

UNIVERSALITY FOR ENSEMBLES OF MATRICES WITH POTENTIAL THEORETIC WEIGHTS ON DOMAINS WITH SMOOTH BOUNDARY

CHRISTOPHER D. SINCLAIR AND MAXIM L. YATTSELEV

ABSTRACT. We investigate a two-dimensional statistical model of N charged particles interacting via logarithmic repulsion in the presence of an oppositely charged compact region K whose charge density is determined by its equilibrium potential at an inverse temperature corresponding to $\beta = 2$. When the charge on the region, s , is greater than N , the particles accumulate in a neighborhood of the boundary of K , and form a determinantal point process on the complex plane. We investigate the scaling limit, as $N \rightarrow \infty$, of the associated kernel in the neighborhood of a point on the boundary under the assumption that the boundary is sufficiently smooth. We find that the limiting kernel depends on the limiting value of N/s , and prove universality for these kernels. That is, we show that, the scaled kernel in a neighborhood of a point $\zeta \in \partial K$ can be succinctly expressed in terms of the scaled kernel for the closed unit disk, and the exterior conformal map which carries the complement of K to the complement of the closed unit disk. When $N/s \rightarrow 0$ we recover the universal kernel discovered by Lubinsky in [13].

1. INTRODUCTION

1.1. Potential Theoretic Weights. Let $K \subseteq \mathbb{C}$ be a compact subset whose boundary $T = \partial K$ is a Jordan curve. We will assume that T is sufficiently nice in a way that will be made precise in the sequel. For such K , there exists a unique measure ω_K , the *equilibrium measure* on K , that minimizes the energy functional $I[\sigma] := -\iint \log|z - u|d\sigma(u)d\sigma(z)$ among all positive probability measures σ supported on K [17]. We define $P_K : \mathbb{C} \rightarrow (0, \infty)$, by

$$(1) \quad P_K(z) := \exp \left\{ I[\omega_K] + \int \log|z - u|d\omega_K(u) \right\}.$$

This function is simply the rescaled exponentiated *equilibrium potential* of K .

It can be verified that ω_K is supported on T , P_K is identically one on K and, as $z \rightarrow \infty$, $P_K(z)/|z| \rightarrow \gamma_K^{-1}$, where $\gamma_K := \exp\{-I[\omega_K]\}$ is the *logarithmic capacity* of K .

In this paper we will be interested in random vectors whose joint density is given by

$$(2) \quad \Omega_N(\lambda) := \frac{1}{Z_N} \left\{ \prod_{n=1}^N \omega(\lambda_n) \right\} \prod_{m < n} |\lambda_n - \lambda_m|^2; \quad \lambda \in \mathbb{C}^N,$$

where

$$(3) \quad \omega(\lambda) = P_K(\lambda)^{-2s}, \quad Z_N := \int_{\mathbb{C}^N} \left\{ \prod_{n=1}^N \omega(\lambda_n) \right\} \prod_{m < n} |\lambda_n - \lambda_m|^2 dA^N(\lambda),$$

and $s > N$ (that is s is sufficiently large to guarantee that Z_N is finite). Here and throughout, A and A^N are Lebesgue measure on \mathbb{C} and \mathbb{C}^N respectively.

We will often refer to the components of such random vectors as eigenvalues, since the joint density (2) can be thought of as a modification of the joint eigenvalue density of the ensemble

Key words and phrases. normal matrix model, universality, scaling limits, exterior asymptotics, area orthogonality.

of matrices with i.i.d. complex Gaussian entries. The eigenvalues of this latter matrix ensemble, originally introduced by Ginibre [10], have joint density given by (2) where $w(\lambda) = e^{-|\lambda|^2}$.

In Section 3 we will give (i) a matrix model whose joint density of eigenvalues is given by (2) with weight given as in (3) for K equal to the closed unit disk, as well as (ii) models for more general K where the components of λ represent the positions of electrostatic particles confined to the plane and in the presence of a field determined by K , and (iii) an ensemble of random polynomials chosen with respect to a height function determined by K whose roots are distributed as in (2).

Our primary goal is to demonstrate that, in the double scaling limit as s and N approach infinity, the local statistics of the eigenvalues near a point on the boundary of K depend only on the limiting ratio of s and N , but are essentially independent of the specifics of K . This will follow from the asymptotic behavior of the reproducing kernel of $L^2(w)$, which in turn follows from the asymptotics of the leading coefficient of the related orthonormal polynomials. When $s = \infty$, w is simply the characteristic function of K and our results collapse to those given by Lubinsky [13] for the universality of reproducing kernels formed with respect to Bergman polynomials for K .

1.2. Eigenvalue Statistics. We briefly review some basic concepts for solvable ensembles of random matrices and how they relate to eigenvalue statistics. In this section we will assume that the joint density of eigenvalues is given by (2) where, for the purposes of this section, $w : \mathbb{C} \rightarrow [0, \infty)$ is any non-negative function such that $0 < Z_N < \infty$.

We will suppose that $\Xi = \{\xi_1, \xi_2, \dots, \xi_N\} \subset \mathbb{C}$ is a random set corresponding to the eigenvalues of a random matrix from our ensemble. (Or, what amounts to the same thing, Ξ is the set corresponding to a random vector sampled from the density Ω_N). Given a set $E \subseteq \mathbb{C}$ we may construct a random variable X given by the cardinality of $\Xi \cap E$. Given disjoint subsets E_1, E_2, \dots, E_n we will let X_1, X_2, \dots, X_n be the corresponding random variables. The n -th *correlation function* of our ensemble is defined to be $R_n : \mathbb{C}^n \rightarrow [0, \infty)$, where

$$E[X_1 X_2 \cdots X_n] := \int_{E_1} \int_{E_2} \cdots \int_{E_n} R_n(\lambda) dA^n(\lambda).$$

It is straightforward to see that $R_N = N! \Omega_N$. A less obvious exercise is to show that for $0 \leq n \leq N$,

$$(4) \quad R_n(\lambda) = \frac{1}{(N-n)!} \int_{\mathbb{C}^{N-n}} R_N(\lambda \vee \mathbf{x}) dA^{N-n}(\mathbf{x}),$$

where $\lambda \vee \mathbf{x} = (\lambda_1, \dots, \lambda_n, x_1, \dots, x_{N-n})$. Many probabilities of interest can be expressed in terms of correlation functions. One particularly important example is the *gap* probability that there are no eigenvalues in E ,

$$(5) \quad \text{Prob}\{X = 0\} = \sum_{n=0}^N \frac{(-1)^n}{n!} \int_{E^n} R_n(\lambda) dA^n(\lambda).$$

Equations (4) and (5) are valid for a wide variety of symmetric measures on \mathbb{C}^N . However, the presence of the square of the Vandermonde determinant which appears in (2) leads to additional structure which may be exploited.

Suppose $\pi_0, \pi_1, \dots, \pi_{N-1}$ are the orthonormal polynomials with respect to the weight w . That is,

$$\int_{\mathbb{C}} \pi_n(z) \overline{\pi_m(z)} w(z) dA = \delta_{nm},$$

where, as usual, δ_{nm} is 1 or 0 depending on whether or not $n = m$. The *kernel* of the ensemble is defined by

$$\tilde{K}_N(z, u) := \sqrt{w(z)w(u)} \sum_{n=0}^{N-1} \pi_n(z) \overline{\pi_n(u)}.$$

(Following Lubinsky's notation, we will reserve the symbol K_N for the unweighted analog of this kernel). In a celebrated result, Mehta and Gaudin [14] were able to express the correlation functions of ensembles with eigenvalue density (2) in terms of determinants of matrices formed from this kernel,

$$(6) \quad R_n(\lambda) = \det \left[\tilde{K}_N(\lambda_j, \lambda_k) \right]_{j,k=1}^n.$$

(See also [22] for a more modern derivation).

1.3. Universality. When N is large we expect that, with high probability, the eigenvalues will accumulate in a neighborhood of ∂K . Slightly more precisely, if $\zeta \in \partial K$, then the number of eigenvalues in a disk of (small) radius ϵ about ζ is proportional to N ; the constant of proportionality is given by the integral of the equilibrium measure over the arc of ∂K contained in the disk. The exact details of this phenomenon will be explored in a subsequent paper, for now we use this only as intuition to guess the proper scale on which we expect \tilde{K}_N to converge.

From (5) and (6), the probability that there are no eigenvalues in a disk of radius ϵ centered at ζ is given by

$$(7) \quad \sum_{n=0}^N \frac{(-1)^n}{n!} \int_{\mathbb{D}^n} \det \left[\epsilon^2 \tilde{K}_{N,s}(\zeta + \epsilon \lambda_j, \zeta + \epsilon \lambda_k) \right]_{j,k=1}^n dA^n(\lambda),$$

where \mathbb{D} is the disk of radius 1 centered at the origin. Here we have made explicit that the kernel is dependent on s as well as N .

Under the assumption that there are $\mathcal{O}(N)$ eigenvalues in a neighborhood of ζ , then we should scale ϵ like $1/N$ in order for (7) to approach a non-trivial limit. That is, the limiting gap probability of there being no eigenvalues in a shrinking neighborhood with radius $\epsilon = 1/N$ is given by

$$(8) \quad \lim_{N \rightarrow \infty} \sum_{n=0}^N \frac{(-1)^n}{n!} \int_{\mathbb{D}^n} \det \left[\frac{1}{N^2} \tilde{K}_{N,s} \left(\zeta + \frac{\lambda_j}{N}, \zeta + \frac{\lambda_k}{N} \right) \right]_{j,k=1}^n dA^n(\lambda).$$

Since $s > N$, this limit also depends on how s scales with N and we will assume that N/s converges to some $\ell \in [0, 1]$.

If it can be shown that there is some limiting kernel $\tilde{H}_{\zeta,\ell}$ so that

$$\frac{1}{N^2} \tilde{K}_{N,s} \left(\zeta + \frac{z}{N}, \zeta + \frac{u}{N} \right) \rightarrow \tilde{H}_{\zeta,\ell}(z, u)$$

uniformly on compact subsets of $\mathbb{C} \times \mathbb{C}$, then (8) converges to

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\mathbb{D}^n} \det \left[\tilde{H}_{\zeta,\ell}(\lambda_j, \lambda_k) \right]_{j,k=1}^n dA^n(\lambda).$$

(See for instance [1, §3.4]). Our primary result here is that $\tilde{H}_{\zeta,\ell}$ exists, and is dependent on ζ and K in only the most trivial manner. More specifically, we will express $\tilde{H}_{\zeta,\ell}$ in terms of the limiting kernel for the ensemble formed from the closed unit disk and the value of a conformal map from $\mathbb{C} \setminus K$ to $\mathbb{C} \setminus \mathbb{D}$ evaluated at ζ . This is what is called *universality* for potential theoretic ensembles.

We will also demonstrate that $\tilde{H}_{\zeta,\ell}$ is a convex combination of $\tilde{H}_{\zeta,0}$ (Lubinsky's limiting kernel) and $\tilde{H}_{\zeta,1}$.

1.4. Potential Theoretic Orthogonal Polynomials. We denote the orthonormal polynomials for the weight P_K^{-2s} , $s > 1$, by $\{\pi_{n,s}\}_{n=0}^{\lfloor s-2 \rfloor}$. That is, these are polynomials with positive leading coefficients that satisfy

$$(9) \quad \int_{\mathbb{C}} \pi_{n,s}(z) \overline{\pi_{m,s}(z)} P_K^{-2s}(z) dA = \delta_{nm}.$$

The reproducing kernel for this system of polynomials is given by

$$(10) \quad K_{N,s}(z, u) := \sum_{n=0}^{N-1} \pi_{n,s}(z) \overline{\pi_{n,s}(u)}, \quad N \leq \lfloor s-1 \rfloor,$$

with the weighted kernel given by

$$(11) \quad \tilde{K}_{N,s}(z, u) := P_K^{-s}(z) P_K^{-s}(u) K_{N,s}(z, u).$$

Our derivation of $\tilde{H}_{\zeta,\ell}$ will follow from the asymptotics of $K_{N,s}$, which in turn will follow from the asymptotics of the orthogonal polynomials. These latter asymptotics are of independent interest, and they provide the other primary results of the paper.

