore cannot be orthonormalized. Given a function \( f \in L^1(\mu) \), we can associate to \( f \) a series

\[
\sum_{i=0}^{\infty} c_i(f;\mu)p_i(x), \quad c_i(f;\mu) := \int f(x)p_i(x)d\mu(x).
\]
Definition. A Frobenius-Padé approximant of type \((m, n)\) in \(\mathbb{N}^2\) to \(f \in L^1(\mu)\) is a rational function \(P_{m,n}/Q_{m,n}\), \(\deg(P_{m,n}) \leq m\), \(\deg(Q_{m,n}) \leq n\), such that
\[
(2) \quad c_i(Q_{m,n}f - P_{m,n}; \mu) = 0, \quad i \in \{0, \ldots, m + n\}.
\]

Frobenius-Padé approximants always exist as finding \(Q_{m,n}\) amounts to solving a linear system
\[
\begin{pmatrix}
c_{m+1}(p_0 f; \mu) & \cdots & c_{m+1}(p_n f; \mu) \\
n_{m+1}(p_0 f; \mu) & \cdots & n_{m+1}(p_n f; \mu)
\end{pmatrix}
\begin{pmatrix}
a_0 \\
a_n
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]
and letting \(Q_{m,n}(x) = \sum_{j=0}^{n} a_j p_j(x)\) (the system has \(n\) equations and \(n + 1\) unknowns), while \(P_{m,n}(x) = \sum_{j=0}^{m} b_j p_j(x)\) uniquely depends on \(Q_{m,n}\) via
\[
\begin{pmatrix}
c_0(p_0 f; \mu) & \cdots & c_n(p_n f; \mu) \\
n_0(p_0 f; \mu) & \cdots & n_n(p_n f; \mu)
\end{pmatrix}
\begin{pmatrix}
a_0 \\
a_n
\end{pmatrix}
= \begin{pmatrix}
b_0 \\
b_m
\end{pmatrix}.
\]

An approximant may not be unique, however, the one corresponding to \(Q_{m,n}\) of the smallest degree is. Hence, if \(\deg(Q_{m,n}) = n\) for all solutions, the approximant is unique.

The main motivation for using Padé approximants of orthogonal expansions is due to their convergence in wider domains than the convergence domains of orthogonal expansions themselves. The problems of convergence of the rows of corresponding tables of the Padé approximants of orthogonal expansions have been investigated by S.P. Suetin [5], [6]. The weak asymptotics and the convergence of the diagonal (i.e. type \((n - 1, n)\)) Padé approximants of orthogonal expansions for Cauchy transforms
\[
(3) \quad \hat{\sigma}(z) := \int \frac{d\sigma(t)}{1 - z}, \quad \sigma'(t) > 0, \quad t \in \Delta_\sigma \subset \mathbb{R}, \quad \Delta_\mu \cap \Delta_\sigma = \emptyset,
\]
have been obtained A.A. Gonchar, E.A. Rakhmanov and S.P. Suetin in [7], [8].

In this paper we investigate the strong asymptotics and convergence properties of the ray sequences (i.e. type \((m, n) : n - 1 \leq m\) and \(n/(n + m) \to c > 0\)) of Frobenius-Padé approximants for Cauchy transforms \((3)\) where \(\sigma\) is, generally speaking, a complex-valued Borel measure. To motivate the forthcoming definitions, let us (following \([7], [8]\)) first heuristically describe the asymptotic behavior of the approximants using the formalism of orthogonal polynomials and potential theory.

For the moment, assume that the measures of \(\mu\) and \(\sigma\) are positive. In this case the linear form
\[
R_{m,n} := Q_{m,n} \sigma - P_{m,n}
\]
is real-valued on \(\Delta_\mu\) and is orthogonal to all polynomials of degree at most \(m + n\) with respect to \(\mu\) by \((2)\). Therefore it must have at least \(m + n + 1\) zeros there. Denote by \(V_{m,n}\) the monic polynomial whose zeros are the zeros of \(R_{m,n}\) on \(\Delta_\mu\). \(\deg(V_{m,n}) \geq m + n + 1\). The expression \(z^k R_{m,n}(z)/V_{m,n}(z)\), \(k \leq \min(n - 1, m)\), is holomorphic off \(\Delta_\sigma\) and is vanishing at infinity with order at least 2. Then it follows from Cauchy’s theorem, Cauchy’s integral formula, and \((3)\) that
\[
(4) \quad \int \frac{x^k Q_{m,n}(x)}{V_{m,n}(x)} d\sigma(x) = 0, \quad k \leq \min(n - 1, m).
\]
When \(n - 1 \leq m\), the number of orthogonality conditions above is equal to \(n\) and therefore \(Q_{m,n}\) must have degree \(n\) since \(d\sigma(x)/V_{m,n}(x)\) is a real measure of constant sign on \(\Delta_\sigma\). In particular, this implies uniqueness of \(Q_{m,n}\) up to a multiplicative factor. On the other hand, similarly to \((4)\), Cauchy integral formula, \((3)\), and orthogonality of \(R_{m,n}\) with respect to \(\mu\) yield that
\[
(5) \quad \int \frac{x^k V_{m,n}(x)}{Q_{m,n}(x)} \left( \int \frac{Q_{m,n}^2(t) d\sigma(t)}{V_{m,n}(t) t - x} \right) d\mu(x) = 0, \quad k \in \{0, \ldots, m + n\}.
\]
Given mutual orthogonality relations (4) and (5), it is well understood [9, 10, 11] which measures describe the limiting behavior of the zeros of \( Q_{m,n} \) and \( V_{m,n} \). Assuming that \( n - 1 \leq m \) and \( n/(n + m) \rightarrow c > 0 \), let \( \tau_{\mu,c}, |\tau_{\mu,c}| = 1 \), and \( \tau_{\sigma,c}, |\tau_{\sigma,c}| = e \), be weak^* limit points of the counting measures of the zeros of \( V_{m,n} \) and \( Q_{m,n} \), respectively, normalized by \( n + m \). Then the pair \((\tau_{\mu,c}, \tau_{\sigma,c})\) can be uniquely identified as follows [10, 11].

**Proposition 1.** Given \( c \in (0,1/2) \), denote by \( \mathcal{M}_c \) the following class of pairs of Borel measures:

\[
\mathcal{M}_c := \{ (\tau_{\mu}, \tau_{\sigma}) : \supp(\tau_{\nu}) \subseteq \Delta_\nu, \ \nu \in \{ \mu, \sigma \}, |\tau_{\mu}| = 1, |\tau_{\sigma}| = c \}.
\]

There exists a pair \((\tau_{\mu,c}, \tau_{\sigma,c}) \in \mathcal{M}_c \) such that\(^1\)

\[
\begin{align*}
2V^{\tau_{\sigma,c}} - V^{\tau_{\mu,c}} &= \min_{\Delta_\nu}(2V^{\tau_{\sigma,c}} - V^{\tau_{\mu,c}}) = 3\ell_{\sigma,c} \text{ on } \supp(\tau_{\sigma,c}), \\
2V^{\tau_{\mu,c}} - V^{\tau_{\sigma,c}} &= \min_{\Delta_\nu}(2V^{\tau_{\mu,c}} - V^{\tau_{\sigma,c}}) = 3\ell_{\mu,c} \text{ on } \supp(\tau_{\mu,c}),
\end{align*}
\]

for some constants \( \ell_{\mu,c} \) and \( \ell_{\sigma,c} \). Moreover, if for some pair \((\tau_{\mu}, \tau_{\sigma}) \in \mathcal{M}_c \) relations analogous to (6) are satisfied, then \((\tau_{\mu}, \tau_{\sigma}) = (\tau_{\mu,c}, \tau_{\sigma,c})\). In addition, it holds that

\[\supp(\tau_{\mu,c}) = \Delta_\mu \text{ and } \supp(\tau_{\sigma,c}) = : \Delta_{\sigma,c} \text{ is an interval.}\]

Furthermore, set

\[
\begin{align*}
D^+_c := \{ z : V^{\tau_{\mu,c}}(z) - 2V^{\tau_{\sigma,c}}(z) + 3\ell_{\sigma,c} > 0 \}, \\
D^-_c := \{ z : V^{\tau_{\mu,c}}(z) - 2V^{\tau_{\sigma,c}}(z) + 3\ell_{\sigma,c} < 0 \}.
\end{align*}
\]

Then \( D^+_c \neq \emptyset \) and \( \Delta_{\sigma,c} \subseteq \partial D^+_c \), \( D_{\sigma,c} \subseteq D^-_{\sigma,c} \) when \( c_1 < c_2 \), \( D^-_{\sigma,c} = \emptyset \) when \( c = c_2 \) and \( \infty \in D^-_{\sigma,c} \neq \emptyset \) otherwise, and \( (\Delta_\sigma \setminus \Delta_{\sigma,c}) \subseteq \partial D^-_{\sigma,c} \), see Figure 1.

