

ASYMPTOTIC UNIQUENESS OF BEST RATIONAL APPROXIMANTS TO COMPLEX CAUCHY TRANSFORMS IN L^2 OF THE CIRCLE

LAURENT BARATCHART AND MAXIM L. YATTSELEV

Dedicated to Guillermo Lopèz Lagomasino on the occasion of his 60-th birthday

ABSTRACT. For all n large enough, we show uniqueness of a critical point in best rational approximation of degree n , in the L^2 -sense on the unit circle, to functions of the form

$$f(z) = \int \frac{d\mu(t)}{z-t} + r(z), \quad d\mu = \dot{\mu} d\omega_{[a,b]},$$

with r a rational function and $\dot{\mu}$ a complex-valued Dini-continuous function on a real segment $[a, b] \subset (-1, 1)$ which does not vanish, and whose argument is of bounded variation. Here $\omega_{[a,b]}$ stands the normalized arcsine distribution on $[a, b]$.

1. INTRODUCTION

Best rational approximation of given degree to a holomorphic function, in the least squares sense on the boundary of a disk included in the domain of analyticity, is a classical issue for which early references are [22, 30, 26, 20, 17]. The interplay between complex and Fourier analysis induced by the circular symmetry confers to such an approximation a natural character, and the corresponding approximants provide one with nice examples of locally convergent sequences of diagonal multipoint Padé interpolants. The problem can be recast as best rational approximation of given degree in the Hardy space H^2 of the unit disk and also, upon reflecting the functions involved across the unit circle, in the Hardy space of the complement of the disk which is the framework we shall really work with.

Because of the natural isometry between Hardy spaces of the disk and the half-plane, that preserves rationality and the degree [25, Ch. 8], the question can equivalently be stated as best rational L^2 -approximation of given degree on the line to a function holomorphic in a half-plane.

Further motivation for this type of approximation stems from Control Theory and Signal Processing. Indeed, the transfer-function of a stable linear control system belongs to the Hardy space of the half-plane or of the complement of the disk, depending whether the setting is in continuous or discrete time, and it is rational if the system is finite-dimensional. Moreover, by Parseval's theorem, the L^2 norm on the line or the circle of this transfer function coincides with the norm of the underlying convolution operator from $L^2[0, \infty)$ to $L^\infty[0, \infty)$ in the time domain [19]. Further, in a stochastic context, it coincides with the variance of the output when the input is white noise [24]. Therefore approximating the transfer function by a rational function of degree n , in L^2 of the line or the circle, is tantamount to identify the best system of order n to model the initial system with respect to the criteria just mentioned. Also, since any stationary regular stochastic process is the output of a linear control system fed with white noise [29, 18], this approximation yields the best ARMA-process to model the initial process while minimizing the variance of the error.

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A thorough discussion of such connections with System Theory, as well as additional references, can be found in [2].

From the constructive viewpoint no algorithm is known to constructively solve the question we raised, and from a computational perspective this is a typical non-convex minimization problem whose numerical solution is often hindered by the occurrence of local minima. It is therefore of major interest in practice to establish conditions on the function to be approximated that ensure uniqueness of a local minimum. This turns out to be difficult, like most uniqueness issues in nonlinear approximation.

New ground for the subject was broken in [1], where a differential-topological method was introduced to approach the uniqueness issue for *critical points*, *i.e.* stationary points of the approximation error. Uniqueness of a critical point implies uniqueness of a local minimum, but is a stronger property which is better suited to analysis. In fact, the above-mentioned method rests on the so-called *index theorem* [5] that provides us with a relation between the Morse indices of the critical points, thereby reducing the proof of uniqueness, which is a global property, to checking that each critical point has Morse index 0, which is a local issue. The latter is in turn equivalent to each critical point being a non-degenerate local minimum.

This approach was taken in [12] to handle the case where the approximated function is of Markov type, that is, the Cauchy transform of a positive measure on a real segment, when that measure is supported within some absolute bounds. Subsequently, in [8], the property of being a local minimum was connected to classical interpolation theory and the technique was applied to prove asymptotic uniqueness of a critical point in best L^2 rational approximation to $e^{1/z}/z$ on the unit circle, as well as in best L^2 rational approximation to generic holomorphic functions over small circles; here, asymptotic uniqueness means uniqueness in degree n for all sufficiently large n . The criterion derived in [8] for being a local minimum was further refined in [11], where it is shown that asymptotic uniqueness of a critical point holds for Markov functions whose defining measure satisfies the Szegő condition. The result is sharp in that the Szegő condition cannot be omitted in general [10]. A general condition on the logarithmic derivative of the approximated function was derived [3] but it only ensures uniqueness in degree 1. A criterion for best approximation in degree $n - 1$ to a rational function of degree n can further be found in [9, Thm. 9.1], based on fast geometric decay of the error in lower degree.

Altogether, these works indicate the fact, perhaps unexpected, that uniqueness of a critical point in best L^2 rational approximation is linked to a regular decrease of the error.

The present paper can be viewed as a sequel to [11]. Indeed, the latter reference expressed hope that the techniques set up there could be adapted to handle more general Cauchy integrals than Markov functions. Below, we take a step towards carrying out this program. Specifically, we consider Cauchy transforms of *complex* measures that are absolutely continuous with respect to the equilibrium distribution on a real segment $[a, b] \subset (-1, 1)$. The density will be required to be Dini-smooth and non-vanishing. In addition, it should admit an argument function of bounded variation on $[a, b]$. Moreover, we handle with little extra-pain the case where a rational function is added to such a Cauchy transform.

For functions of this kind, we establish an analog to [11, Thm. 1.3], namely asymptotic uniqueness of a critical point in best rational approximation for the L^2 -norm on the unit circle. This is the first uniqueness result in degree greater than 1 for Cauchy integrals with complex densities, more generally for non-rational functions without conjugate symmetry. In contrast, say, to [12, Thm. 3], it is only fair to say that such a statement is not really constructive in that no estimate is provided for the degree beyond which uniqueness prevails. However, considering our restricted knowledge on uniqueness in non-linear complex approximation, our result sheds considerable

light on the behaviour of best rational approximants to Cauchy transforms, and it is our hope that suitable refinements of the technique will eventually produce effective bounds.

Our method of proof follows the same pattern as [11]. Namely, the index theorem is invoked to reduce the question of uniqueness to whether each critical point is a non-degenerate local minimum. Next, a criterion for being a local minimum is set up, based on a comparison between the error function generated by the critical point under examination and the error function attached to a particular multipoint Padé interpolant of lower degree; we call it for short the *comparison criterion*. Finally, the fact this criterion applies when the degree is sufficiently large depends on strong asymptotic formulas for the error in rational interpolation to functions of the type we consider, that were recently obtained in [15, 13, 31], and on a specific design of interpolation nodes to build the particular multipoint Padé interpolant of lower degree that we need.

With respect to [11], however, two main differences arise. The first is that the comparison criterion was set up there for conjugate-symmetric functions only, *i.e.* for those functions having real Fourier coefficients. Because we now address Cauchy transforms of complex densities on a segment, we handle complex Fourier coefficients as well. Although the corresponding changes are mostly mechanical, they have to be carried out thoroughly for they impinge on the computation of the Hessian quadratic form and on the nondegeneracy thereof.

The second difference causes more serious difficulties. Indeed, the construction of the special interpolant of lower degree needed to apply the comparison criterion requires rather precise control on the poles of multipoint Padé interpolants to the approximated function. For Markov functions, it is known that such poles are the zeros of certain orthogonal polynomials with respect to a positive measure on the segment $[a, b]$, therefore they lie on that segment. But for Cauchy transforms of complex measures, the poles are the zeros of some *non-Hermitian* orthogonal polynomial with respect to a complex measure on $[a, b]$, and their behaviour does not lend itself to analysis so easily. We resort here to the work in [4] on the geometry of non-Hermitian orthogonal polynomials to overcome this difficulty, and this is where the boundedness of the variation of the density's argument becomes important.

Let us briefly indicate some generalizations of our results that were not included here. Firstly, asymptotic uniqueness of a critical point extends to Cauchy transforms of absolutely continuous measures whose density with respect to *Lebesgue* (rather than equilibrium) measure satisfies similar assumptions, *e.g.* non-vanishing and Hölder-smoothness; in fact, densities with respect to *any* Jacobi weight could be handled the same way under suitable regularity requirements. We did not mention them, however, because such an extension depends on asymptotics for non-Hermitian orthogonal polynomials with respect to this type of weight which are yet unpublished [14].

Secondly, finitely many zeros in the density would still be acceptable, provided they are of power type with sufficiently small exponent. Developing the precise estimates would make the paper heavier (compare [13, Thm. 4] and [31, Thm. 5]), so we felt better omitting this stronger version.