2. STATEMENT OF RESULTS

In what follows, we assume that $T := \partial K$ is a rectifiable Jordan curve which is either analytic or of class $C^{p+1,\alpha}$, where p is a nonnegative integer and $\alpha \in (0, 1)$. That is, the arclength function of T is p times continuously differentiable as a periodic function on the real line and its p -th derivative is α -Hölder continuous. Denote by Φ the conformal map of $O := \overline{\mathbb{C}} \setminus K$ onto $\mathbb{D} := \overline{\mathbb{C}} \setminus \mathbb{D}$ such that $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. In the case where T is an analytic Jordan curve we denote by $\rho(T) < 1$ a number such that Φ^{-1} has a univalent extension into $\{|w| > \rho(T)\}$. Moreover, we put $O_\rho := \Phi^{-1}(\{|w| > \rho\})$ for each $\rho > \rho(T)$.

It is known that $|\Phi|$ is identically equal to P_K on O and therefore $\Phi'(\infty) = \gamma_K^{-1}$. Hence, orthogonality relations (9) can be rewritten as

$$(12) \quad \int_D \pi_{n,s}(z) \overline{\pi_{m,s}(z)} dA + \int_O \pi_{n,s}(z) \overline{\pi_{m,s}(z)} |\Phi(z)|^{-2s} dA = \delta_{nm},$$

$n, m \leq \lfloor s-2 \rfloor$, where D is the interior domain of T . Since $|\Phi| > 1$ in O , we can formally set $\pi_{n,\infty}$ to be polynomials satisfying

$$\int_D \pi_{n,\infty}(z) \overline{\pi_{m,\infty}(z)} dA = \delta_{nm}.$$

In a sense, potential theoretic polynomials $\pi_{n,s}$ can be considered as perturbations of $\pi_{n,\infty}$. The latter were initially studied by Carleman [2] who derived their exterior asymptotics (asymptotics in O) for the case T being an analytic Jordan curve. The results in [2] were subsequently extended by Suetin [21] to include $C^{p+1,\alpha}$ Jordan curves. Other aspects of the behavior of $\pi_{n,\infty}$, such as zero distribution and interior asymptotics, were investigated in [15, 6, 7]. The following theorem provides an analog of [21, Theorem 1.2] for potential theoretic polynomials $\pi_{n,s}$.

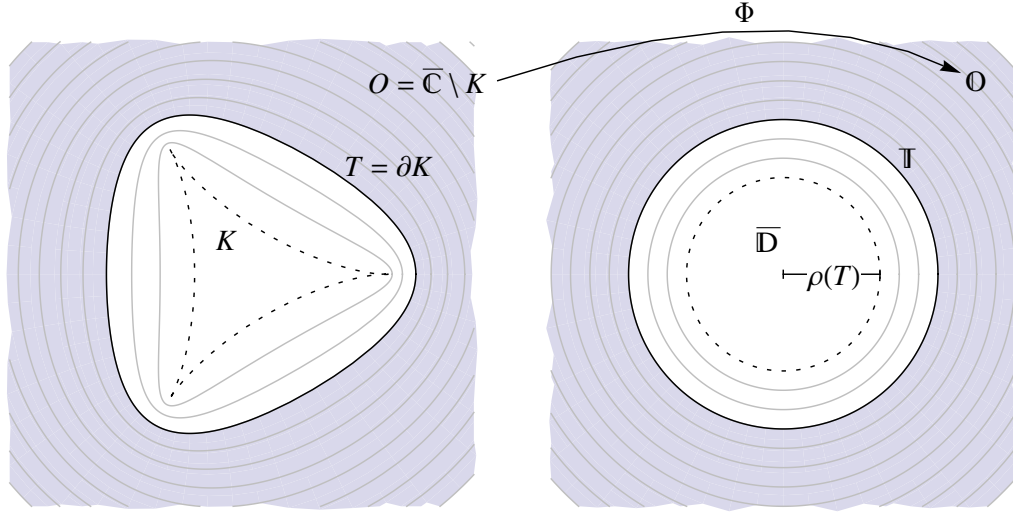


FIGURE 1. In this figure, K (respectively $\bar{\mathbb{D}}$) is the region enclosed by the black contour. Here, K has analytic boundary, and the dashed contour on the left is the inner-most contour outside of which we can find a univalent extension of Φ^{-1} . \bar{O}_ρ is represented by the region outside of the contour corresponding to radius ρ . The curves outside of T are level lines of P_K .

Theorem 1. *Let $T = \partial K$ be a Jordan curve of class $C^{p+1,\alpha}$, $p+\alpha > 1/2$, and $\{\pi_{n,s}\}_{n=0}^{\lfloor s-2 \rfloor}$ be a sequence of polynomials satisfying orthogonality relations (9). Then, as $n, s \rightarrow \infty$, the leading coefficient $\kappa_{n,s}$ of $\pi_{n,s}$ satisfies*

$$(13) \quad \kappa_{n,s} = \frac{1}{\gamma_K^{n+1}} \sqrt{\frac{n+1}{\pi} \left(1 - \frac{n+1}{s}\right)} \left[1 + \mathcal{O}\left(\frac{1}{n^{2(p+\alpha)}}\right)\right].$$

Moreover, if T is an analytic Jordan curve, then the error terms in (13) can be replaced by $\mathcal{O}(\rho^{2n})$ for any $\rho(T) < \rho < 1$. It also holds that

$$(14) \quad \pi_{n,s} = \sqrt{\frac{n+1}{\pi} \left(1 - \frac{n+1}{s}\right)} \Phi^n \Phi' [1 + \mathcal{O}(\Sigma_n)]$$

uniformly on \bar{O} as $n, s \rightarrow \infty$, where Σ_n is given by Table 1.

Remark 1.1. When $p = 0$ and $\limsup_{n,s \rightarrow \infty} n/s = 1$ the authors were unable to show that $\Sigma_n \rightarrow 0$ as $n \rightarrow \infty$ (the employed method yields $\Sigma_n = n^{2(1-\alpha)}$), which is the reason this case is not included in Table 1.

In general, the location of the zeros of $\pi_{n,s}$ depends on s (as well as, obviously, K and n). However, as the following proposition shows, this is not the case for a family of ellipses which interpolate between the unit circle and the interval $[-2, 2]$.

T	Analytic	$C^{p+1,\alpha}$			
	$\rho(T) < \rho < 1$	$p \geq 2$	$\limsup_{n,s \rightarrow \infty} n/s < 1$		$\limsup_{n,s \rightarrow \infty} n/s = 1$
			$p = 1$	$p = 0$	$p = 1$
Σ_n	ρ^n	$\frac{\log n}{n^{p+\alpha}}$	$\frac{\log n}{n^{1+\alpha}}$	$n^{1-2\alpha}$	$n^{-2\alpha}$

TABLE 1. The error term Σ_n depending on the smoothness of T .

Proposition 2. Let $q \in [0, 1)$ and define $\phi(w) := \Phi^{-1}(w) = w + \frac{q}{w}$ so that K is the ellipse bounded by $\phi(\mathbb{T})$. Then, for all $n \leq \lfloor s - 2 \rfloor$ and all s including $s = \infty$,

$$\pi_{n,s} = \sqrt{\frac{n+1}{\pi} \left(1 - \frac{n+1}{s}\right) / \left(1 - q^{2n+2} \frac{s-n-1}{s+n+1}\right)} \Phi^n \Phi' \left(1 - \frac{q^{n+1}}{\Phi^{2n+2}}\right).$$

That is, the polynomials $\pi_{n,s}$ are the renormalized Chebyshev polynomials of the second kind for the interval $[-2\sqrt{q}, 2\sqrt{q}]$, where $\pm 2\sqrt{q}$ are the foci of T .

Remark 2.1. In the proposition above all the ellipses have unit logarithmic capacity (i.e., $\gamma_K = 1$).

Remark 2.2. If $q = 0$, then $T = \mathbb{T}$ ($K = \mathbb{D}$) and

$$\pi_{n,s}(z) = \sqrt{\frac{n+1}{\pi} \left(1 - \frac{n+1}{s}\right)} z^n.$$

Remark 2.3. If $q = 1$, then $T = K = [-2, 2]$ and

$$(15) \quad \pi_{n,s}(z) = \sqrt{\frac{s^2 - (n+1)^2}{2\pi s} \frac{1}{\sqrt{z^2 - 4}}} \left[\left(\frac{z + \sqrt{z^2 - 4}}{2}\right)^{n+1} - \left(\frac{z - \sqrt{z^2 - 4}}{2}\right)^{n+1} \right].$$

It is easy to see that the asymptotic behavior of the normalizing constant in (15) is different from the one in (13). However, the case $K = [-2, 2]$ is not covered by Theorem 1.

Theorem 1 is the essential building block in proving results on asymptotic behavior of kernels $K_{N,s}$ and $\tilde{K}_{N,s}$ defined in (10) and (11), respectively.

Proposition 3. Let $N \leq \lfloor s - 1 \rfloor$. Under the conditions of Theorem 1, it holds that

$$(16) \quad \begin{aligned} K_{N,s}(z, w) &= \frac{\Phi'(z)\overline{\Phi'(w)}}{\pi} \left[\left(1 - \frac{N+1}{s}\right) \left(-(N+1) \frac{[\Phi(z)\overline{\Phi(w)}]^N}{1 - \Phi(z)\overline{\Phi(w)}} + \frac{1 - [\Phi(z)\overline{\Phi(w)}]^{N+1}}{[1 - \Phi(z)\overline{\Phi(w)}]^2} \right) \right. \\ &\quad \left. + \frac{1}{s} \left((N+2) \frac{1 + [\Phi(z)\overline{\Phi(w)}]^{N+1}}{[1 - \Phi(z)\overline{\Phi(w)}]^2} - 2 \frac{1 - [\Phi(z)\overline{\Phi(w)}]^{N+2}}{[1 - \Phi(z)\overline{\Phi(w)}]^3} \right) \right] \\ &\quad + \mathcal{O}(\max\{1, N^2 \Sigma_N\}) \end{aligned}$$

uniformly for $z, w \in \bar{O}$, $z \neq w$, $\text{dist}(z, \partial K) \leq \text{const.}/N$ and $\text{dist}(w, \partial K) \leq \text{const.}/N$. Moreover, it holds that

$$(17) \quad K_{N,s}(z, z) = \frac{|\Phi'(z)|^2}{\pi} \left[\frac{N(N+1)}{2} \left(1 - \frac{N+1}{s} \right) + \frac{N(N+1)(N+2)}{6s} \right] + \mathcal{O} \left(\max \{1, N^2 \Sigma_N\} \right)$$

uniformly for $z \in \partial K$.

To continue, denote by A_D^2 the Hilbert space of holomorphic functions on D whose moduli are square-integrable with respect to the area measure. We equip A_D^2 with the norm induced by the inner product

$$(18) \quad \langle f, g \rangle := \int_D f(z) \overline{g(z)} dA.$$

Denote by $K_D(z, w)$, $z, w \in D$, the reproducing kernel¹ for A_D^2 . That is,

$$(19) \quad f(z) = \int_D f(w) K_D(z, w) dA$$

for any $f \in A_D^2$. It is known [8, Theorem I.5.2] that $K_{N,\infty}$ is the reproducing kernel for the set of polynomials of degree at most $N-1$ in the sense of (19) and that

$$(20) \quad |K_D(z, w) - K_{N,\infty}(z, w)| \rightarrow 0 \quad \text{as } N \rightarrow \infty$$

locally uniformly for $z, w \in D$.