The domains \( D^+_c \) and \( D^-_c \) are significant for our analysis as we shall prove that the approximants do converge to \( \hat{\sigma} \) in \( D^+_c \) and diverge to infinity in \( D^-_c \).

As shown in [12], one can describe the weak asymptotics of the polynomials \( Q_{m,n} \) and \( V_{m,n} \) using the logarithmic potentials of the measures \( \tau_{\sigma,c} \) and \( \tau_{\mu,c} \). As we aim at strong (Szegő) asymptotics we shall omit such a description, which was addressed in [8] for the diagonal case \( m = n - 1 \). Let us point out that relations (6) are stated differently in [8]. There, see [8, Equation (2.1)], it shown that there exists a unique probability measure \( \lambda \), \( \supp(\lambda) = \Delta_\mu \), and a constant \( w \) such that

\[
G^\lambda - 3V^\lambda = w \quad \text{on } \Delta_\mu,
\]

where \( G^\lambda \) is the Green potential of \( \lambda \) relative to \( \mathbb{C} \setminus \Delta_\sigma \). To rewrite the above relation as system (6), recall that \( G^\lambda = 0 \) on \( \Delta_\sigma \) and that \( G^\lambda = V^\lambda - \hat{w} \) in \( \mathbb{C} \) where \( \hat{w} \) is some constant and \( \lambda \) is the balayage of \( \lambda \) onto \( \Delta_\sigma \), see [25, Theorem II.5.1]. Therefore,

\[
\begin{align*}
2V^{\lambda/2} - V^{\lambda} &= \hat{w} \quad \text{on } \Delta_\sigma, \\
2V^{\lambda} - 2V^{\lambda/2} &= (w + \hat{w})/2 \quad \text{on } \Delta_\mu.
\end{align*}
\]

The last equations clearly show that \( \tau_{\mu,1/2} = \lambda \) and \( \tau_{\sigma,1/2} = \hat{\lambda}/2 \).

\(^1\)In what follows, \( V^\nu(z) = -\int \log |z - w| d\nu(w) \) is the logarithmic potential of the measure \( \nu \).
2 MAIN RESULTS

After the work of J. Nuttall \cite{13}, it is well understood that in order to identify strong limits of orthogonal polynomials one needs to replace the potential-theoretic extremal problem with a boundary value problem on a certain Riemann surface. To this end, let $c \in (0,1/2]$ and $\Delta_{\sigma,c}$ be as in Proposition 1. We define the Riemann surface corresponding to $c$, say $\mathcal{R}_c$, through its realization in the following way. Take 3 copies of $\mathbb{C}$. Cut one of them along the interval $\Delta_{\sigma,c}$, which henceforth is denoted by $\mathcal{R}_c^{(0)}$, cut the second one, $\mathcal{R}_c^{(1)}$, along $\Delta_{\mu} \cup \Delta_{\sigma,c}$, and the last one, $\mathcal{R}_c^{(2)}$, along $\Delta_{\mu}$. To finish the construction, glue the banks of the corresponding cuts crosswise, see Figure 2.

![Figure 2: Riemann surface $\mathcal{R}_c$ and its branch points $a_\mu, b_\mu, a_{\sigma,c}, b_{\sigma,c}$.](image)

We denote by $\pi$ the natural projection from $\mathcal{R}_c$ to $\mathbb{C}$. We shall employ the notation $z$ for a generic point of $\mathcal{R}_c$ and use the convention $\pi(z) = z$. If we want to specify the sheet of the surface, we write $z^{(i)}$ for a point on $\mathcal{R}_c^{(i)}$ with $\pi(z^{(i)}) = z$. This notation is well defined everywhere outside of the cycles $\Delta_{\mu} := \pi^{-1}(\Delta_{\mu})$ and $\Delta_{\sigma,c} := \pi^{-1}(\Delta_{\sigma,c})$. Given a function $F[z]$ defined on a subset of $\mathcal{R}_c$, we set $F^{(i)}(z) := F(z^{(i)})$ to be the pull-back from the $i$-th sheet.

Among all such surfaces, the ones with $c = \frac{n}{n+m}$ are especially important to us. We shall denote them by $\mathcal{R}_{m,n}$. Observe that any $\mathcal{R}_c$ has genus 0. Thus, one can arbitrarily prescribe zero/pole multisets of rational functions on them as long as the multisets have the same cardinality. In what follows, we denote by $\Phi_{m,n}$ the rational function on $\mathcal{R}_{m,n}$ with the divisor $^2(\sigma_{2m}^{(1)} - \sigma_{2m}^{(0)} - \sigma_{2m}^{(1)})$ and the normalization

\begin{equation}
\Phi_{m,n}(z)\Phi_{m,n}(1)\Phi_{m,n}(z) \equiv 1.
\end{equation}

Such a normalization is indeed possible since the function $\log \prod_{k=0}^2 |\Phi_{m,n}^{(k)}|$ extends to a harmonic function on $\mathbb{C}$ which has a well defined limit at infinity. Hence, it is a constant. Therefore, if (8) holds at one point, it holds throughout $\overline{\mathbb{C}}$. It is a simple argument using Schwarz reflection principle, equilibrium relations (6), and the fact that only bounded harmonic function on $\mathcal{R}_c$ are constants to show that

\begin{equation}
\frac{1}{n+m} \log |\Phi_{m,n}(z)| = \begin{cases} 
V^{-\tau_{\sigma,c}}(z) + \ell_{\mu,c} + 2\ell_{\sigma,c}, & z \in \mathcal{R}_c^{(0)}_{m,n}, \\
V^{\tau_{\sigma,c} - \tau_{\mu,c}}(z) + \ell_{\mu,c} - \ell_{\sigma,c}, & z \in \mathcal{R}_c^{(1)}_{m,n}, \\
V^{\tau_{\mu,c}}(z) - 2\ell_{\mu,c} - \ell_{\sigma,c}, & z \in \mathcal{R}_c^{(2)}_{m,n}, 
\end{cases}
\end{equation}

\footnote{The divisor is a formal expression that describes all the zeros (preceded by positive integer indicated multiplicity) and poles (preceded by negative integer also indicating multiplicity) of the function.}
where \( c = \frac{n}{n+m} \). Representation (9) is not the only way to understand functions \( \Phi_{m,n} \).

Define

\[
h_{m,n}(z) := \begin{cases} 
\int \frac{d\tau_{\mu,c}(x)}{z-x}, & z \in \mathcal{R}_{m,n}^{(0)}, \\
\int \frac{d(\tau_{\mu,c} - \tau_{0,c})(x)}{z-x}, & z \in \mathcal{R}_{m,n}^{(1)}, \\
-\int \frac{d\tau_{\mu,c}(x)}{z-x}, & z \in \mathcal{R}_{m,n}^{(2)},
\end{cases}
\]

(10)

where, again, \( c = \frac{n}{n+m} \). One can readily observe that

\[ h_{m,n}(z) = 2\partial_z \left( \frac{1}{n+m} \log |\Phi_{m,n}(z)| \right) \]

by (9) and (10), where \( 2\partial_z := \partial_x - i\partial_y \). As \( \partial_z \)-derivative of a harmonic function is holomorphic, \( h_{m,n} \) is a rational function on \( \mathcal{R}_{m,n} \). It also follows from the above relation, that \( h_{m,n} \) is the logarithmic derivative of \( \Phi_{m,n} \) and therefore

\[ \Phi_{m,n}(z) = \exp \left\{ (n+m) \int h_{m,n}(x)dx \right\}, \]

(11)

where the initial bound for integration is chosen so (8) holds. Moreover, we also can describe the divisor of \( h_{m,n} \).

**Proposition 2.** Given \( n \leq m \), let \( c = \frac{n}{n+m} \) and \( h_{m,n} \) be defined by (10). Denote the endpoints of \( \Delta_\nu \) by \( a_\nu \) and \( b_\nu \), \( \nu \in (\mu, \sigma) \), and arrange them so that

- either \( b_\mu < a_\mu < a_\sigma < b_\sigma \) or \( b_\sigma < a_\sigma < a_\mu < b_\mu \).

If, using the same convention, we denote the endpoints of \( \Delta_{\sigma,c} \) by \( a_{\sigma,c} \) and \( b_{\sigma,c} \), then \( a_{\sigma,c} = a_\sigma \).

Moreover, the divisor of \( h_{m,n} \) is given by

\[ \infty^{(0)} + \infty^{(1)} + \infty^{(2)} + z_{m,n} - a_\mu - b_\mu - a_\sigma - b_{\sigma,c}, \]

where \( a_\mu, b_\mu, a_\sigma, b_{\sigma,c} \) are the branch points of \( \mathcal{R}_{m,n} \) with the corresponding projections \( a_\mu, b_\mu, a_\sigma, b_{\sigma,c} \), and \( z_{m,n} \in \mathcal{R}_{m,n}^{(1)} \)

\[ \pi(z_{m,n}) \in \begin{cases} 
[b_{\sigma,c}, \infty) & \text{if } a_\sigma < b_\sigma, \\
(-\infty, b_{\sigma,c}) & \text{if } b_\sigma < a_\sigma.
\end{cases} \]

Furthermore, \( z_{m,n} = b_{\sigma,c} \) if and only if \( b_{\sigma,c} \in \partial \mathcal{D}_{\sigma,c} \), see (7) and Figure 1, that is, if and only if the domain \( \mathcal{D}_{\sigma,c} \) touches the interval \( \Delta_{\sigma,c} \) (observe also that \( b_{\sigma,c} = b_\sigma \) if \( b_{\sigma,c} \notin \partial \mathcal{D}_{\sigma,c} \) since \( \Delta_\sigma \setminus \Delta_{\sigma,c} \subset \mathcal{D}_{\sigma,c} \) by Proposition 1).