The organization of the article is as follows. In Section 2 we present the rational approximation problem under study and we state the main result of the paper, which is Theorem 1. Section 3 introduces the critical points in H^2 -rational approximation and develops their interpolating properties; this part is adapted to complex Fourier coefficients from [8]. The index theorem and the comparison criterion are expounded in section 4, paralleling the treatment for real Fourier coefficients given in [11]. Section 5 recalls the necessary material on interpolation from [15, 13, 31], which is needed to carry out the comparison between critical points and interpolants of lower degree required in the comparison criterion. Finally, elaborating on [11], we prove Theorem 1 in Section 6.

2. NOTATION AND MAIN RESULTS

Let \mathbb{T} be the unit circle and \mathbb{D} the unit disk. We let L^2 (resp. L^∞) stand for the space of square-integrable (resp. essentially bounded) measurable functions on \mathbb{T} . Denote by H^2 the familiar Hardy space of the unit disk, consisting of those L^2 -functions whose Fourier coefficients of strictly negative index are zero. The space H^2 identifies with traces on \mathbb{T} of those holomorphic functions in \mathbb{D} whose L^2 -means on circles centered at zero with radii less than 1 are uniformly bounded above. In fact, any such function has non-tangential boundary values almost everywhere on \mathbb{T} that define a member of L^2 from which the function can be recovered by means of a Cauchy integral [21]. The space of bounded holomorphic functions on \mathbb{D} is denoted by H^∞ and is endowed with the L^∞ norm of the trace.

Let \bar{H}_0^2 be the orthogonal complement of H^2 in L^2 with respect to the standard scalar product

$$(2.1) \quad \langle f, g \rangle := \int_{\mathbb{T}} f(\tau) \overline{g(\tau)} \frac{|d\tau|}{2\pi}, \quad f, g \in L^2.$$

The space \bar{H}_0^2 , in turn, identifies with traces of those holomorphic functions in $\bar{\mathbb{C}} \setminus \bar{\mathbb{D}}$ that vanish at infinity and whose L^2 -means on circles centered at zero with radii greater than 1 are uniformly bounded above. In what follows, we denote by $\|\cdot\|_2$ the norm on L^2 , H^2 , and \bar{H}_0^2 induced by the scalar product (2.1). On one occasion, we shall refer to the Hardy space $\bar{H}_0^2(\bar{\mathbb{C}} \setminus \bar{\mathbb{D}}_\rho)$, which is defined similarly except that \mathbb{D} gets replaced by $\mathbb{D}_\rho := \{|z| < \rho\}$ where $\rho > 0$.

Set \mathcal{P}_n for the space of algebraic polynomials of degree at most n and \mathcal{M}_n for the subset of monic polynomials with exactly n zeros in \mathbb{D} . Note that $q \in \mathcal{M}_n$ if and only if $1/q \in \bar{H}_0^2$ and $1/q(z) \sim z^{-n}$ at infinity. From the differential viewpoint, we regard \mathcal{P}_n as \mathbb{C}^{n+1} and \mathcal{M}_n as an open subset of \mathbb{C}^n , upon taking the coefficients as coordinates (except for the leading coefficient in \mathcal{M}_n which is fixed to unity).

Define

$$(2.2) \quad \mathcal{R}_n := \left\{ \frac{p(z)}{q(z)} = \frac{p_{n-1}z^{n-1} + p_{n-2}z^{n-2} + \cdots + p_0}{z^n + q_{n-1}z^{n-1} + \cdots + q_0} : p \in \mathcal{P}_{n-1}, q \in \mathcal{M}_n \right\}.$$

It is easy to check that \mathcal{R}_n consists of those rational functions of degree at most n that belong to \bar{H}_0^2 , and we endow it with the corresponding topology. Coordinatizing \mathcal{P}_{n-1} and \mathcal{M}_n as above, it is straightforward to see that the canonical surjection $J : \mathcal{P}_{n-1} \times \mathcal{M}_n \rightarrow \mathcal{R}_n$ is smooth (i.e., infinitely differentiable) when viewed as a \bar{H}_0^2 -valued map. Note that J is not injective, due to possible cancellation between p and q , but it is a local homeomorphism at every pair (p, q) such that p, q are coprime.

We shall be concerned with the following problem.

Problem 1: Given $f \in \bar{H}_0^2$ and $n \in \mathbb{N}$, find $r \in \mathcal{R}_n$ to minimize $\|f - r\|_2$.

Let us point out two equivalent formulations of Problem 1 that account for early discussion made in the introduction.

Firstly, it is redundant to assume that r lies in \bar{H}_0^2 , as is subsumed in the definition of \mathcal{R}_n . Indeed, by partial fraction expansion, a rational function of degree at most n in L^2 can be written as $r_1 + r_2$, where $r_1 \in \bar{H}_0^2$ and $r_2 \in H^2$ have degree at most n . By orthogonality of \bar{H}_0^2 and H^2 , we get $\|f - r\|_2^2 = \|f - r_1\|_2^2 + \|r_2\|_2^2$ so that r_1 is a better candidate approximant than r , showing that Problem 1 is in fact equivalent to best rational approximation of given degree to f in L^2 .

Secondly, composing with $z \mapsto 1/z$, which is an L^2 -isometry mapping \tilde{H}_0^2 onto zH^2 while preserving rationality and the degree, Problem 1 transforms to best approximation in H^2 of functions vanishing at the origin by rational functions of degree at most n that vanish at the origin as well. However, by Parseval's identity, any best rational approximant to g in H^2 has value $g(0)$ at 0. Therefore Problem 1 is equivalent to

Problem 2: *Given $g \in H^2$ and $n \in \mathbb{N}$, find a rational r of degree at most n in H^2 to minimize $\|f - r\|_2$.*

Problem 1 is the one we shall work with, and we refer to it as the *rational \tilde{H}_0^2 -approximation problem to f in degree n* . It is well-known (see [3, Prop. 3.1] for a proof and further bibliography on the subject) that the minimum is attained and that a minimizing r , called a *best rational approximant of degree n to f* , lies in $\mathcal{R}_n \setminus \mathcal{R}_{n-1}$ unless $f \in \mathcal{R}_{n-1}$. Uniqueness of such an approximant is a delicate matter. Generically, there is only one best approximant by a theorem of Stechkin on Banach space approximation from approximately compact sets [16]. However, the proof is non-constructive and does not allow us to determine which functions have a unique best approximant and which functions do not. Moreover, from the computational viewpoint, the main interest lies not so much with uniqueness of a best approximant, but rather with uniqueness of a local best approximant for such places are all what a numerical search can usually spot. By definition, a *local best approximant* is a function $r_l \in \mathcal{R}_n$ such that $\|f - r_l\|_2 \leq \|f - r\|_2$ for all $r \in \mathcal{R}_n$ in some neighborhood of r_l . Like best approximants, local best approximants lie in $\mathcal{R}_n \setminus \mathcal{R}_{n-1}$ unless $f \in \mathcal{R}_{n-1}$ [3].

Still more general is the notion of a *critical point*, which is defined as follows. Fix $f \in \tilde{H}_0^2$ and put

$$(2.3) \quad \begin{aligned} \Phi_{f,n} : \mathcal{R}_n &\rightarrow [0, \infty) \\ r &\mapsto \|f - r\|_2^2. \end{aligned}$$

A pair $(p, q) \in \mathcal{P}_{n-1} \times \mathcal{M}_n$ is called *critical* if all partial derivatives of $\Phi_{f,n} \circ J$ vanish at (p, q) . Subsequently, a rational function $r_c \in \mathcal{R}_n$ is said to be a *critical point* of $\Phi_{f,n}$ if there is a critical pair (p_c, q_c) such that $r_c = p_c/q_c$. Critical points fall into two classes: they are termed *irreducible* if they have exact degree n , and *reducible* if they have degree strictly less than n . Note that r_c is irreducible if and only if p_c and q_c are coprime in some (hence any) representation $r_c = p_c/q_c$. Clearly a local best approximant is a particular instance of a critical point, and it is irreducible unless $f \in \mathcal{R}_{n-1}$.

In the present work, we dwell on a differential topological approach to uniqueness of a critical point introduced in [1] and further developed in [5, 2]. In this approach, global uniqueness is deduced from local analysis of the map $\Phi_{f,n}$. Specifically, to conclude there is only one critical point, which is therefore the unique local minimum (and *a fortiori* the global minimum as well), one needs to show that each critical point is irreducible, does not interpolate the approximated function on \mathbb{T} , and is a nondegenerate local minimum; here *nondegenerate* means that the second derivative is a nonsingular quadratic form. This method turns out to be fruitful when studying rational approximation to Cauchy transforms of measures supported in $(-1, 1)$, *i.e.*, functions of the form

$$(2.4) \quad f_\mu(z) := \int \frac{d\mu(t)}{z - t}, \quad \text{supp}(\mu) \subset (-1, 1).$$

The first result in this direction was obtained in [12, Thm. 3] when f_μ is a *Markov function*, meaning that μ in (2.4) is a positive measure. It goes as follows.