Theorem 4. *Under the conditions of Theorem 1, (20) holds with $K_{N,\infty}$ replaced by $K_{N,s}$ as $N, s \rightarrow \infty$, $N \leq [s-1]$.*

Remark 4.1. The original proof of Theorem 4 as devised by the authors used the full strength of Theorem 1 and therefore excluded the case $p = 0$ and $\ell = 1$. The authors are grateful to the anonymous referee who pointed out that only the asymptotics of Carleman polynomials $\pi_{n,\infty}$ on \bar{O} is needed to prove the theorem and thus allowing all $p \geq 0$ and $\ell \in [0, 1]$.

To describe the asymptotic behavior of kernels (10) and (11) near the boundary ∂K , it is convenient to introduce the following notation. Set

$$H_0(\tau) := 2 \frac{e^\tau(\tau-1)+1}{\tau^2} \quad \text{and} \quad H_1(\tau) := 6 \frac{e^\tau(\tau-2)+\tau+2}{\tau^3},$$

and define H_ℓ to be the convex combination,

$$(21) \quad H_\ell(\tau) := \frac{3-3\ell}{3-2\ell} H_0(\tau) + \frac{\ell}{3-2\ell} H_1(\tau), \quad \ell \in (0, 1).$$

Note that the value at the origin for each of these functions is determined by taking a limit; that is, $H_\ell(0) = 1$ for all $\ell \in [0, 1]$. The following theorem is an analog of [13, Theorem 2.1].

Theorem 5. *Let $N \leq [s-1]$, $z \in \partial K$, $\tau(a, z) := a\Phi'(z)\overline{\Phi(z)}$, $\ell := \lim_{N,s \rightarrow \infty} Ns^{-1} \in [0, 1]$ and for $\ell > 0$, set*

$$(22) \quad \omega(a, z) := \begin{cases} \exp\{-\text{Re}(\tau(a, z))/\ell\}, & \text{Re}(\tau(a, z)) > 0, \\ 1, & \text{otherwise.} \end{cases}$$

¹ $K_D(z, w) = \frac{1}{\pi} \frac{\psi'(z)\overline{\psi'(w)}}{(1-\psi(z)\overline{\psi(w)})^2}$, where ψ is any conformal map from D onto \mathbb{D} [8, §1.5].

Under the conditions of Theorem 1, assuming $p > 0$ when $\ell = 1$, it holds that

$$(23) \quad \lim_{N,s \rightarrow \infty} \frac{K_{N,s}(z + \frac{a}{N}, z + \frac{b}{N})}{K_{N,s}(z, z)} = H_\ell \left(\tau(a, z) + \overline{\tau(b, z)} \right).$$

Moreover, if $\ell > 0$, then

$$(24) \quad \lim_{N,s \rightarrow \infty} \frac{\tilde{K}_{N,s}(z + \frac{a}{N}, z + \frac{b}{N})}{\tilde{K}_{N,s}(z, z)} = \omega(a, z)\omega(b, z)H_\ell \left(\tau(a, z) + \overline{\tau(b, z)} \right),$$

and if $\ell = 0$, then

$$(25) \quad \lim_{N,s \rightarrow \infty} \frac{\tilde{K}_{N,s}(z + \frac{a}{N}, z + \frac{b}{N})}{\tilde{K}_{N,s}(z, z)} = \begin{cases} H_0 \left(\tau(a, z) + \overline{\tau(b, z)} \right) & \operatorname{Re}(\tau(a, z)), \operatorname{Re}(\tau(b, z)) < 0; \\ 0 & \text{otherwise.} \end{cases}$$

The convergence in (23)–(25) is uniform for a, b in compact subsets of \mathbb{C} and $z \in \partial K$.

Remark 5.1. The argument of $\tau(a, z)$ is equal to the angle between a and $\Phi(z)/\Phi'(z)$, the outward normal to T at z .

Remark 5.2. As is clear from (11), the function $\omega(a, z)$ is designed to describe the limit of $P_K^{-s}(z + a/N)$ as $N, s \rightarrow \infty$. This limit depends on whether or not the points $z + a/N$ belong to O for N large enough. The case $\operatorname{Re}(\tau(a, z)) = 0$ corresponds to the situation when the sequence $\{z + a/N\}$ approaches $z \in \partial K$ tangentially to the boundary. This does not cause a problem in (22) as this function is continuous with respect to a . However, when $\ell = 0$ formula (22) cannot be used as the limit is described by a discontinuous function of a and the convexity of the boundary ∂K at z starts to play a role.

Remark 5.3. Observe that by putting $s = \infty$ (that is, $\ell = 0$), formulae (16)–(23) specialize to the asymptotic formulae obtained in [13] for Carleman polynomials. Notice also that when $s = N + 1$ ($\ell = 1$), the first summands in (16)–(23) disappear and only the second ones remain. For general ℓ , formulae (16)–(23) turn out to be convex combination of these two extreme cases.

3. THREE MODELS OF POTENTIAL THEORETIC ENSEMBLES

Before proceeding to the proofs of our main results, we will present three models, a matrix model, an electrostatic model and a polynomial model, whose joint density of eigenvalues, particles and roots coincide with the potential theoretic ensembles we are considering.

3.1. Entropic Normal Matrix Ensembles. The *entropy* of a self-map T on a metric space X is a measure of how the distance between nearby points is stretched under iteration of T . In the case where \mathbf{Z} is an $N \times N$ complex matrix acting on \mathbb{C}^N , the entropy of \mathbf{Z} is given by

$$h(\mathbf{Z}) = \sum_{n=1}^N \log \max\{1, |\lambda_n|\},$$

where $\lambda_1, \lambda_2, \dots, \lambda_N$ are the eigenvalues of \mathbf{Z} [23]. We may use this to create a probability measure on normal $N \times N$ complex matrices, which we will denote by $\mathcal{N}_N(\mathbb{C})$.

There exists a canonical measure on $\mathcal{N}_N(\mathbb{C})$ induced by the standard metric on $\mathbb{C}^{N \times N}$ and we may define a probability density with respect to this measure by writing

$$P_N(\mathbf{Z}) = \frac{1}{Z_N} e^{-2s h(\mathbf{Z})},$$

where Z_N is a normalization constant and $s > N$ is a real number necessary so that the probability measure is actually finite.

This probability measure on normal matrices induces a symmetric probability measure on \mathbb{C}^N as identified with vectors of eigenvalues. This measure is absolutely continuous with respect to Lebesgue measure and its density is given as in (2) with $w(\lambda) = \max\{1, |\lambda|\}^{-2s}$ [4, 16]. Normal matrix ensembles, and in particular the statistics of their eigenvalues, were first considered in [3] and [4].

The function $\lambda \mapsto \log \max\{1, |\lambda|\}$ is the logarithmic (equilibrium) potential of the closed unit disk, and the weight for the entropic ensemble is formed from this in the obvious manner. We therefore see that the eigenvalue statistics of the entropic normal matrix ensemble coincides with the potential theoretic ensemble with $K = \overline{\mathbb{D}}$.

3.2. Two-Dimensional Electrostatics. In two-dimensional electrostatics, charged particles are identified with points in the extended complex plane. The potential energy of a system of two like charged particles located at $z, w \in \mathbb{C}$ is proportional to $-\log|z - w|$. More generally, if z_1, z_2, \dots, z_N are the locations of N identically charged particles, then \mathbf{z} determines the *state* of the system and the potential energy of this state is given by

$$-\sum_{m < n} \log|z_n - z_m|.$$

The energy is minimized when the particles are all at ∞ . In order for the system to be found in a state where the particles are at finite positions, there needs to be a potential (or other obstructions) which repels the particles from ∞ . We represent this field by V so that the interaction energy between a particle located at z and the field is given by $V(z)$. The total potential energy of the system comprised of the N particles in the field is given by

$$E(\mathbf{z}) = \sum_{n=1}^N V(z_n) - \sum_{m < n} \log|z_n - z_m|.$$

The system is assumed to be in contact with a heat reservoir so that the energy of the system is variable, but the temperature is fixed. In this setting, β denotes the reciprocal of the temperature, and the Boltzmann factor for the state \mathbf{z} is given by

$$e^{-\beta E(\mathbf{z})} = \left\{ \prod_{n=1}^N e^{-\beta V(z_n)} \right\} \prod_{m < n} |z_n - z_m|^\beta.$$

This quantity gives the relative density of states, so that the probability (density) of finding the system in state \mathbf{z} is given by

$$\frac{1}{Z_N} e^{-\beta E(\mathbf{z})} \quad \text{where} \quad Z_N = \int_{\mathbb{C}^N} e^{-\beta E(\mathbf{z})} dA^N(\mathbf{z}).$$

Comparing with (2) we see that, when $\beta = 2$ the density of states is identical with the density of eigenvalues of the normal matrix ensemble with weight $w(z) = e^{-2V(z)}$.

In this model, a compact set K is identified with a conducting region. A charge supported on K will distribute itself to minimize its potential energy, and this distribution, suitably normalized, leads to the equilibrium measure on K . In this way, we can think of the function $-s \log P_K(z)$ as the potential energy felt by an oppositely charged particle at z when placed in the field given by the minimal energy configuration formed by placing a total charge of s on K . In this situation where our system consists of N charged particles, the condition that $s > N$ is required to make ∞ repulsive (or rather to make K sufficiently attractive so that the particles do not flee to ∞).

It follows that the statistics of particles in this model agree with those of the potential theoretic ensemble for K .

3.3. Roots of Random Polynomials. The *Mahler measure* of a polynomial $f(x) \in \mathbb{C}[x]$ is given by

$$M(f) = \exp \left\{ \int_0^1 \log |f(e^{2\pi i \theta})| d\theta \right\},$$

is an example of *height function*; that is a function which measures the complexity of arithmetic objects, in this case polynomials². One type of problem of interest to number theorists is to provide asymptotic estimates for the number of arithmetic objects whose height is bounded by C as $C \rightarrow \infty$. For instance, for the Mahler measure, such estimates for the number of integer polynomials of fixed degree and Mahler measure bounded by C as $C \rightarrow \infty$ was given by Chern and Vaaler in [5]. They also gave a similar estimate for the number of polynomials with Gaussian integer ($\mathbb{Z}[i]$) coefficients.

In the latter case, the main term in their estimate came from the calculation of the Lebesgue measure of the set of polynomials of degree N with complex coefficients whose Mahler measure is at most 1. A key aspect of their proof is to show that this volume is equal to

$$\frac{\pi}{N+1} \int_{\mathbb{C}^N} \left\{ M \left(x^N + \sum_{n=1}^N a_n z^{N-n} \right) \right\}^{-2N-2} dA^N(\mathbf{a}),$$

That is the volume is proportional to an integral of a (negative) power of the Mahler measure of monic polynomials with respect to Lebesgue measure on the non-leading coefficients of such polynomials. Moreover, after the change of variables from coefficients to roots of polynomials, this volume reduces to

$$(26) \quad \frac{\pi}{N+1} \int_{\mathbb{C}^N} \left\{ \prod_{n=1}^N \exp \left\{ \int_0^1 \log |\alpha_n - e^{2\pi i \theta}| d\theta \right\}^{-2s} \right\} \prod_{m < n} |\alpha_n - \alpha_m|^2 dA^N(\boldsymbol{\alpha}); \quad s = N + 1.$$

That is, this volume, up to the factor of $\pi/(N+1)$ is equal to the normalization constant Z_N for the potential theoretic ensemble for the unit circle for the value $s = N + 1$. In fact, Chern and Vaaler were able to show that this normalization constant as a function of s is a rational function in s with poles at positive integers $\leq N$. This striking result can be seen as a consequence of determinantal nature of the correlation functions.