We prove Proposition 2 in Section 3.1. We clearly see from Proposition 2 that the function \( h_{m,n} \) is algebraic. More precisely, Proposition 2 yields the following.

**Corollary 3.** If \( b_{\sigma,c} \notin \partial \mathcal{D}_{\sigma,c} \), in which case \( b_{\sigma,c} = b_\sigma \), then \( h_{m,n} \) is the solution of the algebraic equation

\[ h^3 - (1 - \varkappa) \frac{P_2(z)}{\Pi(z)} h - \varkappa \frac{P_1(z)}{\Pi(z)} = 0, \quad \varkappa = c - c^2, \]

(13)

where \( c = \frac{n}{n+m} \), \( \Pi(z) = (z-a_\mu)(z-b_\mu)(z-a_\sigma)(z-b_\sigma) \), and the polynomials \( P_j \) are monic and of degree \( j \), \( j = 1, 2 \). The three zeros of the polynomials \( P_1 \) and \( P_2 \) are determined by the three conditions that the discriminant of (13), i.e.,

\[ \frac{1}{\Pi^3(z)} \left[ \left( \frac{1 - \varkappa}{3} P_2(z) \right)^3 - \left( \frac{\varkappa}{2} P_1(z) \right)^2 \Pi(z) \right], \]

has zeros of even multiplicity only and that the Riemann surface of the solution of (13) must be as on Figure 2.
If \( b_{\sigma,c} \in \partial D_{\sigma,c} \), in which case \( z_{m,n} = b_{\sigma,c} \), then \( h_{m,n} \) is the solution of the algebraic equation

\[
(14) \quad h^3 - (1 - \frac{\rho}{z}) \frac{\tilde{p}_1(z)}{\Pi(z)} h - \frac{\rho}{z} \frac{\tilde{p}_1(z)}{\Pi(z)} = 0,
\]

where this time \( \Pi(z) = (z - a_{\mu})(z - b_{\mu})(z - \alpha_{\sigma}) \) and the only zero of the monic polynomial \( \tilde{p}_1 \) is such that the discriminant of \( (14) \), whose numerator is a cubic polynomial, vanishes at \( b_{\sigma,c} \) and has a double zero, and the Riemann surface of \( (14) \) must be as on Figure 2.

If we take \( a_{\mu} = a_{\sigma} = a \) in \( (14) \), then \( \tilde{p}_1(z) = z - a \) and \( (14) \) becomes

\[
(15) \quad h^3 - \frac{(1 - \frac{\rho}{z})}{(z - b_{\mu})(z - a)} h - \frac{\rho}{z} \frac{\tilde{p}_1(z)}{(z - b_{\mu})(z - a)^2} = 0.
\]

The only zero of the discriminant of \( (15) \) is exactly \( b_{\sigma,c} \) and is equal to

\[
(16) \quad b_{\sigma,c} = \frac{(1 - \frac{\rho}{z})^3 a - \left(\frac{\rho}{z}\right)^2 b_{\mu}}{(1 - \frac{\rho}{z})^3 - \left(\frac{\rho}{z}\right)^2}.
\]

Explicit expression \( (15) \) allows us to numerically compute the boundary \( \partial D_{\sigma,c} \), which is the trajectory \( \Re \left[ h_{m,n}^{(0)}(z) - h_{m,n}^{(1)}(z) \right] \) = 0 emanating from \( b_{\sigma,c} \), see Figure 3.

Let us now specify which measures \( \mu \) and \( \sigma \) we consider. We shall assume that

\[
(17) \quad d\nu(x) = \frac{\rho_\nu(x)}{2\pi i} \frac{dx}{w_\nu^+(x)}, \quad \nu \in \{\mu, \sigma\},
\]

where \( \rho_\nu \) is a non-vanishing and holomorphic function in some neighborhood of \( \Delta_\nu \) and

\[
w_\nu(z) := \sqrt{(z - a_\nu)(z - b_\nu)}
\]

is the branch holomorphic in \( \mathbb{C} \setminus \Delta_\mu \) and normalized so that \( w_\nu(z)/z \to 1 \) as \( z \to \infty \). We define \( w_{\sigma,c}(z) \) analogously.

As expected from the classical theory of orthogonal polynomials, we need to introduce an appropriate analog of the Szegő function for the measures \( \mu \) and \( \sigma \). This is precisely the content of Proposition 4 below. Its statement is a direct application of \([14, \text{Proposition } 4]\) with \( \rho_1 = w_\sigma^+/(\rho_\sigma w_\sigma^+)^2 \) and \( \rho_2 = \rho_\mu/(\rho_\mu w_\sigma^+)^2 \) (one needs to notice that the labeling of the sheets \( \Re^{(0)} \) and \( \Re^{(1)} \) is reversed there and the restriction \( \alpha_{ij} > 1 \) in \([14, \text{Eq. (23)}]\) is needed to make \( \rho_1 \) integrable and is not important for \([14, \text{Proposition } 4] \) itself).

**Proposition 4.** There exists a holomorphic and non-vanishing function on \( \Re_c \setminus (\Delta_\mu \cup \Delta_{\sigma,c}) \), say \( S_c \), that has continuous traces on \( \Delta_\mu \cup \Delta_{\sigma,c} \setminus \{a_{\mu}, b_{\mu}, a_{\sigma}, b_{\sigma,c}\} \), satisfies

\[
(18) \quad S_c^{(1)}(x) = \begin{cases} 
S_c^{(0)}(x)/\rho_\sigma w_\sigma^+/w_\sigma^+(x), & x \in \Delta_{\sigma,c}^0, \\
S_c^{(2)}(x)/\rho_\mu w_{\sigma,c}/w_\sigma^+(x), & x \in \Delta_\mu^0,
\end{cases}
\]

![Figure 3](image-url)
where $\Delta^n$ is the interior of the closed interval $\Delta$, is bounded around $a_\mu, b_\mu, a_\sigma$ as well as $b_{\sigma,c}$ when $b_{\sigma,c} = b_\sigma$, and behaves like $|S_c^{(1)}(z)| \sim |S_c^{(0)}(z)|^{-1} \sim |z-b_{\sigma,c}|^{1/4}$ as $z \to b_{\sigma,c} \neq b_\sigma$. Moreover, it holds that $S_c^{(0)}(z)^{1/2} \equiv 1$.

Now we are ready to state our main result.

**Theorem 5.** Let $\mu$ and $\sigma$ be of the form (17) and assume that $\mu$ possesses the full system of orthonormal polynomials. Assume further that $n-1 < m$ and $\frac{n}{m} \to c > 0$ as $n \to \infty$. Then for all $n$ large, $(m,n)$-th Frobenius-Padé approximant $P_m,n/Q_{m,n}$ is unique and $\deg(Q_{m,n}) = n$. Moreover, if $K \subset \mathbb{C} \setminus \Delta_\sigma$ is closed, then

$$Q_{m,n} = [1 + o(1)] \Phi^{(0)}_{m+1,n}S_c^{(0)}$$

on $K$,

$$Q_{m,n} = [1 + o(1)] \Phi^{(0)+}_{m+1,n}S_c^{(0)+} + [1 + o(1)] \Phi^{(0)-}_{m+1,n}S_c^{(0)-}$$

on $\Delta_{\sigma,c}$,

where $o(1)$ is uniform on $K$ and locally uniform on $\Delta_{\sigma,c}$; however, if $\Delta_{\sigma,c} \cap \partial D_{\sigma,c} = \emptyset$, then $\Delta_{\sigma,c} = \Delta_\sigma$ and $o(1) = O(C_{\mu,n})$ for some constant $C_{\mu,n} > 1$ with the second equality holding uniformly on $\Delta_\sigma$. Furthermore, if $K \subset \mathbb{C} \setminus (\Delta_{\sigma} \cup \Delta_\mu)$ is closed, then

$$w_{\sigma,c} R_{m,n} = [1 + o(1)] \Phi^{(1)}_{m+1,n}S_c^{(1)}$$

on $K$,

$$w_{\sigma,c} R_{m,n}^+ = [1 + o(1)] \Phi^{(1)+}_{m+1,n}S_c^{(1)+}$$

on $\Delta_{\sigma,c}^+$,

$$w_{\sigma,c} R_{m,n}^- = [1 + o(1)] \Phi^{(1)-}_{m+1,n}S_c^{(1)-}$$

on $\Delta_{\mu}$,

where $R_{m,n} = Q_{m,n} \Phi - P_{m,n}$ and $o(1)$ has the same properties as in (19).