Theorem A. *Let μ be a positive measure supported on $[a, b] \subset (-1, 1)$ where a and b satisfy $b - a \leq \sqrt{2}(1 - \max\{a^2, b^2\})$. Assume further that μ has at least n points of increase, i.e., $f_\mu \notin \mathcal{R}_{n-1}$. Then there is a unique critical point in rational \bar{H}_0^2 -approximation of degree n to f_μ .*

Removing the restriction on the size of the support makes the situation more difficult. The following theorem [11, Thm. 1.3] asserts that rational approximants are asymptotically unique for Markov functions whose defining measure is sufficiently smooth. Hereafter, we denote by $\omega_{[a,b]}$ the normalized arcsine distribution on $[a, b]$ given by $d\omega_{[a,b]}(t) = dt/(\pi\sqrt{(t-a)(b-t)})$.

Theorem B. *Let μ be a positive measure supported on $[a, b] \subset (-1, 1)$ and let us write $d\mu = \mu' dt + d\mu_s$, where μ_s is singular and μ' is integrable on $[a, b]$. If μ satisfies the Szegő condition: $\int \log \mu' d\omega_{[a,b]} > -\infty$, then there is a unique critical point in rational \bar{H}_0^2 -approximation of degree n to f_μ for all n large enough.*

As an additional piece of information, the following negative result [10, Thm. 5] shows that the asymptotic nature of the previous theorem is indispensable.

Theorem C. *For each $n_0 \in \mathbb{N}$ there exists a positive measure μ satisfying the Szegő condition such that for each odd n between 1 and n_0 there exist at least two different best rational approximants of degree n to f_μ .*

Our goal is to extend Theorem B to a class of *complex measures* which is made precise in the definition below. Recall that a function h with modulus of continuity ω_h is said to be *Dini-continuous* if $\omega_h(t)/t$ is integrable on $[0, \epsilon]$ for some (hence any) $\epsilon > 0$.

Definition (Class \mathbf{M}). *A measure μ is said to belong to the class \mathbf{M} if $\text{supp}(\mu) \subset (-1, 1)$ is an interval, say $[a, b]$, and $d\mu = \dot{\mu} d\omega_{[a,b]}$, where $\dot{\mu}$ is a Dini-continuous non-vanishing function with an argument of bounded variation on $[a, b]$.*

Observe that we deal here with $\dot{\mu}$, the Radon-Nikodym derivative of μ with respect to the arcsine distribution, rather than with μ' , the Radon-Nikodym derivative with respect to the Lebesgue measure. Our main result is:

Theorem 1. *Let $\mathfrak{f} := f_\mu + r$, where $\mu \in \mathbf{M}$ and $r \in \mathcal{R}_m$ has no poles on $\text{supp}(\mu)$. Then there is a unique critical point in rational \bar{H}_0^2 -approximation of degree n to \mathfrak{f} for all n large enough.*

Before we can prove the theorem, we must study in greater detail the structure of critical points, which is the object of Sections 3 and 4 to come.

3. CRITICAL POINTS

The following theory was developed in [5, 6, 7, 8] when the function f to be approximated is conjugate-symmetric, i.e., $f(\bar{z}) = \overline{f(z)}$, and the rational approximants are sought to be conjugate-symmetric as well. In other words, when a function with real Fourier-Taylor expansion at infinity gets approximated by a rational function with real coefficients. Surprisingly enough, this is *not* subsumed in Problem 1 in that conjugate-symmetric functions need not have a best approximant out of \mathcal{R}_n which is conjugate-symmetric. For Markov functions, though, it is indeed the case [4]. Below, we develop an analogous theory for Problem 1, that is, without conjugate-symmetry assumptions. This involves only technical modifications of a rather mechanical nature.

Hereafter, for any $f \in L^2$, we set $f^\sigma(z) := (1/z)\overline{f(1/\bar{z})}$. Clearly, $f \rightarrow f^\sigma$ is an isometric involution mapping H^2 onto \bar{H}_0^2 and vice-versa. Further, for any $p \in \mathcal{P}_k$, we set $\check{p}(z) := \overline{p(1/\bar{z})}$ and define its reciprocal polynomial (in \mathcal{P}_k) to be $\tilde{p}(z) := z^k \check{p}(z) = z^k \overline{p(1/\bar{z})}$. Note that \tilde{p} has the same modulus as p on \mathbb{T} and its zeros are reflected from those of p across \mathbb{T} .

3.1. Critical Points as Orthogonal Projections. Fix $f \in H_0^2$ and let $\Phi_n := \Phi_{f,n}$ be given by (2.3). It will be convenient to use complex partial derivatives with respect to $p_j, q_k, \bar{p}_j, \bar{q}_k$, where, for $0 \leq j, k \leq n-1$, the symbols p_k and q_k refer to the coefficients of $p \in \mathcal{P}_{n-1}$ and $q \in \mathcal{M}_n$ as in equation (2.2). By complex derivatives we mean the standard Wirtinger operators, e.g., if $p_j = x_j + iy_j$ is the decomposition into real and imaginary part then $\partial/\partial p_j = (\partial/\partial x_j - i\partial/\partial y_j)/2$ and $\partial/\partial \bar{p}_j = (\partial/\partial x_j + i\partial/\partial y_j)/2$. The standard rules for derivation are still valid, obviously $\partial g(p_j)/\partial \bar{p}_j = 0$ if g is holomorphic, and it is straightforward that $\overline{\partial h/\partial p_j} = \partial \bar{h}/\partial \bar{p}_j$ for any function h . In particular, since Φ_n is real,

$$(3.1) \quad \frac{\partial \Phi_n}{\partial \bar{p}_j} = \overline{\left(\frac{\partial \Phi_n}{\partial p_j} \right)} \quad \text{and} \quad \frac{\partial \Phi_n}{\partial \bar{q}_k} = \overline{\left(\frac{\partial \Phi_n}{\partial q_k} \right)}.$$

Thus, writing $\Phi_n \circ J(p, q) = \langle f - p/q, f - p/q \rangle$ and differentiating under the integral sign, we obtain that a critical pair (p_c, q_c) of $\Phi_n \circ J$ is characterized by the relations

$$(3.2) \quad \frac{\partial \Phi_n}{\partial p_j}(p_c, q_c) = \left\langle \frac{z^j}{q_c}, f - \frac{p_c}{q_c} \right\rangle = 0, \quad j \in \{0, \dots, n-1\},$$

$$(3.3) \quad \frac{\partial \Phi_n}{\partial q_k}(p_c, q_c) = - \left\langle \frac{z^k p_c}{q_c^2}, f - \frac{p_c}{q_c} \right\rangle = 0, \quad k \in \{0, \dots, n-1\}.$$

Equation (3.2) means that p_c/q_c is the orthogonal projection of f onto V_{q_c} , where for any $q \in \mathcal{M}_n$ we let $V_q := \{p/q : p \in \mathcal{P}_{n-1}\}$ to be the n -dimensional linear subspace of \bar{H}_0^2 consisting of rational functions with denominator q . In what follows, we consistently denote the orthogonal projection of f onto V_q by L_q/q , where $L_q \in \mathcal{P}_{n-1}$ is uniquely characterized by the fact that

$$(3.4) \quad \left\langle f - \frac{L_q}{q}, \frac{p}{q} \right\rangle = 0 \quad \text{for any } p \in \mathcal{P}_{n-1}.$$

Taking equation (3.3) into account, we conclude from what precedes that critical points of Φ_n are precisely \mathcal{R}_n -functions of the form L_{q_c}/q_c , where $q_c \in \mathcal{M}_n$ satisfies

$$(3.5) \quad \left\langle \frac{z^k L_{q_c}}{q_c^2}, f - \frac{L_{q_c}}{q_c} \right\rangle = 0, \quad k \in \{0, \dots, n-1\}.$$

Now, it is apparent from (3.4) that L_q is a smooth function of q , therefore we define a smooth map Ψ_n on \mathcal{M}_n by setting

$$(3.6) \quad \begin{aligned} \Psi_n = \Psi_{f,n} : \mathcal{M}_n &\rightarrow [0, \infty) \\ q &\mapsto \|f - L_q/q\|_2^2. \end{aligned}$$

By construction, Φ_n attains a local minimum at $r = L_{q_l}/q_l$ if and only if Ψ_n attains a local minimum at q_l , and the assumed values are the same. More generally, $r \in \mathcal{R}_n$ is a critical point of Φ_n if and only if $r = L_{q_c}/q_c$, where $q_c \in \mathcal{M}_n$, is a critical point of Ψ_n . This is readily checked upon

comparing (3.5) with the result of the following computation:

$$\begin{aligned}
\frac{\partial \Psi_n}{\partial q_k}(q) &= \left\langle \frac{(\partial L_q / \partial q_k)q - z^k L_q}{q^2}, f - \frac{L_q}{q} \right\rangle + \left\langle f - \frac{L_q}{q}, -\frac{(\partial L_q / \partial \bar{q}_k)}{q} \right\rangle \\
(3.7) \quad &= -\left\langle \frac{z^k L_q}{q^2}, f - \frac{L_q}{q} \right\rangle, \quad k \in \{0, \dots, n-1\},
\end{aligned}$$

where we applied (3.4) using that the derivatives of L_q lie in \mathcal{P}_{n-1} and that $\partial q / \partial \bar{q}_k = 0$. For simplicity, we drop from now on the subscript “ c ” we used so far as a mnemonic for “critical”. Altogether we proved the following result:

Proposition 1. *For $f \in \bar{H}_0^2$, let Φ_n and Ψ_n be defined by (2.3) and (3.6), respectively. Then $r \in \mathcal{R}_n$ is a critical point of Φ_n if and only if $r = L_q / q$ and $q \in \mathcal{M}_n$ is a critical point of Ψ_n .*

In view of Proposition 1, we shall extend to Ψ_n the terminology introduced for Φ_n and say that a critical point $q \in \mathcal{M}_n$ of Ψ_n is irreducible if L_q and q are coprime.