The derivation of (26) shows that the roots of a polynomial chosen randomly from the volume of complex polynomials of degree N and Mahler measure at most 1 obey the same statistics as those of the potential theoretic ensemble for the disk. This gives a polynomial model for these statistics.

The computation of the normalization constant of this polynomial model for potentials for certain other compact regions (in particular the ellipses considered in Proposition 2) is given in [19], while a more general treatment for more general potentials is given in [20]. The special case where the family of ellipses degenerates to the interval $[-2, 2]$ on the real axis, and its application to the estimation of counting reciprocal polynomials with bounded Mahler measure is given in [18] and [19].

²Traditionally Mahler measure is used as a height of polynomials in $\mathbb{Q}[x]$, or more generally $\overline{\mathbb{Q}}[x]$. However, there is no obstruction in defining it for polynomials in $\mathbb{C}[x]$

4. PROOFS

To prove Theorem 1, we use the method of normal moments in which we rely on the results in [21, Ch. 1]. We must therefore discuss Faber polynomials before proceeding to the proof of Theorem 1. We start by stating several auxiliary facts that will be useful later.

4.1. Auxiliary Facts. If g is holomorphic in \mathbb{O} and vanishes at ∞ , then for each $r \in [1, \infty)$ the restriction $\overline{g(r\tau)}$, $\tau \in \mathbb{T} := \{|\tau| = 1\}$, can be interpreted as the trace on \mathbb{T} of $\overline{g(r/\overline{w})}$, $w \in \mathbb{D}$, which is holomorphic and vanishes at the origin. The latter implies that $\int_{\mathbb{T}} \tau^k \overline{g(r\tau)} |d\tau| = 0$ for all integers $k \geq 0$. If, in addition, h is a positive function on $[1, \infty)$ and $\left| \int_{\mathbb{T}} \tau^j \overline{g(r\tau)} |d\tau| \right|$ is integrable with respect to dA for some $j \geq 0$, then the Fubini-Tonelli theorem yields that

$$(27) \quad \int_{\mathbb{O}} w^j \overline{g(w)} h(|w|) dA = \int_1^\infty \left[\int_{\mathbb{T}} \tau^j \overline{g(r\tau)} |d\tau| \right] r^{j+1} h(r) dr = 0.$$

Furthermore, since $dA(w) = |\Phi'(z)|^2 dA(z)$, where $w = \Phi(z)$, it holds that

$$(28) \quad \int_{\mathbb{O}} \Phi^j(z) \Phi'(z) \overline{G(z)} h(|\Phi(z)|) dA = 0$$

for any G holomorphic in O and vanishing at ∞ by (27) applied with $g = (G \circ \phi) \phi'$, where ϕ is the inverse of Φ (granted $|\Phi^j \Phi' G h(|\Phi|)|$ is integrable with respect to dA).

In another connection, the Cauchy-Green identity for the domain D [12, Thm. 1.2.1] says that

$$(29) \quad \int_D g(z) \overline{h'(z)} dA = \frac{1}{2i} \oint_T g(z) \overline{h(z)} dz$$

whenever g and h' are holomorphic functions in D that continuously extend to T , where \oint always means integration in the counter-clockwise direction unless specified otherwise. Now, assume that g and h are holomorphic functions in O such that g has at least a double zero at infinity, and g, h and h' continuously extend to T . Then by using the transformation $z \mapsto 1/z$ and (29), one can show that the Cauchy-Green identity for O assumes the form

$$(30) \quad \int_{\mathbb{O}} g(z) \overline{h'(z)} dA = -\frac{1}{2i} \oint_T g(z) \overline{h(z)} dz.$$

4.2. Faber Polynomials. Denote by F_n the n -th Faber polynomial for D associated with Φ' . That is,

$$F_n(z) = \oint_T \frac{\Phi^n(t) \Phi'(t)}{t-z} \frac{dt}{2\pi i}, \quad z \in D.$$

In other words, F_n is the polynomial part of $\Phi^n \Phi'$. Then it follows from Plemelj-Sokhotski formulae [9] that

$$(31) \quad F_n = \Phi^n \Phi' + E_n \quad \text{in } O,$$

where E_n is a holomorphic function vanishing at infinity with integral representation

$$(32) \quad E_n(z) := \oint_T \frac{\Phi^n(t) \Phi'(t)}{t-z} \frac{dt}{2\pi i}, \quad z \in O.$$

We would like to point out that

$$(33) \quad E_n(z) = \mathcal{O}\left(\frac{1}{z^2}\right) \quad \text{as } z \rightarrow \infty$$

for all integers $n \geq 0$. Indeed, consider \tilde{F}_{n+1} , the $(n+1)$ -st Faber polynomial associated with 1. In this case (31) gets replaced by $\tilde{F}_{n+1} = \Phi^{n+1} + \tilde{E}_{n+1}$, where \tilde{E}_{n+1} has an integral representation similar to (32). By differentiating both sides of the last equality, we get that $(n+1)F_n = \tilde{F}'_{n+1}$ and $(n+1)E_n = \tilde{E}'_{n+1}$. As \tilde{E}_{n+1} is holomorphic and vanishing at infinity, E_n has at least a double zero there.

We are interested in the asymptotic behavior of

$$(34) \quad m_{k,j}^s := \int_{\mathbb{C}} F_j \overline{F_k} P_K^{-2s} dA = \int_D F_j \overline{F_k} dA + \int_O F_j \overline{F_k} |\Phi|^{-2s} dA, \quad j, k \leq [s-2],$$

where we used (12) for the second representation.

It was shown in [21, Equation (1.32) combined with (30) above] that the first integral on the right-hand side of (34) can be written as

$$(35) \quad \frac{\pi}{k+1} \left(\delta_{jk} - \frac{k+1}{\pi} \int_O E_j \overline{E_k} dA \right) =: \frac{\pi}{k+1} (\delta_{kj} + I_D),$$

where the integral over O is well defined (finite) by (33). Moreover, it was also obtained there, see [21, Equation (1.45) and Lemma 1.5], that

$$(36) \quad |I_D| \leq \frac{\text{const.}}{(j+1)^{p+\alpha}(k+1)^{p+\alpha}} \quad \text{or} \quad |I_D| \leq \text{const.} \rho^{j+k},$$

where both constants are independent of j and k , but depend on T and ρ (in the analytic case). Hereafter, by stating a double estimate of the form (36), we always assume that the first bound is given for T of class $C^{p+1,\alpha}$ and the second one for T analytic with $\rho(T) < \rho < 1$.

On the other hand, the second integral on the right-hand side of (34) can be written with the help of (31) as

$$(37) \quad \int_O \Phi^j \overline{\Phi^k} |\Phi'|^2 |\Phi|^{-2s} dA + \int_O \Phi^j \overline{\Phi^k} \overline{E_k} |\Phi|^{-2s} dA + \int_O E_j \overline{\Phi^k} \Phi' |\Phi|^{-2s} dA + \int_O E_j \overline{E_k} |\Phi|^{-2s} dA.$$

It can be immediately computed by conformality of Φ that the first integral in (37) is equal to

$$\int_O \Phi^j \overline{\Phi^k} |\Phi'|^2 |\Phi|^{-2s} dA = \int_{\mathbb{O}} \omega^j \overline{\omega^k} |\omega|^{-2s} dA = \frac{\pi}{s-(k+1)} \delta_{kj}.$$

The second integral in (37) is zero by (28) applied with $G = E_k$ and $h(r) = r^{-2s}$, $r \in [1, \infty)$. Conjugating the third integral in (37), analogous reasoning shows it is zero as well. Thus, similar to (35), the second integral on the right-hand side of (34) can be written as

$$(38) \quad \frac{\pi}{s-(k+1)} \left(\delta_{jk} + \frac{s-(k+1)}{\pi} \int_O E_j \overline{E_k} |\Phi|^{-2s} dA \right) =: \frac{\pi}{s-(k+1)} (\delta_{kj} + I_O).$$

We claim that $|I_O|$ satisfies (36) as well, namely,

$$(39) \quad |I_O| \leq \frac{\text{const.}}{(j+1)^{p+\alpha}(k+1)^{p+\alpha}} \quad \text{or} \quad |I_O| \leq \text{const.} \rho^{j+k}.$$

Indeed, to prove (36), it was shown in [21, (1.44) and the following paragraph, (1.45)] that

$$(40) \quad \int_{\mathbb{T}} |(E_n \circ \phi)(\tau) \phi'(\tau)|^2 |d\tau| \leq \frac{\text{const.}}{(n+1)^{2p+2\alpha}} \quad \text{or} \quad \leq \text{const.} \rho^{2n}$$

for all integers $n \geq 0$, where ϕ is the inverse of Φ . Moreover, the monotonicity of L^2 -norms implies that if the integrand in (40) is evaluated at $r\tau$ rather than at τ for any fixed $r \in (1, \infty)$, the estimate remains valid. Then on account of

$$\left| \int_O E_j \overline{E_k} |\Phi|^{-2s} dA \right| \leq \int_1^\infty \left[\int_{\mathbb{T}} |(E_j E_k) \circ \phi)(r\tau)| |\phi'(r\tau)|^2 |d\tau| \right] r^{1-2s} dr$$

and the Cauchy-Schwarz inequality, $|I_O|$ is bounded by

$$\frac{s - (k+1)}{\pi} \left(\int_1^\infty r^{1-2s} dr \right) \left(\int_{\mathbb{T}} |(E_k \circ \phi)(\tau) \phi'(\tau)|^2 |d\tau| \right)^{1/2} \left(\int_{\mathbb{T}} |(E_j \circ \phi)(\tau) \phi'(\tau)|^2 |d\tau| \right)^{1/2}.$$

Clearly, (39) follows now from (40).

Finally, gathering together (35) and (38), we get that

$$(41) \quad m_{k,j}^s = \frac{s\pi}{(k+1)(s-(k+1))} \left(\delta_{kj} + \epsilon_{k,j}^s \right),$$

where

$$(42) \quad \epsilon_{k,j}^s := \frac{s-(k+1)}{s} I_D + \frac{k+1}{s} I_O = -\frac{k+1}{\pi} \left(1 - \frac{k+1}{s} \right) \int_O E_j \overline{E_k} (1 - |\Phi|^{-2s}) dA$$

and

$$(43) \quad |\epsilon_{k,j}^s| \leq \frac{\text{const.}}{(j+1)^{p+\alpha}(k+1)^{p+\alpha}} \quad \text{or} \quad |\epsilon_{k,j}^s| \leq \text{const.} \rho^{j+k}$$

by (36) and (39).

4.3. The Von Koch-Riesz Algebra. Denote by \mathcal{D} the algebra of all operators defined on $\ell_2(\mathbb{N})$ by matrices $\mathbf{A} = [a_{k,j}]_{j,k=0}^\infty$ with respect to the standard basis for which

$$\|\mathbf{A}\|_{\mathcal{D}} := \max \left\{ \sum_{k=0}^\infty |a_{k,k}|, \left(\sum_{j=0}^\infty \sum_{k=0}^\infty |a_{k,j}|^2 \right)^{1/2} \right\} < \infty.$$

It is known [11, Theorem II.2.1] that if $\{\mathbf{A}_n\}$ is a sequence in \mathcal{D} converging to $\mathbf{A} \in \mathcal{D}$ (with respect to $\|\cdot\|_{\mathcal{D}}$), then the determinant of $\mathbf{I} + \mathbf{A}_n$ (\mathbf{I} being the identity operator) converges to the determinant of $\mathbf{I} + \mathbf{A}$.