**Remark.** Polynomial $Q_{m,n}$ is defined up to a multiplicative constant. However, choosing $Q_{m,n}$ uniquely determines $P_{m,n}$, and respectively $R_{m,n}$. Polynomials $Q_{m,n}$ in (19) are normalized so that the leading coefficient is equal to the coefficient of $\Phi^{(0)}_{m+1,n}S_c^{(0)}$ next to $z^n$ when the latter function is developed into a power series at infinity.

**Remark.** The proof of Theorem 5 follows the framework of Riemann-Hilbert analysis for orthogonal polynomials formulated by Fokas, Its, and Kitaev [15, 16], in which Frobenius-Padé approximants are characterized via a certain Riemann-Hilbert problem whose solution is obtained using a variation of Deift and Zhou steepest descent method [17]. In this realm of ideas it is well understood that one can introduce Fisher-Hartwig singularities into (17). That is, (17) can be replaced by

$$d\nu(x) = \rho(x) \prod_{i=0}^{l_\nu} \left| x - x_{i,\nu} \right|^{\alpha_{i,\nu}} \prod_{i=1}^{l_\nu} \left\{ 1, \begin{array}{ll} x < x_{i,\nu} \\ \beta_{i,\nu} \end{array} \right. \begin{array}{ll} x > x_{i,\nu} \end{array} \right\} dx,$$

where $\rho(x)$ is as before, $\alpha_{i,\nu} = -\sigma_{i,\nu} < x_{i,\nu} < \cdots < x_{i,\nu-1,\nu} < x_{i,\nu+1,\nu} = b_{\nu,\nu}, \alpha_{i,\nu} > -1$, and $\beta_{i,\nu} \notin (-\infty, 0]$, [18, 19, 20, 21, 14]. Implementing such a modification is rather lengthy as details are very technical and does not provide any additional insight on the behavior of the approximants. Thus, we opted to consider only the measures of the form (17).

**Remark.** As was noticed in [22], in the case of positive measures $\mu$ and $\sigma$ in (1), (3), the statement of Theorem 5 for the diagonal sequence $(n-1,n)$ follows from the theorem on strong asymptotics of multiple orthogonal polynomials from [23].

**Remark.** By the definition of the linear forms $R_{m,n}$, (19)–(20), and (9), it holds that the error of approximation by Frobenius-Padé approximants behaves like

$$|\sigma - P_{m,n}/Q_{m,n}| \sim \left| \Phi^{(1)}_{m+1,n}/\Phi^{(0)}_{m+1,n} \right| = \exp \left\{ -(n + m + 1) \left( V_{\sigma,c}^+ - 2V_{\sigma,c}^+ + 3t_{\sigma,c} \right) \right\}.$$

Hence, the approximants converge to $\sigma$ uniformly on compact subsets of $D_{\sigma,c}^+$ and diverge uniformly on compact subsets of $D_{\sigma,c}^-$. It also follows from Proposition 1, that they converge locally uniformly in $\mathbb{C} \setminus \Delta_\sigma$ only if $m = n + o(n)$.
3  PROOFS

3.1 Functions $h_{m,n}$ and $\Phi_{m,n}$

For the proof of Proposition 2, put $c = \frac{n}{n+m}$. To find the divisor of $h_{m,n}$, observe that $h_{m,n}$ is holomorphic everywhere outside of the four branch points of $\mathcal{R}_{m,n}$ and at each of these points it can have at most a simple pole since $\Phi_{m,n}$ is bounded there. Clearly, $h_{m,n}$ has three simple zeros, one at each $\infty(k)$, $k \in \{0,1,2\}$. If $b_{\sigma,c}$ is not a pole, then the remaining three branch points of $\mathcal{R}_{m,n}$ must be poles (the number of poles must be equal to the number of zeros) and there cannot be any more poles and/or zeros. In this case we put $z_{m,n} = b_{\sigma,c}$, which verifies (12). If $b_{\sigma,c}$ is a pole of $h_{m,n}$, then

$$h^{(1)}_{m,n}(x) \to (-1)^{d_{m,n}} \text{ as } x \to b_{\sigma,c}, \ R \ni x \notin \Delta_{\sigma,c}, \quad \iota := \begin{cases} 0, & a_\sigma < b_\sigma, \\ 1, & b_\sigma < a_\sigma, \end{cases}$$

by the very definition of $h^{(1)}_{m,n}$ in (10) including the positivity of $\tau_{\sigma,c}$ and our labeling convention for the endpoints of $\Delta_{\sigma,c}$. On the other hand, it holds that

$$h^{(1)}_{m,n}(z) = \frac{|\tau_{\mu,c}| - |\tau_{\sigma,c}|}{z} + O \left( z^{-2} \right) \text{ as } z \to \infty,$$

again, by the very definition of $h_{m,n}$. As $|\tau_{\mu,c}| - |\tau_{\sigma,c}| = 1 - c > 0$, we have that

$$(-1)^{d_{m,n}} h^{(1)}_{m,n}(x) > 0 \text{ as } R \ni x \to (-1)^{d_{m,n}} \infty.$$

Therefore there indeed exists $z_{m,n}$ between $b_{\sigma,c}$ and $(-1)^{d_{m,n}}$ such that $h^{(1)}_{m,n}(z_{m,n}) = 0$. Since $h_{m,n}$ has three more zeros, the rest of the branch points must be poles as claimed.

Assume that $b_{\sigma,c}$ is a pole of $h_{m,n}$ (equivalently $b_{\sigma,c} \notin z_{m,n}$). Since $h_{m,n}$ has square root branching at $b_{\sigma,c}$, it follows from (21) and the fact that the sum $h^{(0)}_{m,n} + h^{(1)}_{m,n}$ is holomorphic around $b_{\sigma,c}$ that

$$h^{(1)}_{m,n}(z) = d_{m,n} (-1)^{d_{m,n}} (z - b_{\sigma,c})^{1/2} + O(1),$$

where $(-1)^{d_{m,n}} < 0$, and the square root is principal. It further follows from the above asymptotics as well as from (9) and (11) that

$$\sqrt{\tau_{\mu,c} - 2 \tau_{\sigma,c}}(z) + 3 \ell_{\sigma,c} = \Re \left( \int_{b_{\sigma,c}}^{z} h^{(0)}_{m,n} - h^{(1)}_{m,n} \right) dy > 0$$

as $z \to b_{\sigma,c}, z \notin \Delta_{\sigma,c}$. Hence, $b_{\sigma,c} \notin \partial \Delta_{\sigma,c}$. On the other hand, if $b_{\sigma,c}$ is not a pole of $h_{m,n}$ (equivalently $b_{\sigma,c} = z_{m,n}$), then

$$h^{(0)}_{m,n}(z) = h_{m,n}(b_{\sigma,c}) + e_{m,n} (-1)^{d_{m,n}} (z - b_{\sigma,c})^{1/2} + O(|z - b_{\sigma,c}|),$$

$$h^{(1)}_{m,n}(z) = h_{m,n}(b_{\sigma,c}) - e_{m,n} (-1)^{d_{m,n}} (z - b_{\sigma,c})^{1/2} + O(|z - b_{\sigma,c}|),$$

as $z \to b_{\sigma,c}, z \notin \Delta_{\sigma,c}$, since $h^{(0)}_{m,n} + h^{(1)}_{m,n}$ is holomorphic around $b_{\sigma,c}$. Moreover, as $h^{(0)}_{m,n}$ satisfies (22) and is monotone between $b_{\sigma,c}$ and $(-1)^{d_{m,n}}$, it holds that $(-1)^{d_{m,n}} e_{m,n} < 0$. Therefore,

$$\sqrt{\tau_{\mu,c} - 2 \tau_{\sigma,c}}(x) + 3 \ell_{\sigma,c} = \int_{b_{\sigma,c}}^{x} h^{(0)}_{m,n} - h^{(1)}_{m,n} \right) dy < 0$$

for $x \to b_{\sigma,c}, R \ni x \notin \Delta_{\sigma,c}$. In particular, $b_{\sigma,c} \in \partial \Delta_{\sigma,c}$. This finishes the proof of the last claim of the proposition. Finally, similar analysis can be used to show that $a_{\sigma,c} \notin \partial \Delta_{\sigma,c}$, which, as noted at the end of Proposition 2, implies the equality $a_{\sigma,c} = a_{\sigma}$.

For the future use let us record several facts. Firstly, it holds that

$$|\Phi^{(2)}_{m,n} / \Phi^{(1)}_{m,n}| < 1 \text{ in } \mathbb{C} \setminus \Delta_{\mu}.$$
Indeed, (23) is equivalent to $V^{\tau_{\sigma,e}} - 2V^{\tau_{\mu,e}} + 3t_{\mu,\sigma} > 0$ by (9). The left-hand side of this inequality is superharmonic in $C \setminus \Delta_{\mu}$, is identically zero on $\Delta_{\mu}$ by (6), and approaches $+\infty$ as $z \to \infty$ since $|\tau_{\sigma,e}| = c < 2 = |\tau_{\mu,e}|$. The desired inequality now follows from the minimum principle for superharmonic functions [24, Theorem 2.3.1].