3.2. Interpolation Properties of Critical Points. If we denote with a superscript “ \perp ” the orthogonal complement in \bar{H}_0^2 , it is elementary to check that

$$(3.8) \quad V_q^\perp = \left\{ \frac{\tilde{q}}{q} u : u \in \bar{H}_0^2 \right\}, \quad \bar{H}_0^2 = V_q \oplus V_q^\perp.$$

Hence, by (3.4), there exists $u_q = u_{f,q} \in \bar{H}_0^2$ such that

$$(3.9) \quad f q - L_q = \tilde{q} u_q.$$

Relation (3.9) means that L_q / q interpolates f at the reflections of the zeros of q across \mathbb{T} . Assume now that $q \in \mathcal{M}_n$ is a critical point of Ψ_n . Then, combining (3.9) and (3.7), we derive that

$$(3.10) \quad 0 = \left\langle \frac{p L_q}{q^2}, f - \frac{L_q}{q} \right\rangle = \left\langle \frac{p L_q}{q^2}, \frac{\tilde{q} u_q}{q} \right\rangle = \int_{\mathbb{T}} \frac{p(\tau) (L_q u_q^\sigma)(\tau) d\tau}{q(\tau) \tilde{q}(\tau) 2\pi i}, \quad p \in \mathcal{P}_{n-1}.$$

Since $L_q u_q^\sigma / \tilde{q} \in H^2$, we see by letting p range over elementary divisors of q and applying the residue formula that (3.10) holds if and only if each zero of q is a zero of $L_q u_q^\sigma$ of the same multiplicity or higher. That is, q is a critical point of Ψ_n if and only if q divides $L_q u_q^\sigma$ in H^2 .

Let $d \in \mathcal{M}_k$ be the monic g.c.d. of L_q and q , with $0 \leq k \leq n-1$. Writing $L_q = d p^*$ and $q = d q^*$, where p^* and q^* are coprime, we deduce that q^* divides u_q^σ in H^2 or equivalently that $u_q = \check{q}^* h$ for some $h \in \bar{H}_0^2$. Besides, it follows from (3.4), applied with $p = d v$ and $v \in \mathcal{P}_{n-k-1}$, that $p^* = L_{q^*}$. Therefore, upon dividing (3.9) by d , we get

$$(3.11) \quad f q^* - L_{q^*} = \frac{\check{q} \check{q}^*}{d} h,$$

implying that $u_{q^*} = \check{d} \check{q}^* h / d$. In particular, q , thus *a fortiori* q^* , divides $u_{q^*}^\sigma$ in H^2 , whence q^* is critical for Ψ_{n-k} by what we said before. Finally, dividing (3.11) by q^* and taking into account the definition of the reciprocal polynomial, we find that we established the following result.

Proposition 2. *Let q be a critical point of Ψ_n and $d \in \mathcal{M}_k$ be the monic g.c.d. of L_q and q with $0 \leq k \leq n-1$. Then $q^* = q/d \in \mathcal{M}_{n-k}$ is an irreducible critical point of Ψ_{n-k} , and L_{q^*}/q^* interpolates f at the zeros of $\check{q}^{*2}\check{d}/z$ in Hermite's sense on $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$, that is, counting multiplicities including at infinity.*

The converse is equally easy: if q^* is an irreducible critical point of Ψ_{n-k} , and L_{q^*}/q^* interpolates f at the zeros of $\check{q}^{*2}\check{d}/z$ in Hermite's sense for some $d \in \mathcal{M}_k$, then $q = q^*d$ is a critical point of Ψ_n and d is the monic g.c.d. of q and L_q . This we shall not need.

It is immediately seen from Proposition 2 that a critical point of $\Phi_{f,n}$ must interpolate f with order 2 at the reflections of its poles across \mathbb{T} ; for best approximants, this property is classical [22, 26].

3.3. Smooth Extension of Ψ_n . One of the advantages of Ψ_n , as compared to Φ_n , is that its domain of definition can be compactified, which is essential to rely on methods from differential topology. To do that, however, we need to place an additional requirement on f .

Let us denote by \bar{H}_0 the subset of \bar{H}_0^2 comprised of functions that extend holomorphically across \mathbb{T} . Hereafter we will suppose that $f \in \bar{H}_0$ and pick $\rho = \rho(f) < 1$ such that f is holomorphic in $\{|z| > \rho - \epsilon\}$ for some $\epsilon > 0$. In particular, f is holomorphic across $\mathbb{T}_\rho := \{|z| = \rho\}$.

Denote by $\overline{\mathcal{M}}_n$ and $\mathcal{M}_n^{1/\rho}$ respectively the closure of \mathcal{M}_n and the set of monic polynomials with zeros in $\mathbb{D}_{1/\rho} := \{|z| < 1/\rho\}$; as usual, we regard these as subsets of \mathbb{C}^n when coordinatized by their coefficients except the leading one. This way $\mathcal{M}_n^{1/\rho}$ becomes an open neighborhood of the compact set $\overline{\mathcal{M}}_n$, which is easily seen to consist of polynomials with zeros in $\overline{\mathbb{D}}$. Also, q lies on the boundary $\partial \overline{\mathcal{M}}_n = \overline{\mathcal{M}}_n \setminus \mathcal{M}_n$ if and only if it is a monic polynomial of degree n having at least one zero of modulus 1 and no zero of modulus strictly greater than 1.

For $q \in \mathcal{M}_n$, since q/\tilde{q} is unimodular on \mathbb{T} , it follows from (3.9) that

$$(3.12) \quad \Psi_n(q) = \left\| f - \frac{L_q}{q} \right\|_2^2 = \|u_q\|_2^2 = \int_{\mathbb{T}} (u_q u_q^\sigma)(\tau) \frac{d\tau}{2\pi i}.$$

In addition, taking into account the Cauchy formula, the analyticity of L_q/\tilde{q} in \mathbb{D} , the analyticity of f across \mathbb{T}_ρ , and the definition of the σ -operation, we obtain

$$(3.13) \quad u_q(z) = u_{f,q}(z) = \frac{1}{2\pi i} \int_{\mathbb{T}_\rho} \frac{f(\tau)q(\tau)}{\tilde{q}(\tau)} \frac{d\tau}{z-\tau}, \quad |z| > \rho,$$

$$(3.14) \quad u_q^\sigma(z) = \frac{1}{2\pi i} \int_{\mathbb{T}_{1/\rho}} \frac{f^\sigma(\tau)\tilde{q}(\tau)}{q(\tau)} \frac{d\tau}{\tau-z}, \quad |z| < 1/\rho$$

$$(3.15) \quad L_q(z) = L_{f,q}(z) = \int_{\mathbb{T}_\rho} \frac{f(\tau)}{\tilde{q}(\tau)} \frac{q(z)\tilde{q}(\tau) - \tilde{q}(z)q(\tau)}{z-\tau} \frac{d\tau}{2\pi i}, \quad |z| > \rho.$$

Now, if $q \in \mathcal{M}_n^{1/\rho}$, then \tilde{q} has all its zeros of modulus greater than ρ , therefore (3.15) and (3.13) are well-defined and smooth with respect to the coefficients of q , with values in \mathcal{P}_{n-1} and $\bar{H}_0^2(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}_\rho)$ respectively. Because evaluation at $\tau \in \mathbb{T}$ is uniformly bounded with respect to τ on $\bar{H}_0^2(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}_\rho)$, Ψ_n in turn extends smoothly to $\mathcal{M}_n^{1/\rho}$ in view of (3.12). Moreover, differentiating under the

integral sign, we obtain

$$(3.16) \quad \frac{\partial \Psi_n}{\partial q_j}(q) = \int_{\mathbb{T}} \left(\frac{\partial u_q}{\partial q_j}(\tau) u_q^\sigma(\tau) + u_q(\tau) \frac{\partial u_q^\sigma}{\partial q_j}(\tau) \right) \frac{d\tau}{2\pi i},$$

with

$$(3.17) \quad \frac{\partial u_q}{\partial q_j}(z) = \frac{1}{2\pi i} \int_{\mathbb{T}_\rho} \frac{\tau^j f(\tau)}{\tilde{q}(\tau)} \frac{d\tau}{z - \tau}, \quad |z| > \rho,$$

$$(3.18) \quad \frac{\partial u_q^\sigma}{\partial q_j}(z) = \frac{1}{2\pi i} \int_{\mathbb{T}_{1/\rho}} \frac{-\tau^j f^\sigma(\tau) \tilde{q}(\tau)}{q^2(\tau)} \frac{d\tau}{\tau - z}, \quad |z| < 1/\rho,$$

for $j = 0, \dots, n-1$. To recap, we have proved:

Proposition 3. *Let $f \in \bar{H}_0$. Then Ψ_n extends to a smooth function in some neighborhood of $\bar{\mathcal{M}}_n$ and so do L_q and u_q with values in \mathcal{P}_{n-1} and $\bar{H}_0^2(\bar{\mathbb{C}} \setminus \bar{\mathbb{D}}_\rho)$ respectively. In addition, (3.16), (3.17), and (3.18) hold.*

We shall continue to denote the extension whose existence is asserted in Proposition 3 by $\Psi_{f,n}$, or simply Ψ_n if f is understood from the context.

3.4. Critical points on the boundary. Having characterized the critical points of Ψ_n on \mathcal{M}_n in Section 3.2, we need now to describe the critical points that it may have on $\partial \bar{\mathcal{M}}_n$. We shall begin with the case where all the roots of the latter lie on \mathbb{T} .

Let $f \in \bar{H}_0$ and assume $v(z) = (z - \xi)^k$, $\xi \in \mathbb{T}$, is a critical point of Ψ_k . It immediately follows from (3.13), the Cauchy formula, and the definition of v that

$$(3.19) \quad \tilde{v} = e^{i\theta} v, \quad \text{where } e^{i\theta} := (-\bar{\xi})^k, \quad L_v \equiv 0, \quad \text{and } u_v = e^{-i\theta} f.$$

In this case, equations (3.17) and (3.18) become

$$\begin{aligned} \frac{\partial u_v}{\partial q_j}(z) &= \frac{e^{-i\theta}}{2\pi i} \int_{\mathbb{T}_\rho} \frac{\tau^j f(\tau)}{v(\tau)} \frac{d\tau}{z - \tau}, \quad |z| > \rho, \\ \frac{\partial u_v^\sigma}{\partial q_j}(z) &= \frac{-e^{i\theta}}{2\pi i} \int_{\mathbb{T}_{1/\rho}} \frac{\tau^j f^\sigma(\tau)}{v(\tau)} \frac{d\tau}{\tau - z}, \quad |z| < 1/\rho. \end{aligned}$$

Plugging these expressions into (3.16), we obtain

$$0 = \frac{\partial \Psi_k}{\partial q_j}(v) = \int_{\partial \mathbb{A}_\rho} \frac{\tau^j (f f^\sigma)(\tau)}{v(\tau)} \frac{d\tau}{2\pi i}, \quad j \in \{0, \dots, k-1\},$$

where $\mathbb{A}_\rho := \{\rho < |z| < 1/\rho\}$ with positively oriented boundary $\partial \mathbb{A}_\rho$, and we used the Fubini-Tonelli theorem. By taking linear combinations of the previous equations, we deduce from the Cauchy formula that

$$0 = \int_{\partial \mathbb{A}_\rho} \frac{(f f^\sigma)(\tau)}{(\tau - \xi)^l} \frac{d\tau}{2\pi i} = \frac{(f f^\sigma)^{(l-1)}(\xi)}{(l-1)!}, \quad l \in \{1, \dots, k\}.$$

Hence v divides $f f^\sigma$, when viewed as a holomorphic function in \mathbb{A}_ρ . Consequently, since $\zeta \in \mathbb{T}$ is a zero of f if and only if it is a zero of f^σ , we get that f vanishes at ζ with multiplicity $\lfloor (k+1)/2 \rfloor$, where $\lfloor x \rfloor$ is the integer part of x .

Next we consider the case where q is a critical point of Ψ_n having exactly one root on \mathbb{T} : $q = vq^*$ with $v(z) = (z - \xi)^k$, $\xi \in \mathbb{T}$, and $q^* \in \mathcal{M}_{n-k}$. Denote by \mathcal{Q} and \mathcal{V} some neighborhoods of q^* and v , in $\mathcal{M}_{n-k}^{1/\rho}$ and $\mathcal{M}_k^{1/\rho}$ respectively, taking them so small that each $\chi \in \mathcal{Q}$ is coprime to each $v \in \mathcal{V}$; this is possible since q^* and v are coprime. Then, $(\chi, v) \mapsto \chi v$ is a diffeomorphism from $\mathcal{Q} \times \mathcal{V}$ onto a neighborhood of q in $\mathcal{M}_n^{1/\rho}$. In particular, the fact that q is a critical point of Ψ_n means that q^* is a critical point of Θ and v a critical point of Ξ , where

$$\begin{aligned} \Theta: \mathcal{Q} &\rightarrow [0, \infty) & \text{and} & \quad \Xi: \mathcal{V} \rightarrow [0, \infty) \\ \chi &\mapsto \Psi_n(v\chi) & & \quad v \mapsto \Psi_n(vq^*). \end{aligned}$$

Since $\widetilde{v\chi} = e^{i\theta} v \widetilde{\chi}$, where $e^{i\theta}$ is as in (3.19), it follows from (3.13) that $u_{v\chi} = e^{-i\theta} u_\chi$, and therefore by (3.12) that $\Theta = \Psi_{n-k|2}$, implying that q^* is a critical point of Ψ_{n-k} . In another connection, shrinking \mathcal{V} if necessary, we may assume there exists $\varrho > \rho$ such that $\mathcal{V} \subset \mathcal{M}_k^{1/\varrho}$. Put for simplicity $w := u_{q^*} = u_{f, q^*}$, which is clearly an element of \bar{H}_0 by (3.13). Computing with the latter formula yields for any $v \in \mathcal{V}$ that

$$\begin{aligned} u_{w, v}(z) &= \frac{1}{2\pi i} \int_{\mathbb{T}_\varrho} \frac{(wv)(\tau)}{\widetilde{v}(\tau)} \frac{d\tau}{z - \tau} = \frac{1}{2\pi i} \int_{\mathbb{T}_\varrho} \frac{1}{2\pi i} \int_{\mathbb{T}_\rho} \frac{(fq^*)(t)}{\widetilde{q}^*(t)} \frac{dt}{\tau - t} \frac{v(\tau)}{\widetilde{v}(\tau)} \frac{d\tau}{z - \tau} \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}_\rho} \frac{(fvq^*)(t)}{\widetilde{(vq^*)}(t)} \frac{dt}{z - t} = u_{f, vq^*}(z), \quad |z| > \varrho, \end{aligned}$$

where we used the Fubini-Tonelli theorem and the Cauchy integral formula. Thus, we derive from (3.12) that

$$\Xi(v) = \Psi_{f, n}(vq^*) = \|u_{f, vq^*}\|_2^2 = \|u_{w, v}\|_2^2 = \Psi_{w, k}(v), \quad v \in \mathcal{V}.$$

As v is a critical point of Ξ , we see that it is also critical for $\Psi_{w, k}$, so by the case previously considered we conclude that $w = u_{f, q^*}$ vanishes at ξ with multiplicity $\lfloor (k+1)/2 \rfloor$. By (3.9), this is equivalent to the fact that $L_{q^*}/q^* = L_q/q$ interpolates f at the zeros of $d(z) = (z - \xi)^{\lfloor (k+1)/2 \rfloor}$ in Hermite's sense.

Finally, the case where q is arbitrarily located on $\partial \overline{\mathcal{M}}_n$ is handled the same way upon writing $q = q^* v_1 \dots v_\ell$, where $v_j(z) = (z - \xi_j)^{k_j}$ for some $\xi_j \in \mathbb{T}$, and introducing a product neighborhood $\mathcal{Q} \times \mathcal{V}_1 \times \dots \times \mathcal{V}_\ell$ of $q^* v_1 \dots v_\ell$ to proceed with the above analysis on each of the corresponding maps $\Theta, \Xi_1, \dots, \Xi_\ell$. Thus, taking into account Proposition 2 and the fact that v_j and \check{v}_j have the same zeros in $\overline{\mathbb{C}}$, we obtain:

Proposition 4. *Let $f \in \bar{H}_0$ and $q = vq^*$, where $v = \prod (z - \xi_j)^{k_j}$, $\xi_j \in \mathbb{T}$, $\deg(v) = k$, and $q^* \in \mathcal{M}_{n-k}$. Assume that q is a critical point of $\Psi_n = \Psi_{f, n}$. Then q^* is a critical point of Ψ_{n-k} . Moreover, if we write $q^* = q_1 d_1$ where d_1 is the monic g.c.d. of L_{q^*} and q^* , then $L_{q^*}/q^* = L_q/q$ interpolates f at the zeros of $\check{q}_1^2 \check{d}_1 \check{d} / z$ in Hermite's sense on $\overline{\mathbb{C}} \setminus \mathbb{D}$, where $d(z) = \prod (z - \xi_j)^{\lfloor (k_j+1)/2 \rfloor}$.*

Again the converse of Proposition 4 is true, namely the properties of q^* and v asserted there imply that $q = q^*v$ is critical for Ψ_n . This is easy to check by reversing the previous arguments, but we shall not use it.