Let $\{s_n\}$ be an increasing sequence of positive reals such that $s_n \rightarrow \infty$ as $n \rightarrow \infty$. Set, for convenience, $\epsilon_{k,j}^{s_n} := 0$ when either j or k is greater than $\lfloor s_n - 2 \rfloor$ and define $\mathbf{E}_{s_n} := [\epsilon_{k,j}^{s_n}]_{j,k=0}^\infty$. We also set $\mathbf{E}_\infty := [\epsilon_{k,j}^\infty]_{j,k=0}^\infty$, where we put

$$(44) \quad \epsilon_{k,j}^\infty := -\frac{k+1}{\pi} \int_O E_j \overline{E_k} dA.$$

Observe that the estimate in (43) is also valid for $s = \infty$. Using this bound, it is simple to verify that

$$\|\mathbf{E}_{s_n}\|_{\mathcal{D}} \leq \text{const.} \sum_{k=1}^\infty \frac{1}{k^{2(p+\alpha)}} \quad \text{or} \quad \|\mathbf{E}_{s_n}\|_{\mathcal{D}} \leq \frac{\text{const.}}{1-\rho^2}$$

for each n including the case $n = \infty$, where the constant $\sum_{k=1}^\infty k^{-2(p+\alpha)}$ is finite as $p + \alpha > 1/2$. Thus, all the operators \mathbf{E}_{s_n} belong to the Von Koch-Riesz algebra \mathcal{D} . Moreover, it holds that

$$(45) \quad \|\mathbf{E}_{s_n} - \mathbf{E}_\infty\|_{\mathcal{D}} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

Indeed, let $\{k_n\}$ be a non-decreasing sequence of integers such that $k_n \rightarrow \infty$ and $k_n/n \rightarrow 0$ as $n \rightarrow \infty$. Then

$$(46) \quad \sum_{k=k_n}^{\infty} \left| \epsilon_{k,k}^{s_n} - \epsilon_{k,k}^{\infty} \right| \leq \sum_{k=k_n}^{\infty} \left(|\epsilon_{k,k}^{s_n}| + |\epsilon_{k,k}^{\infty}| \right) \leq \frac{\text{const.}}{(k_n + 1)^{p+\beta}} \quad \text{or} \quad \sum_{k=k_n}^{\infty} \left| \epsilon_{k,k}^{s_n} - \epsilon_{k,k}^{\infty} \right| \leq \text{const.} \rho^{2k_n}$$

by (43). Furthermore, we can readily deduce from (42) and (44) using the notation of (35) and (38) that

$$\left| \epsilon_{k,k}^{s_n} - \epsilon_{k,k}^{\infty} \right| = \frac{k+1}{n} |I_O - I_D| \leq \frac{k_n}{n} \frac{\text{const.}}{(k+1)^{2(p+\alpha)}} \quad \text{or} \quad \left| \epsilon_{k,k}^{s_n} - \epsilon_{k,k}^{\infty} \right| \leq \frac{k_n}{n} \text{const.} \rho^{2k}$$

by (36) and (39) for all $k \in \{0, \dots, k_n - 1\}$. Therefore, it holds that

$$(47) \quad \sum_{k=0}^{k_n-1} \left| \epsilon_{k,k}^{s_n} - \epsilon_{k,k}^{\infty} \right| \leq \text{const.} \frac{k_n}{n}.$$

Combining (46) and (47), we deduce that

$$(48) \quad \sum_{k=0}^{\infty} \left| \epsilon_{k,k}^{s_n} - \epsilon_{k,k}^{\infty} \right| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$

by the choice of the sequence $\{k_n\}$. Analogously, one can show that

$$(49) \quad \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left| \epsilon_{k,j}^{s_n} - \epsilon_{k,j}^{\infty} \right|^2 \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,$$

which finishes the proof of (45).

Naturally [11, Section I.1], it holds that $\det(\mathbf{I} + \mathbf{E}_{s_n}) = \det \left[\delta_{kj} + \epsilon_{k,j}^{s_n} \right]_{j,k=0}^{\lfloor s_n-2 \rfloor}$ and therefore we deduce from the remark made at the beginning of this section that

$$(50) \quad \det(\mathbf{I} + \mathbf{E}_{s_n}) \rightarrow \det(\mathbf{I} + \mathbf{E}_{\infty}) > 0 \quad \text{as} \quad n \rightarrow \infty,$$

where the last inequality was shown in [21, Section I.4].

4.4. Proof of Theorem 1. Since $\{F_n\}$ is a complete system of polynomials, each $\pi_{n,s}$ can be expressed as a linear combination of F_0, \dots, F_n with the coefficients determined via the orthogonality relations (9). In fact, it holds that

$$(51) \quad \pi_{n,s}(z) = \frac{1}{\sqrt{D_{n-1,s} D_{n,s}}} \begin{bmatrix} m_{0,0}^s & m_{0,1}^s & \dots & m_{0,n}^s \\ m_{1,0}^s & m_{1,1}^s & \dots & m_{1,n}^s \\ \vdots & \vdots & \ddots & \vdots \\ m_{n-1,0}^s & m_{n-1,1}^s & \dots & m_{n-1,n}^s \\ F_0(z) & F_1(z) & \dots & F_n(z) \end{bmatrix},$$

where the moments $m_{k,j}^s$ are defined in (34) and $D_{n,s} := \det[m_{k,j}^s]_{j,k=0}^n$.

Set $\Delta_{n,s} := \det[\delta_{kj} + \epsilon_{k,j}^s]_{j,k=0}^n$ and observe that

$$(52) \quad D_{n,s} = \Delta_{n,s} \prod_{k=0}^n \frac{s\pi}{(k+1)(s-(k+1))} \quad \text{and} \quad D_{n,s}(j) = \Delta_{n,s}(j) \prod_{k=0, k \neq j}^n \frac{s\pi}{(k+1)(s-(k+1))}$$

by (41), where the determinants $D_{n,s}(j)$ and $\Delta_{n,s}(j)$ are obtained from the same matrices as $D_{n,s}$ and $\Delta_{n,s}$ only with the last row and the $(j+1)$ -st column removed. Given (43), it is a straightforward algebraic computation using Hadamard's inequality, see [21, Lemma 1.7], to derive that

$$(53) \quad \Delta_{n,s}(j) \leq \frac{\text{const.}}{(j+1)^{p+\beta}(n+1)^{p+\beta}} \quad \text{or} \quad \Delta_{n,s}(j) \leq \text{const.} \rho^{j+n}$$

for any $j \in \{0, 1, \dots, n-1\}$.

On the other hand, the family $\{\Delta_{n,s}\}$ is bounded away from zero. Indeed, as mentioned just before (12), $\Phi'(\infty) = \gamma_K^{-1}$ and therefore the leading coefficient of F_n is equal to γ_K^{-n-1} . Hence, we get from (51) and (52) that

$$(54) \quad \kappa_{n,s} \gamma_K^{n+1} = \sqrt{\frac{D_{n-1,s}}{D_{n,s}}} = \sqrt{\frac{n+1}{\pi} \left(1 - \frac{n+1}{s}\right) \frac{\Delta_{n-1,s}}{\Delta_{n,s}}}.$$

Recall that any monic orthogonal polynomial has the smallest L^2 -norm with respect to the weight of orthogonality among all monic polynomials of the same degree. In particular,

$$\frac{1}{\kappa_{n,s}^2} = \int_{\mathbb{C}} |\pi_{n,s}/\kappa_{n,s}|^2 |P_K|^{-2s} dA \leq \int_{\mathbb{C}} |\gamma_K^{n+1} F_n|^2 |P_K|^{-2s} dA = \gamma_K^{2n+2} m_{n,n}^s.$$

Therefore, it follows from (54), (41), and (42) that

$$\begin{aligned} \Delta_{n,s} &= \frac{n+1}{\pi} \left(1 - \frac{n+1}{s}\right) \frac{\Delta_{n-1,s}}{\gamma_K^{2n+2} \kappa_{n,s}^2} \leq \frac{n+1}{\pi} \left(1 - \frac{n+1}{s}\right) m_{n,n}^s \Delta_{n-1,s} \\ &= (1 + \epsilon_{n,n}^s) \Delta_{n-1,s} < \Delta_{n-1,s}. \end{aligned}$$

Hence, it holds that

$$(55) \quad \inf_s \min_{1 \leq n \leq [s-2]} \Delta_{n,s} = \inf_s \Delta_{[s-2],s} > 0$$

by (50) since $\Delta_{[s-2],s} = \det(\mathbf{I} + \mathbf{E}_s)$, which proves the claim.

Thus, expanding the determinant $\Delta_{n,s}$ by the last row, we get that

$$(56) \quad \Delta_{n,s} = (1 + \epsilon_{n,n}^s) \Delta_{n-1,s} + \sum_{j=0}^{n-1} (-1)^{n+j} \epsilon_{n,j}^s \Delta_{n,s}(j).$$

Dividing both sides of the equality above by $\Delta_{n-1,s}$ and using (43), (53), and (55) yields

$$(57) \quad \frac{\Delta_{n,s}}{\Delta_{n-1,s}} = 1 + \mathcal{O}\left(\frac{1}{n^{2(p+\alpha)}}\right) \quad \text{or} \quad \frac{\Delta_{n,s}}{\Delta_{n-1,s}} = 1 + \mathcal{O}(\rho^{2n}).$$

Clearly, we get (13) by taking the reciprocal of (57) and substituting it into (54).

Now, expanding the determinant in (51) by the last row as in (56) yields

$$\pi_{n,s} = \sqrt{\frac{n+1}{\pi} \left(1 - \frac{n+1}{s}\right)} \sqrt{\frac{\Delta_{n-1,s}}{\Delta_{n,s}}} \left(F_n + \sum_{j=0}^{n-1} (-1)^{n+j} \frac{(j+1)(s-j-1)}{(n+1)(s-n-1)} \frac{\Delta_{n,s}(j)}{\Delta_{n-1,s}} F_j \right).$$

Hence, by factoring out $\Phi^n \Phi'$ and using (31), the error term in (14) can be written as

$$\sqrt{\frac{\Delta_{n-1,s}}{\Delta_{n,s}}} - 1 + \sqrt{\frac{\Delta_{n-1,s}}{\Delta_{n,s}}} \left[\frac{E_n}{\Phi^n \Phi'} + \sum_{j=0}^{n-1} (-1)^{n+j} \frac{(j+1)(1 - \frac{j+1}{s})}{(n+1)(1 - \frac{n+1}{s})} \frac{\Delta_{n,s}(j)}{\Delta_{n-1,s}} \left(1 + \frac{E_j}{\Phi^j \Phi'}\right) \frac{1}{\Phi^{n-j}} \right].$$

Since $|\Phi| > 1$ and $|\Phi'|$ is bounded away from zero in O , we get from (40), (53), (55), and (57) that the error term in (14) is of order

$$(58) \quad \frac{\log(n+1)}{(n+1)^{p+\alpha}} + \frac{1}{(n+1)^{p+\alpha}} \sum_{j=0}^{n-1} \frac{(j+1)(1-\frac{j+1}{s})}{(n+1)(1-\frac{j+1}{s})} \frac{1}{(j+1)^{p+\alpha}}.$$

If $\limsup_{n,s \rightarrow \infty} n/s < 1$, then the fractions $(1-\frac{j+1}{s})/(1-\frac{j+1}{s})$ are uniformly bounded above and it easily follows from (58) that the error term in (14) is of order

$$(59) \quad \frac{\log(n+1)}{(n+1)^{p+\alpha}} + \frac{1}{(n+1)^{2(p+\alpha)-1}},$$

where the first summand is larger for all $p \geq 1$ and the second one is larger when $p = 0$. Clearly, the estimate for the error term in the case of analytic curve, can be derived in a similar fashion. On the other hand, if $\limsup_{n,s \rightarrow \infty} n/s = 1$, then we use the estimate

$$1 - \frac{n+1}{s} \geq 1 - \frac{n+1}{n+2} = \frac{1}{n+2},$$

which is valid since $n \leq \lfloor s-2 \rfloor$. In this case (59) gets replaced by

$$\frac{\log(n+1)}{(n+1)^{p+\alpha}} + \frac{1}{(n+1)^{2(p+\alpha)-2}},$$

where the first summand is larger for all $p \geq 2$ and the second one is larger when $p = 1$ (we exclude $p = 0$ as in this case the above bound grows as $n^{2(1-\alpha)}$). Analogous estimate shows that the error term in (14) is of order ρ^{-n} when T is an analytic curve. This finishes the proof of Theorem 1. \square

4.5. Proof of Proposition 2. Let U_n be the monic Chebyshev polynomial of the second kind for the interval $[-2\sqrt{q}, 2\sqrt{q}]$. That is,

$$U_n = \Phi^n \Phi' \left(1 - \frac{q^{n+1}}{\Phi^{2n+2}} \right), \quad \Phi(z) = \frac{z + \sqrt{z^2 - 4q}}{2}, \quad z \in O.$$

It can be readily checked that the inverse of Φ is indeed $\phi(w) = w + q/w$, Φ is the conformal map of the complement of $[-2\sqrt{q}, 2\sqrt{q}]$ onto $\{w : |w| > \sqrt{q}\}$ with positive derivative at infinity, and the level lines of Φ are ellipses with foci $\pm 2\sqrt{q}$.