Secondly, let $\{c_n\}$ be a sequence such that $c_n \to c > 0$ as $n \to \infty$, $c_n \leq 1/2$. Then

\begin{equation}
\tau_{\mu,c_n} \rightharpoonup \tau_{\mu,c} \quad \text{and} \quad \tau_{\sigma,c_n} \rightharpoonup \tau_{\sigma,c},
\end{equation}

where $\rightharpoonup$ stands for the weak* convergence of measures. Indeed, besides (6), the pair $(\tau_{\mu,c}, \tau_{\sigma,c})$ is characterized as the unique minimizers in $M_c$ of the energy functional

\begin{equation}
J(\tau_{\mu}, \tau_{\sigma}) := I(\tau_{\mu}, \tau_{\mu}) + I(\tau_{\sigma}, \tau_{\sigma}) - I(\tau_{\mu}, \tau_{\sigma}),
\end{equation}

where $I(\nu, \lambda) := -\int \log |z - w| d\nu(z) d\lambda(w)$, see [9, 10]. Let $\tau_{\mu}$ and $\tau_{\sigma}$ be weak* limit points of $\{\tau_{\mu,c_n}\}$ and $\{\tau_{\sigma,c_n}\}$, respectively. Clearly, $(\tau_{\mu,c}, \tau_{\sigma,c}) \in C$. Then

\begin{equation}
J(\tau_{\mu,c}, \tau_{\sigma,c}) = \lim_{n \to \infty} I(\tau_{\mu,c}, \frac{c_n}{c} \tau_{\sigma,c}) \geq \lim_{n \to \infty} \inf I(\tau_{\mu,c_n}, \tau_{\sigma,c_n}) = J(\tau_{\mu}, \tau_{\sigma}),
\end{equation}

where the first inequality follows from the fact that $(\tau_{\mu,c_n}, \tau_{\sigma,c_n})$ is the minimizer of the $J$-functional in $M_{c_n}$ and the second inequality is the consequence of the principle of descent [25, Theorem 1.6.8], i.e., $\lim \inf I(\tau_{\mu,c_n}, \tau_{\sigma,c_n}) \geq I(\tau_{\mu}, \tau_{\sigma})$, and the fact that $\Delta_{\mu} \cap \Delta_{\sigma} = \emptyset$ (in this case the kernel $\log |z - w|$ is continuous on $\Delta_{\mu} \times \Delta_{\sigma}$ and therefore $I(\tau_{\mu,c_n}, \tau_{\sigma,c_n}) \to I(\tau_{\mu}, \tau_{\sigma})$ by weak* convergence of measures). As $(\tau_{\mu,c}, \tau_{\sigma,c})$ is the unique minimizer of the $J$-functional in $M_c$, (24) follows.

Finally, let us point out that in the above setting $b_{\sigma,c_n} \to b_{\sigma,c}$, and

\begin{equation}
V^{\tau_{\nu,c_n}} \to V^{\tau_{\nu,c}} \quad \text{locally uniformly in} \quad \mathbb{C} \setminus \Delta_{\nu,c}, \quad \nu \in \{\mu, \sigma\},
\end{equation}

as $n \to \infty$, which is an immediate consequence of (24).

3.2 Riemann-Hilbert Problem for Frobenius-Padé Approximants

Given such a pair of integers $(m, n)$, $n - 1 \leq m$, we are interested in finding a $3 \times 3$ matrix-valued function $Y$ that solves the following Riemann-Hilbert Problem (RHP-$Y$):

\begin{enumerate}
\item $Y$ is analytic in $\mathbb{C} \setminus (\Delta_{\mu} \cup \Delta_{\sigma})$ and
\begin{equation}
\lim_{z \to \infty} Y(z) \diag(z^{-n}, z^{-m-1}, z^{n+m+1}) = I,
\end{equation}

where $\diag(\cdot, \cdot, \cdot)$ is the diagonal matrix and $I$ is the identity matrix;
\item $Y$ has continuous traces on $\Delta_{\mu}^0 \cup \Delta_{\sigma}^0$ that satisfy
\begin{equation}
Y_+ = Y_- \Delta^0_v \begin{pmatrix}
1 & \rho_v / \omega_v \\
0 & 1
\end{pmatrix}
\end{equation}

on $\Delta^0_v$, $v \in \{\mu, \sigma\}$, where transformations $\Delta_{\mu}$ and $\Delta_{\sigma}$ act on $2 \times 2$ matrices in the following fashion:

\begin{equation}
\Delta_{\mu} A := \begin{pmatrix}
1 & 0 & 0 \\
0 & \mathcal{A}_{11} & \mathcal{A}_{12} \\
0 & \mathcal{A}_{21} & \mathcal{A}_{22}
\end{pmatrix}
\quad \text{and} \quad
\Delta_{\sigma} A := \begin{pmatrix}
\mathcal{A}_{11} & \mathcal{A}_{12} & 0 \\
\mathcal{A}_{21} & \mathcal{A}_{22} & 0 \\
0 & 0 & 1
\end{pmatrix};
\end{equation}
\item the entries of $Y$ are bounded except for the second column around the endpoints of $\Delta_{\sigma}$ and the third column around the endpoint of $\Delta_{\mu}$ where they behave as $O(|z - e|^{-1/2})$ with $e$ being the corresponding endpoint.
\end{enumerate}

To see how RHP-$Y$ is connected to Frobenius-Padé approximants, observe that the linear form $R_{m,n}$ is a holomorphic function in $\mathbb{C} \setminus \Delta_{\sigma}$ with a pole of degree at most $m$ at infinity. Moreover, it follows from Plemelj-Sokhotski formulæ [26, Section I.4.2] and (17) that

\begin{equation}
R_{m,n}^+ - R_{m,n}^- = Q_{m,n} \rho_\sigma / \omega_\sigma^+ \quad \text{on} \quad \Delta_{\sigma}^0.
\end{equation}
It is also known from the theory of boundary behavior of Cauchy integrals [26, Section I.8] that \( R_{m,n}(z) \sim |z - e|^{-1/2} \) as \( z \to e \in \{(a, b)\} \). As mentioned before, condition (2) implies that \( R_{m,n} \) is orthogonal to all polynomials of degree at most \( m + n \) with respect to \( \mu \), i.e.,

\[
\int x^i R_{m,n}(x) \, d\mu(x) = 0, \quad i \in \{0, \ldots, m + n\}.
\]

Orthogonality relations (27) imply that the Cauchy transform of \( R_{m,n} \) vanishes at infinity with order at least \( m + n + 2 \). That is, the function

\[
C_{m,n}(z) := \int \frac{R_{m,n}(x)}{x - z} \, d\mu(x), \quad z \in \mathbb{C} \setminus \Delta_\mu,
\]

is a holomorphic function in \( \mathbb{C} \setminus \Delta_\mu \), has a zero of order at least \( m + n + 2 \) at infinity, and satisfies

\[
C_{m,n}^+ - C_{m,n}^- = R_{m,n} \rho_\mu / \omega_\mu^+ \quad \text{on} \quad \Delta_\mu^0.
\]

As in the case of \( R_{m,n} \), we can conclude that \( C_{m,n}(z) \sim |z - e|^{-1/2} \) as \( z \to e \in \{(a, b)\} \).

**Lemma 6.** Let \( n - 1 \leq m \). If \((m, n)\)-th Frobenius-Padé approximant is unique and \( \deg(Q_{m,n}) = n \), then \( R_{m+1,n-1}(z) \sim z^{m+1} \) and \( C_{m,n-1}(z) \sim z^{-(n+m+1)} \) as \( z \to \infty \) for any \((m+1, n-1)\)-st and \((m, n-1)\)-st approximants, respectively.

**Proof.** Assume to the contrary that there is \((m, n-1)\)-st approximant such that \( C_{m,n-1}(z) \sim z^{-(n+m+j+1)} \) as \( z \to \infty \) for some \( j > 0 \). It can be readily verified that in this case the corresponding linear form \( R_{m,n-1} \) is orthogonal to all polynomials of degree at most \( m + n + j \). Then we can conclude from (2) that this \((m, n-1)\)-st approximant is also \((m+j, n-1+j_2)\)-th approximant for any choice of \( j_1, j_2 \geq 0 \). By taking \( j_1 = 1 \) and \( j_2 = 0 \), we see that there exists \((m+1, n-1)\)-st approximant for which \( R_{m+1,n-1}(z) \sim z^{m+1+1} \) for some \( i > 0 \) (recall that \( n - 1 \leq m \)). This implies that \( \deg(P_{m+1,n-1}) = m+1-i \leq m \), and respectively, this Frobenius-Padé approximant also corresponds to the index \((m, n)\) and its denominator has degree at most \( n - 1 \). □

Assuming \((m, n)\)-th approximant is the unique and \( \deg(Q_{m,n}) = n \), define

\[
Y_{m,n} := C_{m,n} \begin{pmatrix}
Q_{m,n} & R_{m,n} & C_{m,n} \\
Q_{m+1,n-1} & R_{m+1,n-1} & C_{m+1,n-1} \\
Q_{m,n-1} & R_{m,n-1} & C_{m,n-1}
\end{pmatrix},
\]

where \( C_{m,n} \) is a diagonal matrix of constants chosen so that \( Y_{m,n} \) satisfies the normalization at infinity from RHP-Y(a). The choice of \( C_{m,n} \) is always possible due to Lemma 6. Then the following lemma holds.