4. A CRITERION FOR LOCAL MINIMA

Let $f \in \bar{H}_0$ and $\Psi_n = \Psi_{f, n}$ be the extended map obtained in Proposition 3, based on (3.12) and (3.13). The latter is a smooth real-valued function, defined on an open neighborhood of $\overline{\mathcal{M}}_n$

identified with a subset of $\mathbb{C}^n \sim \mathbb{R}^{2n}$ by taking as coordinates all coefficients but the leading one. By definition, a critical point of Ψ_n is a member of $\overline{\mathcal{M}}_n$ at which the gradient $\nabla\Psi_n$ vanishes. This notion is of course independent of which coordinates are used, and so is the signature of the second derivative, the so called *Hessian* quadratic form¹. A critical point q is called nondegenerate if the Hessian form is nonsingular at q , and then the number of negative eigenvalues of this form is called the *Morse index* of q denoted by $M(q)$. Observe that nondegenerate critical points are necessarily isolated.

From first principles of differential topology [23] it is known that $(-1)^{M(q)}$, which is called the *index* of the nondegenerate critical point q , is equal to the so-called Brouwer degree of the vector field $\nabla\Psi_n/\|\nabla\Psi_n\|_e$ on any sufficiently small sphere centered at q , where $\|\cdot\|_e$ is the Euclidean norm in \mathbb{R}^{2n} .

One can show that $\partial\overline{\mathcal{M}}_n$ is a compact manifold², so if Ψ_n has no critical points on $\partial\overline{\mathcal{M}}_n$ and only nondegenerate critical points in \mathcal{M}_n , then the sum of the indices of the critical points is equal to the Brouwer degree of $\nabla\Psi_n/\|\nabla\Psi_n\|_e$ on $\partial\overline{\mathcal{M}}_n$. The surprising fact is that the latter is *independent of f* (see [1], [5, Sec. 5], and [2, Thm. 2]) and is actually equal to 1. Altogether, the following analogue of the Poincaré-Hopf theorem holds in the present setting.

The Index Theorem. *Let $f \in \tilde{H}_0$ and $\mathcal{C}_{f,n}$ be the set of the critical points of $\Psi_{f,n}$ in $\overline{\mathcal{M}}_n$. Assume that all members of $\mathcal{C}_{f,n}$ are nondegenerate, and that $\mathcal{C}_{f,n} \cap \partial\overline{\mathcal{M}}_n = \emptyset$. Then*

$$\sum_{q \in \mathcal{C}_{f,n}} (-1)^{M(q)} = 1.$$

To us, the value of the index theorem is that if can show every critical point is a nondegenerate local minimum and none of them lies on $\partial\overline{\mathcal{M}}_n$, then the critical point is unique. To see this, observe that local minima have Morse index 0 and therefore index 1.

To make this criterion effective, we need now to analyze the Morse index of a critical point, starting with the computation of the Hessian quadratic form.

Let q be a critical point of Ψ_n . It is easy to check that the Hessian quadratic form of Ψ_n at q is given by

$$(4.1) \quad \mathcal{Q}(v) = 2\operatorname{Re} \left(\sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \left(v_j v_k \frac{\partial^2 \Psi_n}{\partial q_k \partial q_j}(q) + v_j \bar{v}_k \frac{\partial^2 \Psi_n}{\partial \bar{q}_k \partial q_j}(q) \right) \right),$$

where we have set $v(z) = \sum_{j=0}^{n-1} v_j z^j$ for a generic element of \mathcal{P}_{n-1} , the latter being naturally identified with the tangent space to \mathcal{M}_n at q , and we continue to consider $q_j, \bar{q}_j, j \in \{0, \dots, n-1\}$, as coordinates on \mathcal{M}_n . Clearly, q is a nondegenerate local minimum if and only if \mathcal{Q} is positive definite, *i.e.*,

$$(4.2) \quad \mathcal{Q}(v) > 0 \quad \text{for } v \in \mathcal{P}_{n-1}, v \neq 0.$$

Let us assume that q is irreducible, hence $q \in \mathcal{M}_n$ by Proposition 4. To derive conditions that ensure the validity of (4.2), we commence by reworking the expression for \mathcal{Q} .

¹This is not true at non-critical points.

²We skim through technical difficulties here, because this manifold is not smooth; the interested reader should consult the references we give.

Any polynomial in \mathcal{P}_{2n-1} can be written $p_1 L_q + p_2 q$ for suitable $p_1, p_2 \in \mathcal{M}_{n-1}$, due to the coprimeness of L_q and q . Therefore

$$(4.3) \quad \left\langle \frac{p}{q^2}, f - \frac{L_q}{q} \right\rangle = 0 \quad \text{for any } p \in \mathcal{P}_{2n-1}$$

by (3.4) and (3.7). In view of (4.3), differentiating (3.4) with respect to q_j and evaluating at q leads us to

$$(4.4) \quad \left\langle \frac{\partial}{\partial q_j} \left(\frac{L_q}{q} \right), \frac{p}{q} \right\rangle = 0, \quad j \in \{0, \dots, n-1\}, \quad p \in \mathcal{P}_{n-1},$$

which means that $\partial(L_q/q)/\partial q_j$ belongs to V_q^\perp . Hence, we get from (3.8) that

$$(4.5) \quad \frac{\partial}{\partial q_j} \left(\frac{L_q}{q} \right) = \frac{q \partial L_q / \partial q_j - z^j L_q}{q^2} =: \frac{\tilde{q} v_j}{q^2}, \quad v_j \in \mathcal{P}_{n-1}, \quad j \in \{0, \dots, n-1\}.$$

As L_q and q are coprime, the polynomials v_j are linearly independent by construction, thus we establish a one-to-one linear correspondence on \mathcal{P}_{n-1} by setting

$$(4.6) \quad v(z) = \sum_{j=0}^{n-1} v_j z^j \quad \leftrightarrow \quad \nu(z) = - \sum_{j=0}^{n-1} v_j v_j(z).$$

Moreover, from Proposition 2 where $d = 1$ and $q^* = q$, we can write (compare (3.11))

$$(4.7) \quad f - \frac{L_q}{q} = \frac{\tilde{q} \check{q}}{q} w_q^\sigma \quad \text{for some } w_q \in H^2.$$

Note that $w_q^\sigma \in \bar{H}_0$ since f does, hence w_q is holomorphic across \mathbb{T} . Now, it follows from (3.7) and (3.1) that

$$(4.8) \quad \begin{aligned} \frac{\partial^2 \Psi_n}{\partial \bar{q}_k \partial q_j}(q) &= - \left\langle \frac{\partial^2}{\partial \bar{q}_k \partial q_j} \left(\frac{L_q}{q} \right), f - \frac{L_q}{q} \right\rangle + \left\langle \frac{\partial}{\partial q_j} \left(\frac{L_q}{q} \right), \frac{\partial}{\partial q_k} \left(\frac{L_q}{q} \right) \right\rangle \\ &= \left\langle \frac{\tilde{q} v_j}{q^2}, \frac{\tilde{q} v_k}{q^2} \right\rangle = \left\langle \frac{v_j}{q}, \frac{v_k}{q} \right\rangle \end{aligned}$$

by (4.3), (4.5), and the fact that \tilde{q}/q is unimodular on \mathbb{T} . Furthermore

$$\begin{aligned} \frac{\partial^2 \Psi_n}{\partial q_k \partial q_j}(q) &= - \left\langle \frac{\partial^2}{\partial q_k \partial q_j} \left(\frac{L_q}{q} \right), f - \frac{L_q}{q} \right\rangle + \left\langle \frac{\partial}{\partial q_j} \left(\frac{L_q}{q} \right), \frac{\partial L_q / \partial \bar{q}_j}{q} \right\rangle \\ &= - \left\langle \frac{\partial^2}{\partial q_k \partial q_j} \left(\frac{L_q}{q} \right), \frac{\tilde{q} \check{q}}{q} w_q \right\rangle \end{aligned}$$