Let us show that polynomials U_n are orthogonal on D with respect to area measure. It follows from the Cauchy-Green identity (29) that

$$(60) \quad \begin{aligned} 2i(k+1) \int_D U_n(z) \overline{z^k} dA &= \oint_T (\Phi^n(z) - q^{n+1}/\Phi^{n+2}(z)) \overline{\Phi'(z) z^{k+1}} dz \\ &= \oint_{\mathbb{T}} (\tau^n - q^{n+1}/\tau^{n+2}) (\tau q + 1/\tau)^{k+1} d\tau = 0 \end{aligned}$$

for $k < n$, where we used the identity $\tau = 1/\bar{\tau}$ on \mathbb{T} and the last equality is a consequence of the facts $\oint_{\mathbb{T}} \tau^n \tau^j d\tau = 0$ for all $j \geq -n$ and $\oint_{\mathbb{T}} \tau^{-n-2} \tau^j d\tau = 0$ for all $j \leq n$.

In another connection, it holds that

$$\begin{aligned} \int_O U_n(z) \overline{z^k} |\Phi(z)|^{-2s} dA &= \int_O (\Phi^n(z) - q^{n+1}/\Phi^{n+2}(z)) \overline{(z^k/\Phi'(z))} |\Phi(z)|^{-2s} |\Phi'(z)|^2 dA \\ &= \int_O (w^n - q^{n+1}/w^{n+2}) \overline{\phi^k(w) \phi'(w)} |w|^{-2s} dA. \end{aligned}$$

It is easy to check using the expressions $\phi(w) = w + q/w$ and $w = r\tau$, $r \in [1, \infty)$ and $\tau \in \mathbb{T}$, that the chain of equalities above can be continued as

$$(61) \quad \int_1^\infty \left[\int_{\mathbb{T}} \left((r\tau)^n - \frac{q^{n+1}}{(r\tau)^{n+2}} \right) \left(\frac{r}{\tau} + \frac{q\tau}{r} \right)^k \left(1 - \frac{q\tau^2}{r^2} \right) |d\tau| \right] \frac{rdr}{r^{2s}} = 0$$

for $k < n$, since the Laurent polynomials in τ integrated over \mathbb{T} does not contain a constant term as the exponents of τ range from $n+k+2$ down to $n-k$ and then from $k-n$ down to $-k-n$.

Altogether, the polynomials U_n are orthogonal over \mathbb{C} with respect to the measure $P_K^{-2s} dA$. In fact, it can be easily shown that they are also the Faber polynomials for this K . It remains to compute the normalizing factor. Evaluating as in (60) and (61), we get that

$$\begin{aligned} \int_D U_n(z) \overline{z^n} dA &= \frac{1}{2i} \frac{1}{n+1} \oint_{\mathbb{T}} \left(w^n - \frac{q^{n+1}}{w^{n+2}} \right) \left((qw)^{n+1} + \dots + \frac{1}{w^{n+1}} \right) dw \\ &= \frac{\pi}{n+1} (1 - q^{2n+2}) \end{aligned}$$

and

$$\begin{aligned} \int_O U_n(z) \overline{z^n} |\Phi(z)|^{-2s} dA &= \int_1^\infty \left[\int_{\mathbb{T}} \left((r\tau)^n - \frac{q^{n+1}}{(r\tau)^{n+2}} \right) \left(-\frac{q^{n+1}\tau^{n+2}}{r^{n+2}} + \dots + \frac{r^n}{\tau^n} \right) |d\tau| \right] \frac{rdr}{r^{2s}} \\ &= \pi \left(\frac{1}{s-(n+1)} + \frac{q^{2n+2}}{s+n+1} \right). \end{aligned}$$

Thus, we deduce that

$$\begin{aligned} \kappa_{n,s}^{-2} &= \int_{\mathbb{C}} |U_n(z)|^2 P_K^{-2s}(z) dA = \int_D U_n(z) \overline{z^n} dA + \int_O U_n(z) \overline{z^n} |\Phi(z)|^{-2s} dA \\ &= \frac{\pi s}{(n+1)(s-n-1)} \left(1 - q^{2n+2} \frac{s-n-1}{s+n+1} \right). \quad \square \end{aligned}$$

4.6. Proof of Proposition 3. Fix $N \leq \lfloor s-1 \rfloor$ and let ζ be a point such that

$$(62) \quad \zeta \in \overline{O} \quad \text{and} \quad \text{dist}(\zeta, K) \leq c/N$$

for some fixed constant c . Further, let $\zeta_0 \in T$ be such that $|\zeta - \zeta_0| = \text{dist}(\zeta, K)$. Since $|\Phi(\zeta_0)| = 1$ and Φ is continuously differentiable in \overline{O} (since T is at least $C^{1,\alpha}$ -smooth) it holds that

$$|\Phi(\zeta)| \leq 1 + |\Phi(\zeta) - \Phi(\zeta_0)| \leq 1 + \mathcal{O}(|\zeta - \zeta_0|) = 1 + \mathcal{O}(N^{-1}),$$

where the estimate $\mathcal{O}(\cdot)$ does not depend on the choice of ζ satisfying (62). Hence,

$$(63) \quad \max_{k \in \{1, \dots, N\}} |\Phi^k(\zeta)| \leq \text{const.}$$

for some absolute constant.

Select z, w satisfying (62) and assume that $z \neq w$. Put, for brevity, $u := \Phi(z) \overline{\Phi(w)}$. Then we get from the definition of $K_{N,s}$, (14), and (63) that

$$\begin{aligned} K_{N,s}(z, w) &= \frac{\Phi'(z) \overline{\Phi'(w)}}{\pi} \sum_{n=0}^{N-1} \left((n+1) - \frac{(n+1)^2}{s} \right) u^n [1 + \mathcal{O}(\Sigma_n)] \\ (64) \quad &= \frac{\Phi'(z) \overline{\Phi'(w)}}{\pi} \left(\sum_{n=0}^{N-1} (n+1) u^n - \frac{1}{s} \sum_{n=0}^{N-1} (n+1)^2 u^n \right) + \mathcal{O}(\max\{1, N^2 \Sigma_N\}). \end{aligned}$$

Since,

$$\sum_{n=0}^{N-1} (n+1)u^n = -(N+1)\frac{u^N}{1-u} + \frac{1-u^{N+1}}{(1-u)^2},$$

and

$$\sum_{n=0}^{N-1} (n+1)^2 u^n = -(N+1)^2 \frac{u^N}{1-u} + (N+1) \frac{1-u^{N+1}}{(1-u)^2} - (N+2) \frac{1+u^{N+1}}{(1-u)^2} + 2 \frac{1-u^{N+2}}{(1-u)^3},$$

the validity of (16) follows. In a similar but simpler fashion, we also get (17).

4.7. Proof of Theorem 4. Recall the Christoffel variational principle:

$$(65) \quad K_{N,s}(z, z) = \max_{\deg(p) < N} \frac{|p(z)|^2}{\int_{\mathbb{C}} |p|^2 P_K^{-2s} dA}, \quad z \in \mathbb{C},$$

and the reproducing property:

$$(66) \quad p(z) = \int_{\mathbb{C}} p(w) K_{N,s}(z, w) P_K^{-2s}(w) dA, \quad \deg(p) < N,$$

that hold for all $s \in (1, \infty]$ and $N \leq [s - 1]$. It can be readily deduced from (65) that

$$(67) \quad K_{N,s}(z, z) \leq K_{N,\infty}(z, z) \leq K_D(z, z),$$

where the second inequality follows from the fact that $K_D(z, w) = \sum_{n=0}^{\infty} \overline{\pi_{n,\infty}(w)} \pi_{n,\infty}(z)$ [8, § 1.5]. Furthermore, (66) together with (67) yield

$$(68) \quad \int_D |K_{N,s}(z, w)|^2 dA \leq K_D(z, z) \quad \text{and} \quad \int_O |K_{N,s}(z, w)| |\Phi(w)|^{-2s} dA \leq K_D(z, z).$$

It follows from (19) and (68) that

$$\begin{aligned} \int_D |K_D(u, w) - K_{N,s}(u, w)|^2 dA &= K_D(w, w) - 2K_{N,s}(w, w) + \int_D |K_{N,s}(u, w)|^2 dA \\ &\leq K_D(w, w) - K_{N,s}(w, w). \end{aligned}$$

Therefore, (19), the Cauchy-Schwarz inequality, and the above estimate yield

$$\begin{aligned} |K_D(z, w) - K_{N,s}(z, w)| &\leq \int_D |K_D(u, w) - K_{N,s}(u, w)| |K_D(z, u)| dA \\ &\leq (K_D(z, z))^{1/2} (K_D(w, w) - K_{N,s}(w, w))^{1/2}. \end{aligned}$$

That is, we only need to demonstrate the convergence in (20) along the diagonal. Moreover, since (20) is valid for $K_{N,\infty}$, it suffices to show only that $K_{N,\infty}(z, z) - K_{N,s}(z, z) \rightarrow 0$ locally uniformly in D as $N, s \rightarrow \infty$. To this end, observe that

$$\begin{aligned} K_{N,\infty}(z, z) &= \int_{\mathbb{C}} K_{N,\infty}(z, u) K_{N,s}(u, z) P_K^{-2s}(u) dA \\ &= K_{N,s}(z, z) + \int_O K_{N,\infty}(z, u) K_{N,s}(u, z) |\Phi(u)|^{-2s} dA \end{aligned}$$

by (66) and the positivity of $K_{N,\infty}(z, z)$. Then we get from the equality above, the Cauchy-Schwarz inequality, and (68) that

$$(69) \quad \begin{aligned} K_{N,\infty}(z, z) - K_{N,s}(z, z) &\leq \left(\int_O \frac{|K_{N,s}(z, u)|^2}{|\Phi(u)|^{2s}} dA \right)^{1/2} \left(\int_O \frac{|K_{N,\infty}(z, u)|^2}{|\Phi(u)|^{2s}} dA \right)^{1/2} \\ &\leq (K_D(z, z))^{1/2} \left(\int_O \frac{|K_{N,\infty}(z, u)|^2}{|\Phi(u)|^{2s}} dA \right)^{1/2}. \end{aligned}$$