**Lemma 7.** Let \( n - 1 \leq m \). If RHP-Y is solvable, then \((m, n)\)-th, \((m, n-1)\)-st, and \((m+1, n-1)\)-st Frobenius-Padé approximants are unique, \( \deg(Q_{m,n}) = n \), and \( Y = Y_{m,n} \).

**Proof.** Assume that RHP-Y is solvable and \( Y \) is a solution. We consider only the first row as the other ones can be analyzed similarly. It follows from RHP-Y(a,b) that \( |Y|_{11} \) must be polynomial of degree \( n \). Further, all three properties RHP-Y(a,b,c) imply that \( |Y|_{12} = |Y|_{11} \delta - P \) for some polynomial \( P \), \( \deg(P) \leq m \). Analogously, we see that \( |Y|_{13} \) must be a Cauchy transform of \(|Y|_{12} \rho_\mu / \omega_\mu^+ \). The vanishing of \(|Y|_{13} \) at infinity with order at least \( m + n + 2 \) implies that \(|Y|_{12} \) is orthogonal to \( x^i \), \( i \in \{0, \ldots, m+n\} \), with respect to \( \mu \). Therefore, \( c_i(|Y|_{11} \delta - P) = 0 \) for such \( i \), and, by definition, \( P/|Y|_{11} \) is an \((m, n)\)-th Frobenius-Padé approximant.

To show uniqueness of the approximants, observe first that the solution of RHP-Y is unique. Indeed, Let \( Y_1 \) and \( Y_2 \) be solutions. As the determinant of the jump matrix in RHP-Y(b) is 1 and \( \det(Y_1) \) can have at most square root singularities at the endpoints of \( \Delta_\mu \) and \( \Delta_\sigma \) by RHP-Y(c), \( \det(Y_1) \) is an entire function such that \( \det(Y_1)(\infty) = 1 \). Hence, \( \det(Y_1) \equiv 1 \) and therefore \( Y_1 \) is invertible. Then \( Y_2 Y_1^{-1} \) is an entire matrix-valued function that is equal to \( I \) at infinity. Thus, \( Y_2 = Y_1 \).
Second, observe that if \((m,n)\)-th approximant is unique and \(\deg(Q_{m,n}) = n\), then \(Y_{m,n}\) solves RHP-Y. Indeed, the fact that \(Y_{m,n}\) satisfies RHP-Y(a,c) easily follows from the analyticity properties and the behavior at infinity of \(Q_{m,n}, R_{m,n}, \) and \(C_{m,n}\), as well as from the choice of \(C_{m,n}\). RHP-Y(b) is an immediate consequence of (26) and (28).

Now, let \(Y\) be the solution. Assume \(Q_{m,n}, R_{m,n}, \) and \(C_{m,n}\) correspond to another \((m,n)\)-th approximant. Without loss of generality we can assume that \(\deg(Q_{m,n}) = n\) (otherwise we should take \([Y]\) instead of \(Q_{m,n}\)). Construct matrix \(Y_1\) by replacing the first row of \(Y\) with \((Q_{m,n},R_{m,n},C_{m,n})\). From the first paragraph we know that the second and third rows of \(Y_1\) correspond to \((m+1,n-1)\)-st and \((m,n-1)\)-st approximants and therefore we deduce from the third paragraph that \(Y_1\) is a solution of RHP-Y. By uniqueness, we get that \(Y_1 = Y\) and therefore \((m,n)\)-th approximant is unique. Thus, we know from Lemma 6 that any \((m+1,n-1)\)-st and \((m,n-1)\)-st must satisfy its conclusions. Hence, if they were not unique, we could replace the second and third rows of \(Y\) by the functions coming from other approximants and obtain a solution of RHP-Y different form \(Y\), which is impossible.

### 3.3 Non-Linear Steepest Descent Analysis in the Case \(\Delta_{\sigma,c} \cap \partial D^{-}_{\sigma,c} = \emptyset\)

Recall that in the considered case \(\Delta_{\sigma,c} = \Delta_{\sigma}\), see Proposition 1. Moreover, it follows from (24) and Proposition 2 that \(\Delta_{\sigma} = \Delta_{\sigma}\) for all \(n\) large enough. In particular, we have that \(\mathfrak{R}_{m,n} = \mathfrak{R}_n =: \mathfrak{R}\) for all such \(n\) and we consider only these indices from now on.

Let \(\Gamma_{\mu}\) and \(\Gamma_{\nu}\) be positively oriented Jordan curves lying exterior to each other and containing \(\Delta_{\mu}\) and \(\Delta_{\nu}\), respectively. We denote by \(\Omega_{\nu}\) the domain delimited by \(\Gamma_{\nu} \cup \Delta_{\nu}, \nu \in \{\mu, \sigma\}\). We assume that \(\rho_{\nu}\) extends holomorphically across \(\Gamma_{\nu}, \nu \in \{\mu, \sigma\}\), and that \(\Gamma_{\sigma} \subset D^{-}_{\sigma,c}\). Observe that in the considered case \(\partial D^{-}_{\sigma,m} \cup \partial D^{-}_{\sigma,m}\) approaches \(D^{-}_{\sigma,c}\) by (25) and therefore \(\Gamma_{\sigma} \subset D^{-}_{\sigma,m}\) is uniformly separated from \(\partial D^{-}_{\sigma,m}\). Define

\[
X = Y \begin{cases} T_{\nu} \begin{pmatrix} 1 & 0 \\ -w_{\nu}/\rho_{\nu} & 1 \end{pmatrix}, & \text{in } \Omega_{\nu}, \nu \in \{\mu, \sigma\}, \\ I, & \text{otherwise.} \end{cases}
\]  

It is easy to verify that \(X\) solves the following Riemann-Hilbert problem (RHP-X):

(a) \(X\) is analytic in \(\mathbb{C} \setminus (\Delta_{\mu} \cup \Delta_{\sigma} \cup \Gamma_{\mu} \cup \Gamma_{\sigma})\) and

\[
\lim_{z \to \infty} X(z) \text{ diag } \left(z^{-n}, z^{-m-1}, z^{n+m+1}\right) = I;
\]

(b) \(X\) has continuous traces on \(\Delta_{\mu}^{*} \cup \Delta_{\sigma}^{*} \cup \Gamma_{\mu} \cup \Gamma_{\sigma}\) that satisfy

\[
X_{+} = X_{-} \begin{cases} T_{\nu} \begin{pmatrix} 0 & \rho_{\nu}/w_{\nu}^{*} \\ -w_{\nu}/\rho_{\nu} & 0 \end{pmatrix} & \text{on } \Delta_{\nu}^{*}, \\ T_{\nu} \begin{pmatrix} 1 & 0 \\ w_{\nu}/\rho_{\nu} & 1 \end{pmatrix} & \text{on } \Gamma_{\nu}, \nu \in \{\sigma, \mu\}; \end{cases}
\]

(c) \(X\) satisfies RHP-Y(c).

Then the following lemma can be easily checked.

**Lemma 8.** RHP-X is solvable if and only if RHP-Y is solvable. When solutions of RHP-X and RHP-Y exist, they are unique and connected by (30).

As typical in the steepest descent analysis of Riemann-Hilbert problems, we ignore the jump of \(X\) on \(\Gamma_{\mu} \cup \Gamma_{\sigma}\) and look for the following approximation to \(X\) (RHP-N):

(a) \(N\) is analytic in \(\mathbb{C} \setminus (\Delta_{\mu} \cup \Delta_{\sigma})\) and

\[
\lim_{z \to \infty} N(z) \text{ diag } \left(z^{-n}, z^{-m-1}, z^{n+m+1}\right) = I;
\]
Then the following lemma holds.

**Lemma 10.** The solution of RHP-Z exists for all \( n \) large enough and satisfies

\[
Z = I + O(C_{\mu, \sigma}^n)
\]

for some constant \( C_{\mu, \sigma} > 1 \), where \( O(\cdot) \) holds uniformly in \( \mathbb{C} \).