by (4.4) and (4.7). Now, a simple computation using (4.5) yields

$$\frac{\partial^2}{\partial q_k \partial q_j} \left(\frac{L_q}{q} \right) = \frac{q(\partial^2 L_q / \partial q_k \partial q_j) - z^k (\partial L_q / \partial q_j) + z^j (\partial L_q / \partial q_k)}{q^2} - 2z^j \frac{\tilde{q} v_k}{q^3},$$

and since the first fraction on the above right-hand side belongs to \mathcal{P}_{2n-1}/q , we deduce from (4.4) and what precedes that

$$(4.9) \quad \frac{\partial^2 \Psi_n}{\partial q_k \partial q_j}(q) = 2 \left\langle \frac{z^j \tilde{q} v_k}{q^3}, \frac{\tilde{q} \check{q}}{q} w_q^\sigma \right\rangle = 2 \left\langle \frac{z^j v_k}{q}, w_q^\sigma \right\rangle,$$

since \tilde{q}/q is unimodular while $\bar{\check{q}} = q$ on \mathbb{T} . So, we get from (4.8), (4.9), and (4.6) that

$$\sum_{j=0}^{n-1} \sum_{k=0}^{n-1} v_j v_k \frac{\partial^2 \Psi_n}{\partial q_k \partial q_j}(q) = -2 \left\langle \frac{v v}{q}, w_q^\sigma \right\rangle = -2 \left\langle \frac{v}{q}, (v w_q)^\sigma \right\rangle$$

and

$$\sum_{j=0}^{n-1} \sum_{k=0}^{n-1} v_j \bar{v}_k \frac{\partial^2 \Psi_n}{\partial \bar{q}_k \partial q_j}(q) = \left\langle \frac{v}{q}, \frac{v}{q} \right\rangle = \left\| \frac{v}{q} \right\|_2^2.$$

Therefore, in view of (4.1) the quadratic form $\mathcal{Q}/2$ can be rewritten as

$$(4.10) \quad \frac{1}{2} \mathcal{Q}(v) = \left\| \frac{v}{q} \right\|_2^2 - 2 \operatorname{Re} \left\langle \frac{v}{q}, (v w_q)^\sigma \right\rangle = \left\| \frac{v}{q} \right\|_2^2 - 2 \operatorname{Re} \left\langle \left(\frac{v}{q} \right)^\sigma, v w_q \right\rangle.$$

To manage the above expression, we assume that L_q/q does not interpolate f on \mathbb{T} , i.e. that w_q has no zeros there, and we let $Q \in \mathcal{M}_l$ have the same zeros as w_q in \mathbb{D} , counting multiplicities. Thus we can write $w_q = o_q Q/\tilde{Q}$, where o_q is holomorphic and zero-free on a neighborhood of $\bar{\mathbb{D}}$, while $|o_q| = |w_q|$ on \mathbb{T} since Q/\tilde{Q} is unimodular there³. Consider now the Hankel operator Γ , with symbol $s_q := L_q/(o_q q \tilde{q})$, i.e.

$$\begin{aligned} \Gamma : H^2 &\rightarrow \bar{H}_0^2 \\ u &\mapsto P_- (s_q u), \end{aligned}$$

where P_- is the orthogonal projection from L_2 onto \bar{H}_0^2 . Observe that Γ is well defined because s_q is bounded on \mathbb{T} , and since the latter is meromorphic in \mathbb{D} with poles at the zeros of q , counting multiplicities. It is elementary [27] that $\Gamma(H^2) = V_q$, that $\operatorname{Ker} \Gamma = (q/\tilde{q})H^2$, and that $\Gamma : V_{\tilde{q}} \rightarrow V_q$ is an isomorphism, where $V_{\tilde{q}} := \mathcal{P}_{n-1}/\tilde{q}$ is readily seen to be the orthogonal complement of $\operatorname{Ker} \Gamma$ in H^2 . Thus, there exists an operator $\Gamma^\# : V_q \rightarrow V_{\tilde{q}}$, which is inverse to $\Gamma|_{V_{\tilde{q}}}$. To evaluate $\Gamma^\#$, observe from (4.5) that

$$\begin{aligned} \Gamma(v o_q) &= P_- \left(\frac{v L_q}{q \tilde{q}} \right) = \sum_{j=0}^{n-1} v_j P_- \left(\frac{z^j L_q}{q \tilde{q}} \right) = \sum_{j=0}^{n-1} v_j P_- \left(\frac{q(\partial L_q / \partial q_j) - \tilde{q} v_j}{q \tilde{q}} \right) \\ &= \sum_{j=0}^{n-1} v_j P_- \left(\frac{-\tilde{q} v_j}{q \tilde{q}} \right) = P_- \left(\frac{v}{q} \right) = \frac{v}{q}, \end{aligned}$$

where we used that $(\partial L_q / \partial q_j) / \tilde{q} \in H^2$ and that $v/q \in \bar{H}_0^2$. Hence we may write

$$(4.11) \quad \Gamma^\# \left(\frac{v}{q} \right) = v o_q + u, \quad \text{with } u \in \frac{q}{\tilde{q}} H^2 = \operatorname{Ker} \Gamma,$$

³The function o_q is none but the *outer factor* of w_q in H^2 , see [21, Thm. 2.8].

and since $w_q/o_q \in H^\infty$ it follows that $uw_q/o_q \in \text{Ker } \Gamma$ as well, entailing by (4.10) and (4.11) that

$$\frac{1}{2} \mathcal{Q}(v) = \left\| \frac{v}{q} \right\|_2^2 - 2 \operatorname{Re} \left\langle \left(\frac{v}{q} \right)^\sigma, \frac{w_q}{o_q} \Gamma^\# \left(\frac{v}{q} \right) \right\rangle$$

because $(v/q)^\sigma \in V_{\tilde{q}} = (\text{Ker } \Gamma)^\perp$. Altogether, we see that

$$\frac{1}{2} \mathcal{Q}(v) \geq \left\| \frac{v}{q} \right\|_2^2 - 2 \left| \left\langle \left(\frac{v}{q} \right)^\sigma, \frac{w_q}{o_q} \Gamma^\# \left(\frac{v}{q} \right) \right\rangle \right| \geq (1 - 2\|\Gamma^\#\|) \left\| \frac{v}{q} \right\|_2^2$$

by the Schwarz inequality and since $|w_q/o_q| = 1$ on \mathbb{T} while the σ operation preserves the norm. The inequalities above imply that \mathcal{Q} is positive definite as soon as $\|\Gamma^\#\| < 1/2$. This last inequality is equivalent to saying that 2 is strictly less than the smallest singular value of $\Gamma|_{V_{\tilde{q}}}$, which is also the n -th singular value of Γ since $V_{\tilde{q}}$ has dimension n and is the orthogonal complement of $\text{Ker } \Gamma$ in H^2 . By the Adamjan-Arov-Krein theorem [27], the singular value in question is equal to the error in L^∞ -best approximation to s_q from H_{n-1}^∞ , where H_{n-1}^∞ stands for the set of functions of the form b/χ where $b \in H^\infty$ and $\chi \in \mathcal{M}_{n-1}$. Let us indicate this approximation number by σ_{n-1} :

$$\sigma_{n-1} := \inf_{g \in H_{n-1}^\infty} \|s_q - g\|_{L^\infty}.$$

As s_q is holomorphic on a neighborhood of $\overline{\mathbb{D}}$, it follows from the Adamjan-Arov-Krein theory that the infimum is uniquely attained at some $g_{n-1} \in H_{n-1}^\infty$ which is holomorphic on a neighborhood of \mathbb{T} , that $|s_q - g_{n-1}|(\xi) = \sigma_{n-1}$ for all $\xi \in \mathbb{T}$, and that $\mathbf{w}_\mathbb{T}(s_q - g_{n-1}) \leq -2n + 1$ as soon as $\sigma_{n-1} > 0$, where $\mathbf{w}_\mathbb{T}$ stands for the usual winding number of a non-vanishing continuous function on \mathbb{T} .

We will appeal to a de la Vallée-Poussin principle for this type of approximation, to the effect that

$$(4.12) \quad \sigma_{n-1} \geq \inf_{\mathbb{T}} |s_q - g|,$$

whenever $g \in H_{n-1}^\infty$ is such that

$$\mathbf{w}_\mathbb{T}(s_q - g) \leq 1 - 2n.$$

This principle is easily deduced from the Rouché theorem, for if (4.12) did not hold then the inequality

$$|(g_{n-1} - g) - (s_q - g)| = |g_{n-1} - s_q| = \sigma_{n-1} < |s_q - g|$$

would imply that $\mathbf{w}_\mathbb{T}(g_{n-1} - g) = \mathbf{w}_\mathbb{T}(s_q - g) \leq 1 - 2n$, which is impossible unless $g_{n-1} = g$ because $g_{n-1} - g$ is meromorphic with at most $2n - 2$ poles in \mathbb{D} .