To estimate the integral in (69), observe that

$$|K_{N,\infty}(z, u)|^2 \leq K_{N,\infty}(z, z) K_{N,\infty}(u, u) \leq K_D(z, z) K_{N,\infty}(u, u)$$

by the Cauchy-Schwarz inequality and (67). Hence,

$$(70) \quad \begin{aligned} \int_O |K_{N,\infty}(z, u)|^2 |\Phi(u)|^{-2s} dA &\leq K_D(z, z) \sum_{n=0}^{N-1} \int_O |\pi_{n,\infty}|^2 |\Phi|^{-2s} dA \\ &\leq \text{const.} K_D(z, z) \sum_{n=0}^{N-1} \int_O |\Phi|^{2n-2s} |\Phi'|^2 dA \\ &\leq \text{const.} K_D(z, z) \sum_{n=0}^{N-1} \frac{n+1}{s-(n+1)} \\ &= \text{const.} K_D(z, z) \frac{N^2}{s-N}, \end{aligned}$$

where we used (14) for $s = \infty$. Clearly, (70), (69), and the reduction process carried out above, prove Theorem 4 under the condition $N^2/s \rightarrow 0$ as $N, s \rightarrow \infty$. The proof for all N readily follows from the obvious inequality $K_{N,s}(z, z) \leq K_{M,s}(z, z)$, $N \leq M$. \square

4.8. Proof of Theorem 5. Since $p > 0$ when $\ell = 1$, it holds that $\Sigma_N \rightarrow 0$ as $N \rightarrow \infty$. Thus, we deduce from (17) that

$$(71) \quad \lim_{N,s \rightarrow \infty} K_{N,s}(z, z) N^{-2} = \frac{|\Phi'(z)|^2}{\pi} \left(\frac{1-\ell}{2} + \frac{\ell}{6} \right) = \frac{|\Phi'(z)|^2}{\pi} \frac{3-2\ell}{6}$$

uniformly for $z \in T$. Fix $c > 0$ and let a, b be such that $|a|, |b| < c$. It follows from the Cauchy-Schwarz inequality that

$$(72) \quad \left| K_{N,s} \left(z + \frac{a}{N}, z + \frac{b}{N} \right) \right| \leq \max |K_{N,s}(\zeta, \zeta)|,$$

where the maximum is taken over all ζ satisfying (62). Furthermore, we get from the Bernstein-Walsh inequality and (63) that

$$(73) \quad |K_{N,s}(\zeta, \zeta)| \leq |\Phi(\zeta)|^{2(N-1)} \max_{w \in T} |K_{N,s}(w, w)| \leq \text{const.} \max_{w \in T} |K_{N,s}(w, w)|$$

for any ζ satisfying (62) with some absolute constant. Combining (72) and (73) with (71), we see that $\{K_{N,s}(z + \frac{a}{N}, z + \frac{b}{N}) N^{-2}\}$ is a normal family for $|a|, |b| < c$, where the functions in this family are indexed by $z \in T$, $N \in \mathbb{N}$, and $s \in [N+1, \infty)$. Therefore, it suffices to prove (23) only for those a, b for which $z + \frac{a}{N}, z + \frac{b}{N} \notin K$ (since T has a tangent at z , it holds that either $z + a/N \in K$ for all N large enough or $z + a/N \notin K$ for all N large enough).

As $|\Phi'|$ is bounded above in \overline{O} , there exists a path $\gamma \subset \overline{O}$ connecting z and $z + \frac{a}{N}$ whose length is proportional to $1/N$. Hence,

$$(74) \quad \Phi\left(z + \frac{a}{N}\right) - \Phi(z) - \frac{a}{N}\Phi'(z) = \int_{\gamma} (\Phi'(t) - \Phi'(z))dt = o\left(\frac{1}{N}\right),$$

where the estimate holds uniformly for $z \in T$ and locally uniformly for $a, b \in \mathbb{C}$. As $|\Phi(z)| = 1$, it holds that

$$(75) \quad \Phi\left(z + \frac{a}{N}\right) \overline{\Phi\left(z + \frac{b}{N}\right)} = 1 + \frac{\tau(a, z) + \overline{\tau(b, z)}}{N} + o\left(\frac{1}{N}\right)$$

and

$$(76) \quad \left[\Phi\left(z + \frac{a}{N}\right) \overline{\Phi\left(z + \frac{b}{N}\right)} \right]^N = \exp\left\{\tau(a, z) + \overline{\tau(b, z)} + o(1)\right\},$$

where $\tau(\cdot, \cdot)$ was defined before (22) and $o(1)$ is again uniform for $z \in T$ and a, b in compact subsets of \mathbb{C} . As before, we can assume without loss of generality that $\tau(a, z) + \overline{\tau(b, z)} \neq 0$. Then we get from (16), (75), (76), and the continuity of Φ' that

$$(77) \quad \lim_{N, s \rightarrow \infty} K_{N, s}\left(z + \frac{a}{N}, z + \frac{b}{N}\right) N^{-2} = \frac{|\Phi'(z)|^2}{\pi} \left(\frac{1-\ell}{2} H_0\left(\tau(a, z) + \overline{\tau(b, z)}\right) + \frac{\ell}{6} H_1\left(\tau(a, z) + \overline{\tau(b, z)}\right) \right)$$

uniformly for $z \in T$ and a, b in compact subsets of \mathbb{C} . The limit in (23) now follows from (71), (77), and (21).

As obvious from (11), to prove (24), it suffices to show that

$$(78) \quad \lim_{N, s \rightarrow \infty} P_K^{-s}(z + a/N) = \omega(a, z)$$

uniformly for $z \in T$ and locally uniformly for $a \in \mathbb{C}$. To this end observe that an outward normal to T at z is given by $\Phi(z)/\Phi'(z)$. Hence, the angle between the vectors a/N and $\Phi(z)/\Phi'(z)$ is less than $\pi/2$ if and only if the vector $a\Phi'(z)/\Phi(z) = \tau(a, z)$ belongs to the right half-plane. That is, if $\operatorname{Re}(\tau(a, z)) > 0$. Hence, the limit in (78) holds for $\operatorname{Re}(\tau(a, z)) < 0$ as $P_K(z + a/N) \equiv 1$ for such a . Moreover, when $\operatorname{Re}(\tau(a, z)) > 0$, (78) follows immediately from (76). The case $\operatorname{Re}(\tau(a, z)) = 0$ can be deduced by continuity and the uniformity of the estimate follows from the uniform character of the estimate in (74). Finally, the same arguments yield (24) for $\ell = 0$. \square

APPENDIX A. PLOTS OF CORRELATION FUNCTIONS

To provide intuition for the results reported here we consider the scaled limit of R_1 and R_2 of the entropic (potential theoretic, with $K = \overline{\mathbb{D}}$) ensemble in a neighborhood of a point on the unit circle. By the radial symmetry of the weight it suffices to restrict ourselves to a neighborhood of 1. In this case $\tau(a, 1) = a$ and $\omega(a, 1) = e^{-\operatorname{Re}(a)/\ell}$ for $\operatorname{Re}(a) > 0$ and $\omega(a, 1) = 1$ otherwise. As before, if $\ell = 0$ and $\operatorname{Re}(a) > 0$ then we take $\omega(a, 1) = 0$. For convenience we define the scaled kernel at $z = 1$ by

$$\tilde{H}_\ell(a, b) = \omega(a, 1)\omega(b, 1)H_\ell(a + \bar{b}).$$

The limiting density of scaled eigenvalues is then given by

$$R_1^\ell(a) = \tilde{H}_\ell(a, a),$$

and the scaling limit of the second correlation function is given by

$$R_2^\ell(a, b) = \tilde{H}_\ell(a, a)\tilde{H}_\ell(b, b) - \tilde{H}_\ell(a, b)\tilde{H}_\ell(b, a)$$

The visualizations provided here are for the cases where a and b are either real or on the imaginary axis.

A.1. Tangent to the curve. The tangent line of the circle at $z = 1$ is parallel to the imaginary axis and the local density of eigenvalues in this direction is given by

$$\tilde{H}_\ell(it, it) = 1.$$

This is expected since the spatial density of eigenvalues on the unit circle must be invariant under rotation (and locally, this rotation is given by translation up the imaginary axis).

Looking at the second correlation function, when a and b are on the imaginary axis, we see that \tilde{H}_ℓ is a function of $t = -i(a + \bar{b})$. Figures 2, 3 and 4 show plots of the second correlation function for various values of ℓ in various regions as a function of t .

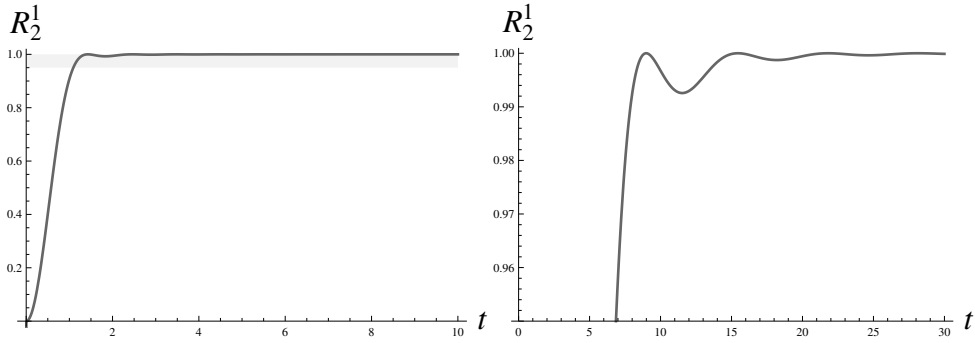


FIGURE 2. Plots of $R_2^1(a, b)$ as a function of $t = -i(a + \bar{b})$. The second plot is an enlargement of the shaded region.

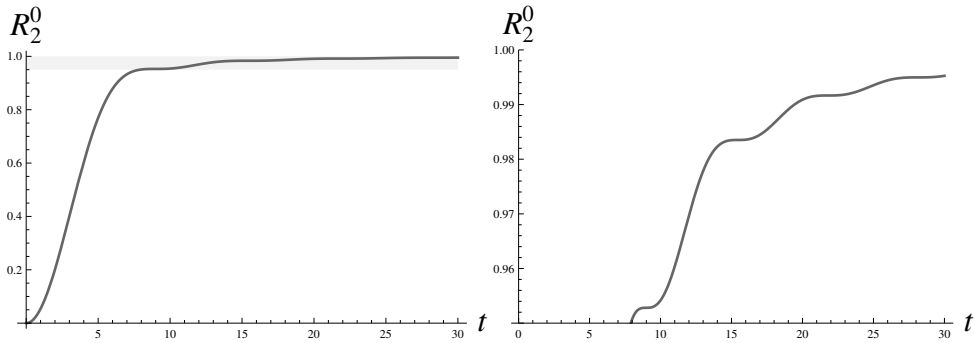


FIGURE 3. Plots of $R_2^0(a, b)$ as a function of $t = -i(a + \bar{b})$.

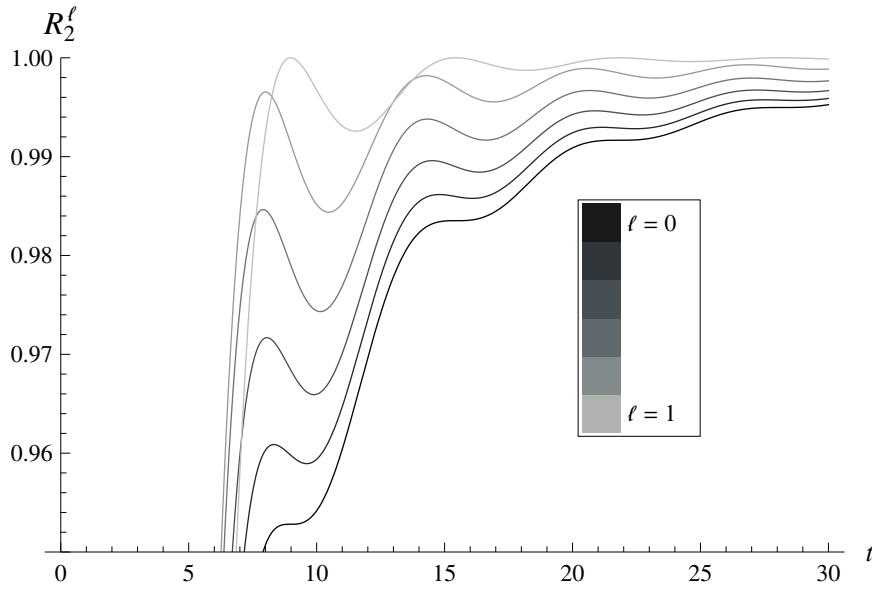


FIGURE 4. Plot of the interpolation between $R_2^0(a, b)$ and $R_2^1(a, b)$ as a function of $t = -i(a + \bar{b})$.