**Proof.** The jump matrix for \( Z \) on \( \Gamma_\sigma \) is equal to

\[
I + \frac{w_\sigma}{\rho_\sigma} \Phi_{m+1,n}^{(1)} \frac{M_{E2}}{M}^{-1}.
\]

}\)
where $E_{ij}$ is the matrix with all zero entries except for $(i, j)$-th, which is $1$. Since $M$ is fixed and has constant determinant, it follows from (9), definition of $D^+_{\sigma,m,n}$ in (7), and the choice of $\Gamma_\sigma$ that the jump of $Z$ on $\Gamma_\sigma$ is of the form $I + O(C^{-\mu}_\sigma)$ for some $C_{\mu,\sigma} > 1$. Similarly, we have that the jump $Z$ on $\Gamma_\mu$ is equal to

$$I + \frac{w_\mu \Phi^{(2)}_{m+1,n}}{\rho_\mu \Phi^{(1)}_{m+1,n}} ME_{32} M^{-1},$$

which is also of the form $I + O(C^{-\mu}_\sigma)$ for some properly adjusted $C_{\mu,\sigma} > 1$ by (23). The conclusion of the lemma follows now from the same argument as in [27, Corollary 7.108] with another adjustment of $C_{\mu,\sigma}$. □

Finally, let $Z$ be a solution of RHP-Z granted by Lemma 10 and $N = CMD$ be as in Lemma 9. Then it can be easily checked that $X = CZMD$ solves RHP-X and therefore

$$Y = CZMD \begin{cases} T_\nu \left( \frac{1}{w_\mu/\rho_\nu} \right), & \text{in } \Omega_\nu, \quad \nu \in [\mu, \sigma], \\ I, & \text{otherwise.} \end{cases}$$

solves RHP-Y.

### 3.4 Non-Linear Steepest Descent Analysis in the Case $\Delta_{\sigma,c} \cap \partial D^-_{\sigma,c} \neq \emptyset$

In the case $\Delta_{\sigma,c} \cap \partial D^-_{\sigma,c} \neq \emptyset$ there no longer exists a Jordan curve $\Gamma_\sigma \subset D^+_{\sigma,c}$ encircling $\Delta_{\sigma,c}$, which prevents us from carrying out the estimate (33). Hence, we shall require the Jordan curve $\Gamma_{\sigma,c}$ to encircle $\Delta_{\sigma,c}$ except for the point $b_{\sigma,c}$, which they have in common. Moreover, given disjoint disks $U_{b_{\sigma,c}}$ and $U_{b_\sigma}$ centered at $b_{\sigma,c}$ and $b_\sigma$, respectively, (unless $b_{\sigma,c} = b_\sigma$ in which case these disks coincide) we also require that $\Gamma_{\sigma,c} \setminus U_{b_{\sigma,c}} \subset D^-_{\sigma,c}$, see Figure 4. To slightly alleviate the notation, let us set

$$b_{m+1,n} := b_{\sigma_{m+1,n}}, \quad \Delta_{m+1,n} := \Delta_{\sigma_{m+1,n}}, \quad \text{and} \quad \Gamma_{m+1,n} := \Gamma_{\sigma_{m+1,n}},$$

where the curves $\Gamma_{m+1,n}$ are selected analogously to $\Gamma_{\sigma,c}$ with the requirement that $\Gamma_{m+1,n} \to \Gamma_{\sigma,c}$ as $n \to \infty$ in Hausdorff metric (recall that $b_{m+1,n} \to b_{\sigma,c}$ as $n \to \infty$, see (25)). We take $\Gamma_\mu$ to be a Jordan curve encircling $\Delta_\mu$, which is disjoint from all $\Gamma_{m+1,n}$. As before, we assume that $\rho_\mu$ is holomorphic across $\Gamma_\mu$ and $\rho_\nu$ is holomorphic across each $\Gamma_{m+1,n}$. We continue to denote by $\Omega_\mu$ and $\Omega_{m+1,n}$ the domains bounded by $\Gamma_\mu \cup \Delta_\mu$ and $\Gamma_{m+1,n} \cup \Delta_{m+1,n}$, respectively.

In what follows, we shall often refer back to Riemann-Hilbert problems formulated in Section 3.3. For each such reference it is understood that when $\Delta_\sigma$, $\Gamma_\sigma$, and $\Omega_\sigma$ occur, they should be replaced by $\Delta_{m+1,n}$, $\Gamma_{m+1,n}$, and $\Omega_{m+1,n}$.

Define $X$ by (30). Then $X$ satisfies RHP-X(a,c) and RHP-X(b) with an additional jump

$$X_+ = X_- T_\sigma \left( \begin{array}{cc} 1 & \rho_\sigma/\omega_\sigma^+ \\ 0 & 1 \end{array} \right) \text{ on } \Delta_\sigma \setminus \Delta_{m+1,n}.$$

Clearly, Lemma 8 remains valid.
Define $D$ as in Lemma 9, where $D_{m+1,n}$ is a rational function on $\mathcal{R}_{m+1,n}$ with the divisor $(n + m + 1)\infty(2) - n\infty(0) - (m + 1)\infty(1)$ and normalized as in (8). Let $S_{m+1,n}$ be the function on $\mathcal{R}_{m+1,n}$ granted by Proposition 4 applied with $c = \frac{n}{m + 1}$ and $\gamma_k := \gamma_{k,m+1,n}$, $k \in \{0, 1, 2\}$, be the rational function on $\mathcal{R}_{m+1,n}$ with the divisor $\infty(0) - \infty(k)$ and the normalization as in (8). Define matrices $M$ and $C$ as in Lemma 9 using the above functions. Then $N = \Psi M D$ again solves RHP-$N$ and it is still true that $\det(N) \equiv 1$. Therefore $\det(M)$ is a constant, but in this case it might depend on $(m, n)$. However, observe that an analogous matrix $M = M_c$ can be defined on the “limiting” surface $\mathcal{R}_c$ as well. Moreover, it was shown in [14, Section 7] that

\[(36) \quad S_{m+1,n} \to S_c \quad \text{and} \quad \gamma_{k,m+1,n} \to \gamma_{k,c} \]

uniformly on $\mathcal{R}_{c,\delta}$ for any $\delta > 0$, where $\mathcal{R}_{c,\delta}$ is obtained from $\mathcal{R}_c$ by removing circular neighborhoods of radius $\delta$ around each branch point of $\mathcal{R}_c$ and functions $S_{m+1,n}$ and $\gamma_{k,m+1,n}$ are carried over to $\mathcal{R}_{c,\delta}$ with the help of natural projections. Hence, $\det(M_{m+1,n}) \to \det(M_c)$, in particular, the determinants $\det(M)$ are uniformly bounded away from zero and infinity with $m$ and $n$.

Let $P_b$, $b \in \{b_{a,c}, b_\sigma\}$, be a matrix-valued function that solves RHP-$X$ inside of $\mathcal{U}_b$ and satisfies

\[(37) \quad P_b = M(I + O(\varepsilon_{m+1,n}))D \]

uniformly on $\partial\mathcal{U}_b$, where $0 < \varepsilon_{m+1,n} \to 0$ as $n \to \infty$. Such matrices do exist. Indeed, when $b_{a,c} \neq b_\sigma$, one can easily check that

\[P_{b,\sigma} = MT_{\sigma} \begin{pmatrix} 1 & \mathcal{E}_{\sigma}(\Phi(0)_{m+1,n}/\Phi(1)_{m+1,n}) \\ 0 & 1 \end{pmatrix} D,\]

where $\mathcal{E}_{\sigma}(z) := \frac{1}{z^m} \int_{\Delta_{\sigma}} \frac{\rho_\sigma(x)}{x-z} \frac{d\sigma(x)}{w_\sigma(x)}$. As the construction of $P_{b,\sigma,c}$ is quite long and is absolutely identical to the one in [14, Sections 9.4 and 9.5], we omit it here. Let us just mention that it is based on the model Riemann-Hilbert problem associated with solutions Panlevé XXXIV equation [14, Section 4.3] (matrix $\Psi_{0,1}$ when $b_{a,c} \neq b_\sigma$ and $\Psi_{1/2,0}$ otherwise), see also [28, 29].

\[\Gamma_{\mu} \quad \Gamma_{m+1,n} \quad U_{b_{a,c}} \quad U_{b_\sigma}\]

**Figure 5.** Lens $\Sigma_Z$ consisting of the curve $\Gamma_{\mu}$, the arc $\Gamma_{m+1,n} \setminus U_{b_{a,c}}$, the interval $\Delta_{\sigma} \setminus (\Delta_{a,c} \cup U_{b_{a,c}} \cup U_{b_\sigma})$, and the circles $\partial U_{b_{a,c}}$, $\partial U_{b_\sigma}$.

Let the contour $\Sigma_Z$ be as depicted on Figure 5. The last matrix-valued function needed to solve RHP-$Y$ is described by the following Riemann-Hilbert problem:

(a) $Z$ is a holomorphic matrix function in $\mathcal{T} \setminus \Sigma_Z$ and $Z(\infty) = I$;

(b) $Z$ has continuous traces on $\Sigma_Z$ except perhaps at its branching points that satisfy

\[Z_+ = Z_- (MD) T_{\nu} \begin{pmatrix} 1 & 0 \\ \rho_{\nu} / \rho_{\sigma} & 1 \end{pmatrix} (MD)^{-1} \]

on $\Gamma_{\mu}$ when $\nu = \mu$ and $\Gamma_{m+1,n} \setminus U_{b_{a,c}}$ when $\nu = \sigma$,

\[Z_+ = Z_- (MD) T_{\sigma} \begin{pmatrix} 1 & \rho_{\sigma} / \rho_{\nu} \\ 0 & 1 \end{pmatrix} (MD)^{-1} \]

on $\Delta_{\sigma} \setminus (\Delta_{a,c} \cup U_{b_{a,c}} \cup U_{b_\sigma})$, and $Z_+ = Z_- P_b (MD)^{-1}$, on $\partial U_b$, $b \in \{b_{a,c}, b_\sigma\}$.

Exactly as in the previous case, the following lemma holds.

**Lemma 11.** The solution of RHP-$Z$ exists for all $n$ large enough and satisfies

\[(38) \quad Z = I + O(\varepsilon_{m+1,n})\]

where $O(\cdot)$ holds uniformly in $\mathcal{T}$ and $\varepsilon_{m+1,n}$ are the constants from (37).
Proof. Recall that the determinants \( \det(M) \) are identically constant as functions of \( z \) and that these constants are uniformly separated from zero and infinity with \( m \) and \( n \). Hence, the estimates of the size of the jumps on \( \Gamma_{\mu} \) and \( \Gamma_{m+1,n} \setminus \bigcup_{b_{\sigma,c}} \) are absolutely analogous to (33) and (34). The jumps in this case are geometrically close to the identity where the constant of proportionality depends on how close \( \Gamma_{m+1,n} \setminus \bigcup_{b_{\sigma,c}} \) is to \( \partial D_{\sigma,c} \) (the latter sets are uniformly separated from each other by our construction of \( \Gamma_{m+1,n} \)). The jump on \( \Delta_{\sigma} \setminus (\Delta_{\sigma,c} \cup \bigcup_{b_{\sigma,c}} \cup \bigcup_{b_{\sigma}}) \) is also geometrically close to the identity since this interval belongs to \( D_{\sigma,c} \) where \( |\Phi_{m+1,n}^{(1)}| > |\Phi_{m+1,n}^{(0)}| \). Finally, we see that the jump on \( \partial U_b \), \( b \in \{b_{\sigma,c}, b_{\sigma}\} \), is \( O(1/\epsilon_{m+1,n}) \) close to the identity by (37) and the normality of \( M \), see (36). The existence of \( Z \) again follows from [27, Corollary 7.108]. The size of the error is proportional to \( 1/n \) as the latter is of order \( 1/n \) at best, see [14, Sections 9.4 and 9.5].

Altogether, the solution of RHP-X is given by

\[
X = CZ \begin{cases} \Phi_{b_{\sigma}} & \text{in } U_b, \ b \in \{b_{\sigma,c}, b_{\sigma}\}, \\ \text{MD} & \text{otherwise,} \end{cases}
\]

and then the solution of RHP-Y is obtained by inverting (30).

3.5 Asymptotic Analysis

Below we write \( S_{m+1,n} \) and \( \Upsilon_{k,m+1,n} \) irrespectively of whether we are in the case of Section 3.3 or Section 3.4. Write \( Z = I + \sum_{i,j=1}^{3} \nu_{m+1,n}^{(i,j)} \), where we know from Lemmas 10 and 11 that

\[
|\nu_{m+1,n}^{(i,j)}| = \mathcal{O}(\epsilon_{m+1,n}) \quad \text{or} \quad |\nu_{m+1,n}^{(i,j)}| = \mathcal{O}(\epsilon_{m+1,n})
\]

uniformly in \( \mathcal{C} \) depending on the considered case (\( \nu_{m+1,n}^{(i,j)}(\infty) = 0 \) as \( Z(\infty) = I \)). Given any closed set \( K \subset \mathcal{C} \setminus \Delta_{\sigma} \), choose \( \Omega_{\sigma} \) or \( \Omega_{m+1,n} \bigcup \bigcup U_b \) so that \( K \) belongs to the complement of its closure. Then we get from (35) and (39) that \( Y = CZMD \) on \( K \) and therefore

\[
[Y]_{11} = \gamma_{m+1,n}^{(0)} \Phi_{m+1,n}^{(0)} S_{m+1,n}^{(0)} \left( 1 + \nu_{m+1,n}^{(1,1)} + \nu_{m+1,n}^{(1,2)} \gamma_{m+1,n}^{(0)} + \nu_{m+1,n}^{(1,3)} \gamma_{m+1,n}^{(0)} \right)
\]

on \( \Delta_{\sigma} \) or \( \Delta_{\sigma,c} \setminus \bigcup_{b_{\sigma,c}} \), depending on the considered case. The second relation in (19) now follows from (29), (36), (40). On the other hand, we get from (35) and (39) that

\[
[Y]_{12} = \gamma_{m+1,n}^{(0)} \Phi_{m+1,n}^{(1)} S_{m+1,n}^{(1)} \left( 1 + \nu_{m+1,n}^{(1,1)} + \nu_{m+1,n}^{(1,2)} \gamma_{m+1,n}^{(1)} + \nu_{m+1,n}^{(1,3)} \gamma_{m+1,n}^{(1)} \right)
\]

and

\[
[Y]_{21} = \gamma_{m+1,n}^{(0)} \Phi_{m+1,n}^{(1)} S_{m+1,n}^{(1)} \left( 1 + \nu_{m+1,n}^{(1,1)} + \nu_{m+1,n}^{(1,2)} \gamma_{m+1,n}^{(1)} + \nu_{m+1,n}^{(1,3)} \gamma_{m+1,n}^{(1)} \right)
\]

on \( \Delta_{\sigma} \) or \( \Delta_{\sigma,c} \setminus \bigcup_{b_{\sigma,c}} \), depending on the considered case. The second relation in (19) now follows from (29), (18), (36), (40), and the fact that \( Y_{k}^{(0)} = Y_{k}^{(1)} \) on \( \Delta_{\sigma,c} \).

Now, let \( K \subset \mathcal{C} \setminus (\Delta_{\mu} \cup \Delta_{\sigma}) \). Adjust the set \( \Omega_{\mu} \cup \Omega_{\sigma} \) or \( \Omega_{\mu} \cup \Omega_{m+1,n} \cup \bigcup U_b \) if necessary so that \( K \) belongs to the complement of its closure. Then \( Y = CZMD \) on \( K \) and therefore

\[
[Y]_{12} = \frac{\gamma_{m+1,n}^{(0)}}{\nu_{m+1,n}^{(1)}} \Phi_{m+1,n}^{(1)} S_{m+1,n}^{(1)} \left( 1 + \nu_{m+1,n}^{(1,1)} + \nu_{m+1,n}^{(1,2)} \gamma_{m+1,n}^{(1)} + \nu_{m+1,n}^{(1,3)} \gamma_{m+1,n}^{(1)} \right).
\]

Even though \( \gamma_{m+1,n}^{(1)} \) has a pole at infinity, the product \( \nu_{m+1,n}^{(1,2)} \gamma_{m+1,n}^{(1)} \) is finite satisfies (40) by (36) and the maximum modulus principle. As \( \nu_{m+1,n}^{-1} \to \nu_{\sigma,c}^{-1} \) locally uniformly in \( \mathcal{C} \setminus (\alpha_{\sigma}, \beta_{\sigma,c}) \), the first relation in (20) follows from (29), (36), and (40). Observe also that the last equality essentially does not change on \( \Delta_{\sigma} \) or \( \Delta_{\sigma,c} \setminus \bigcup_{b_{\sigma,c}} \), i.e., we simply need...
to replace the functions by their boundary values. This yields the second formula in (20). Finally, we get that

\[ \Phi_{12}^{(0)} = \sum_{m=1, n} \sum_{1 \leq k \leq \nu} \frac{1}{m+n} \left( 1 + \nu_{1; m+n}^{(1,1)} + \nu_{2; m+n}^{(1,2)} \right) + \sum_{m=1, n} \sum_{1 \leq k \leq \nu} \frac{1}{m+n} \left( 1 + \nu_{1; m+n}^{(2,1)} + \nu_{2; m+n}^{(2,2)} \right) \]

on \( \Delta_\mu \). The third relation in (20) now follow from (29), (18), (36), and (40). This finishes the proof of Theorem \( \delta \) since the uniformity of the estimates in the case \( b_{\gamma, c} \notin \mathcal{D}_{\gamma, c} \) follows from the fact \( S_{m+n} = S_{c} \) and \( \gamma_{k; m+n} = \gamma_{k} \) for all \( n \) large enough and therefore we do not need to use (36).

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Keldysh Institute of Applied Mathematics, Russian Academy of Science, Moscow, 125047, Russia
E-mail address: aptekaas@keldysh.ru

Pirogov Russian National Research Medical University, Ostrovitianov str. 1, Moscow, 117997, Russia
E-mail address: bogolub@gmail.com

Department of Mathematical Sciences, Indiana University-Purdue University Indianapolis, 402 North Blackford Street, Indianapolis, IN 46202
E-mail address: maxyatts@iupui.edu