By way of comparison we also provide plots of the second scaled correlation function for ensembles with the sine kernel. Specifically,

$$S(a, b) = 2 \frac{\sin((a-b)/2)}{(a-b)} \quad \text{and} \quad R_2^{\sin}(a, b) = 1 - S(a, b)^2.$$

(The slightly unusual normalization given by the superfluous appearing factors of 2 in the first equation arises in the scaling limit when we take the expected distance between eigenvalues to be 2π —this allows for the most accurate comparison with our other figures).

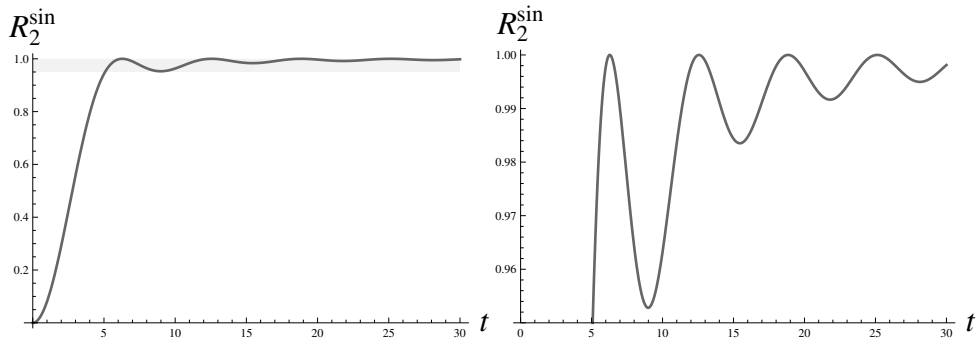


FIGURE 5. plots of $R_2^{\sin}(a, b)$ as a function of $t = a - b$.

A.2. Normal to the curve. In the regime where a and b are real, we are looking in a neighborhood of $z = 1$ in a direction perpendicular to that where the density of eigenvalues becomes constant. That is, the first scaled correlation function should decay as a moves away from 0. Negative a corresponds to moving into K (where the potential is constant) whereas positive a corresponds to moving away from K where the potential acts to make ∞ repulsive. As ℓ decreases to 0, the field increases in strength until at $\ell = 0$ there is no possibility that an eigenvalue can be outside K . That is, when $\ell = 0$ the first correlation function vanishes for $a > 0$.

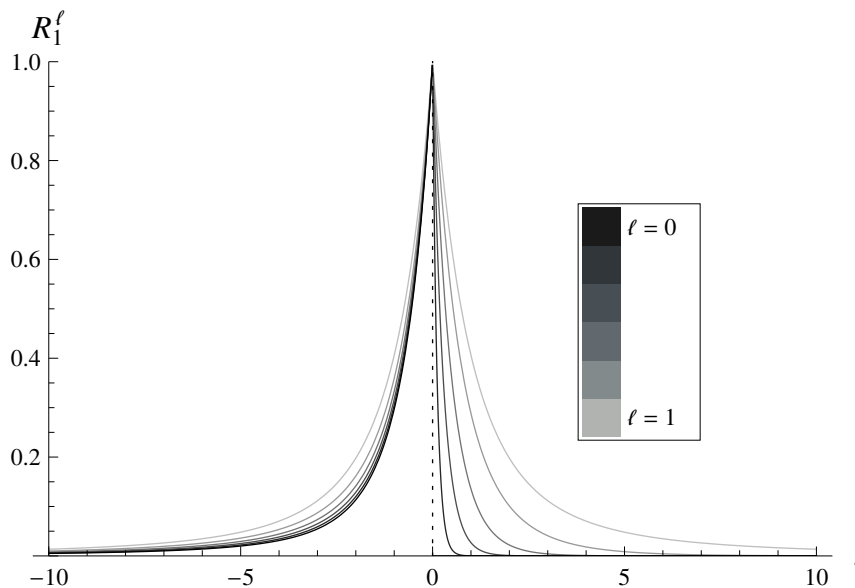


FIGURE 6. Plot of the interpolation between $R_1^0(a)$ and $R_1^1(a)$ as a function of $t = -\text{Re}(a)$. When $\ell = 0$ there is a sharp cutoff at $t = 0$.

When a and b are real, $R_2^\ell(a, b)$ is no longer a function of a linear combination of a and b , and we plot this as a surface for $\ell = 0$ and $\ell = 1$.

REFERENCES

- [1] G.W. Anderson, A. Guionnet, and O. Zeitouni. *An introduction to random matrices*, volume 118 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010.
- [2] T. Carleman. Über die Approximation analytischer Funktionen durch lineare Aggregate von vorgegebenen Potenzen. *Ark. Mat. Astr. Fys.*, 17(9):1–30, 1922.
- [3] L.-L. Chau and Y. Yu. Unitary polynomials in normal matrix models and wave functions for the fractional quantum hall effects. *Physics Letters A*, 167(5-6):452 – 458, 1992.
- [4] L.-L. Chau and O. Zeitouni. On the structure of correlation functions in the normal matrix model. *Comm. Math. Phys.*, 196(1):203–247, 1998.
- [5] S.-J. Chern and J.D. Vaaler. The distribution of values of Mahler’s measure. *J. Reine Angew. Math.*, 540:1–47, 2001.
- [6] P. Dragnev and E. Miña-Díaz. Asymptotic behavior and zero distribution of Carleman orthogonal polynomials. *J. Approx. Theory*, 162(11):1982–2003, 2010.
- [7] P. Dragnev and E. Miña-Díaz. On series representation for Carleman orthogonal polynomials. *Proc. Amer. Math. Soc.*, 138(12):4271–4279, 2010.
- [8] D. Gaier. *Lectures on Complex Approximation*. Birkhäuser Boston, Inc., 1987.
- [9] F.D. Gakhov. *Boundary Value Problems*. Dover Publications, Inc., New York, 1990.
- [10] J. Ginibre. Statistical ensembles of complex, quaternion, and real matrices. *J. Mathematical Phys.*, 6:440–449, 1965.

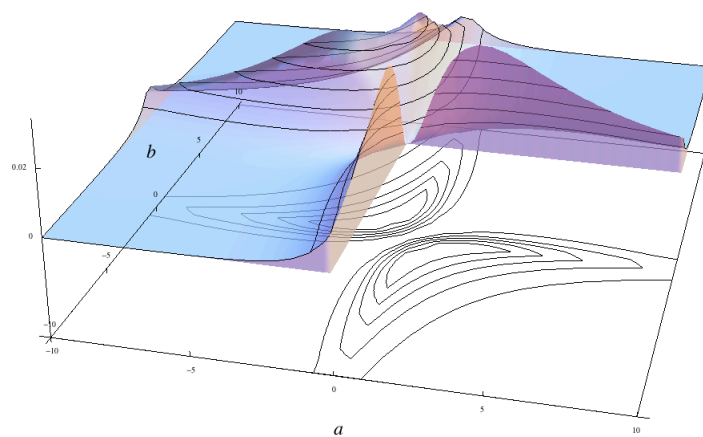


FIGURE 7. Plot of $R_2^1(a, b)$ when a and b are real, with part of the surface removed to see the cross-section.

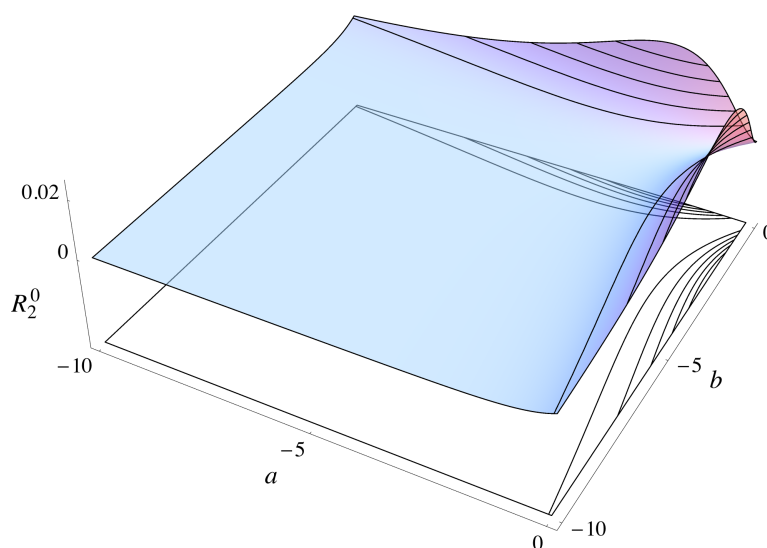


FIGURE 8. Plot of $R_2^0(a, b)$ when a and b are real. Note that $R^0(a, b)$ is identically zero if either a or b is greater than 0.

- [11] I. Gohberg, S. Goldberg, and N. Krupnik. *Traces and Determinants of Linear Operators*, volume 116 of *Operator Theory Advances and Applications*. Birkhäuser, Boston, 2000.
- [12] L. Hörmander. *An Introduction to Complex Analysis in Several Variables*. The University Series in Higher Mathematics. D. Van Nostrand Company, Inc., Princeton, New Jersey, 1966.
- [13] D.S. Lubinsky. Universality type limits for Bergman orthogonal polynomials. *Comput. Methods Funct. Theory*, 10:135–154, 2010.
- [14] M.L. Mehta and M. Gaudin. On the density of eigenvalues of a random matrix. *Nuclear Physics*, 18:420 – 427, 1960.
- [15] E. Miña-Díaz. An asymptotic integral representation for Carleman orthogonal polynomials. *Int. Math. Res. Not. IMRN*, (16), 2008. Art. ID rnn065, 38pp.

- [16] G. Oas. Universal cubic eigenvalue repulsion for random normal matrices. *Phys. Rev. E*, 55(1):205–211, Jan 1997.
- [17] T. Ransford. *Potential Theory in the Complex Plane*, volume 28 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1995.
- [18] C.D. Sinclair. The distribution of Mahler’s measures of reciprocal polynomials. *Int. J. Math. Math. Sci.*, (49-52):2773–2786, 2004.
- [19] C.D. Sinclair. The range of multiplicative functions on $\mathbb{C}[x]$, $\mathbb{R}[x]$ and $\mathbb{Z}[x]$. *Proc. Lond. Math. Soc. (3)*, 96(3):697–737, 2008.
- [20] C.D. Sinclair. *Multiplicative distance functions*. ProQuest LLC, Ann Arbor, MI, 2005. Thesis (Ph.D.)–The University of Texas at Austin.
- [21] P.K. Suetin. *Polynomials Orthogonal over a Region and Bieberbach Polynomials*, volume 100 of *Proc. Steklov Inst. Math.* Amer. Math. Soc. Translations, 1974.
- [22] C.A. Tracy and H. Widom. Correlation functions, cluster functions, and spacing distributions for random matrices. *J. Statist. Phys.*, 92:809–835, 1998.
- [23] S.A. Yuzvinskii. Computing the entropy of a group of endomorphisms. *Siberian Mathematical Journal*, 8:172–178, 1967. 10.1007/BF01040581.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OR, 97403
E-mail address: csinclai@uoregon.edu

CORRESPONDING AUTHOR, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OR, 97403
E-mail address: maximy@uoregon.edu