Hence, with our assumptions, that q is an irreducible critical point and that $f - L_q/q$ has no zero on \mathbb{T} , we find that \mathcal{Q} will be positive definite if there exists $\Pi_q \in \mathcal{R}_{n-1}$ such that

$$(4.13) \quad 2|f - L_q/q| < |\Pi_q - L_q/q| \quad \text{on } \mathbb{T} \quad \text{and} \quad \mathbf{w}_\mathbb{T}(f - \Pi_q) \leq 1 - 2n.$$

Indeed, in this case, we will get

$$(4.14) \quad 2 < \left| \frac{L_q/q - \Pi_q}{\tilde{q}\tilde{q}^\vee w_q^\sigma/q} \right| = \left| \frac{L_q}{o_q q \tilde{q}} - \frac{\Pi_q}{\tilde{q} o_q} \right| = \left| s_q - \frac{\Pi_q}{\tilde{q} o_q} \right| \quad \text{on } \mathbb{T},$$

because $|\check{q}/q| \equiv 1$ and $|w_q^\sigma| \equiv |w_q| \equiv |o_q|$ on \mathbb{T} . Moreover, it follows from (4.13) and the triangle inequality that $|f - L_q/q| < |\Pi_q - f|$, and therefore

$$\left| (L_q/q - \Pi_q) - (f - \Pi_q) \right| < \left| f - \Pi_q \right|$$

so that $\mathbf{w}_{\mathbb{T}}(L_q/q - \Pi_q) = \mathbf{w}_{\mathbb{T}}(f - \Pi_q)$ by Rouché's theorem. Consequently

$$\mathbf{w}_{\mathbb{T}} \left(s_q - \frac{\Pi_q}{\check{q}o_q} \right) = \mathbf{w}_{\mathbb{T}} \left(\frac{L_q/q - \Pi_q}{o_q\check{q}} \right) = \mathbf{w}_{\mathbb{T}}(L_q/q - \Pi_q) = \mathbf{w}_{\mathbb{T}}(f - \Pi_q),$$

where we used that $o_q\check{q}$ does not vanish on $\bar{\mathbb{D}}$. So we see that $\Pi_q/\check{q}o_q$ can be used as g in (4.12) to bound σ_{n-1} from below by a quantity which, by (4.14), is strictly bigger than 2. Altogether, rewriting (4.13) in the equivalent form (4.15) below, we proved:

Theorem 2 (Comparison Criterion). *Let $f \in \bar{H}_0$ and $q \in \mathcal{M}_n$ be an irreducible critical point of $\Psi_{f,n}$ such that $f - L_q/q$ does not vanish on \mathbb{T} . Then q is a nondegenerate local minimum as soon as there exists $\Pi_q \in \mathcal{R}_{n-1}$ satisfying*

$$(4.15) \quad 2 < \left| 1 - \frac{f - \Pi_q}{f - L_q/q} \right| \quad \text{on } \mathbb{T} \quad \text{and} \quad \mathbf{w}_{\mathbb{T}}(f - \Pi_q) = 1 - 2n.$$

In order to use this criterion, we need an appraisal of the error in interpolation to f by members of \mathcal{R}_n and \mathcal{R}_{n-1} . In the next section, we gather the necessary estimates for the class of functions introduced in Theorem 1 after the works [13, 31].

5. ERROR IN RATIONAL INTERPOLATION

Let us recall the notion of a diagonal multipoint Padé approximant. Conceptually, a diagonal (multipoint) Padé approximant to a function g holomorphic in a domain $\Omega \subset \bar{\mathbb{C}}$, is a rational function of type (n, n) ⁴ that interpolates g in a prescribed system of $2n + 1$ points of Ω , counting multiplicities. However, such a definition may not work and it is best to adopt a linearized one as follows. Without loss of generality, we normalize one interpolation point to be infinity and assume that $g(\infty) = 0$. The remaining $2n$ interpolation points, finite or infinite, form a set \mathcal{S}_{2n} accounting for multiplicities with repetition. Let Q_{2n} be a polynomial vanishing exactly at the finite points of \mathcal{S}_{2n} . Then:

Definition (Padé Approximant). *The n -th diagonal (multipoint) Padé approximant to g associated with \mathcal{S}_{2n} is the rational function $\Pi_n = p_n/\ell_n$ satisfying:*

- $\deg p_n \leq n$, $\deg \ell_n \leq n$, and $\ell_n \not\equiv 0$;
- $(\ell_n(z)g(z) - p_n(z))/Q_{2n}(z)$ is analytic in Ω ;
- $(\ell_n(z)g(z) - p_n(z))/Q_{2n}(z) = O(1/z^{n+1})$ as $z \rightarrow \infty$.

The conditions for p_n and ℓ_n amount to solving a system of $2n + 1$ homogeneous linear equations with $2n + 2$ unknown coefficients, and clearly no solution can be such that $\ell_n \equiv 0$. Moreover it is plain to see that all pairs (p_n, ℓ_n) define the same rational function p_n/ℓ_n , thus a Padé approximant indeed exists uniquely with the above definition. As a result of our normalization, note that the third condition in the above definition entails at least one interpolation condition at infinity, therefore Π_n is in fact of type $(n - 1, n)$.

⁴A rational function is said to be of type (n_1, n_2) if it can be written as the ratio of a polynomial of degree at most n_1 by a polynomial of degree at most n_2 .

For our present purpose, we shall be interested only in the case where $g = f$ is as in Theorem 1, and \mathcal{S}_{2n} consists of n points, each of which appears with multiplicity 2. In other words, we let E_n consist of n points, repeated according to their multiplicities, in the analyticity domain of f , say, $D_f = \overline{\mathbb{C}} \setminus ([a, b] \cup \Lambda)$ where $[a, b] := \text{supp}(\mu)$ and Λ is the set of poles of r . We further let v_n be the monic polynomial whose roots are the finite points of E_n , and we obtain Π_n from the previous definition where $g = f$ and $Q_{2n} = v_n^2$.

Next, we let n range over \mathbb{N} and we put $\mathcal{E} = \{E_n\}$ for the *interpolation scheme*, i.e. the sequence of sets of interpolation points. By definition, the *support* of \mathcal{E} is $\text{supp}(\mathcal{E}) := \bigcap_{n \in \mathbb{N}} \overline{\bigcup_{k \geq n} E_k}$. We also introduce the *probability counting measure* of E_n to be the measure with mass $1/n$ at each point of E_n , repeating according to multiplicities.

We shall need strong asymptotics on the behaviour of Π_n as $n \rightarrow \infty$. To describe them, we need some more notation. Let us denote by

$$w(z) := \sqrt{(z-a)(z-b)}, \quad w(z)/z \rightarrow 1,$$

the holomorphic branch of the square root outside of $[a, b]$ which is positive on $(b, +\infty)$, and by

$$(5.1) \quad \phi(z) := \frac{2}{b-a} \left(z - \frac{b+a}{2} - w(z) \right)$$

the conformal map of $D := \overline{\mathbb{C}} \setminus [a, b]$ into \mathbb{D} such that $\phi(\infty) = 0$ and $\phi'(\infty) > 0$. Note that ϕ is conjugate-symmetric.

Recall that the logarithmic energy of a positive Borel measure σ , compactly supported in \mathbb{C} , is given by $-\int \int \log|z-t| d\sigma(z) d\sigma(t)$, which is a real number or $+\infty$.

Definition (Admissibility). *An interpolation scheme \mathcal{E} is called admissible if the sums $\sum_{e \in E_n} |\phi(e) - \phi(\bar{e})|$ are uniformly bounded with n , $\text{supp}(\mathcal{E}) \subset D_f$, and the probability counting measure of E_n converges weak* to some Borel measure σ with finite logarithmic energy⁵.*

The weak* convergence in the above definition is understood upon regarding complex measures on \mathbb{C} as the dual space of continuous functions with compact support.

To an admissible scheme \mathcal{E} , we associate a sequence of functions on D_f by putting

$$(5.2) \quad R_n(z) = R_n(\mathcal{E}; z) := \prod_{e \in E_n} \frac{\phi(z) - \phi(e)}{1 - \phi(z)\phi(e)}, \quad z \in D.$$

Each R_n is holomorphic in D , has continuous boundary values from both sides of $[a, b]$, and vanishes only at points of E_n . Note from the conjugate-symmetry of ϕ that

$$\frac{\phi(z) - \phi(e)}{1 - \phi(z)\phi(e)} = \frac{\phi(z) - \overline{\phi(e)}}{1 - \phi(z)\phi(e)} \left(1 + \frac{\phi(\bar{e}) - \phi(e)}{\phi(z) - \phi(\bar{e})} \right).$$

Thus, R_n is a Blaschke product with zero set $\overline{\phi(E_n)}$ composed with ϕ , times an infinite product which is boundedly convergent on any curve separating $[a, b]$ from $\text{supp}(\mathcal{E})$ by the admissibility conditions. In particular, $\{R_n\}$ converges to zero locally uniformly in D .

⁵Note that σ may not be compactly supported. In this case, pick $z_0 \in \mathbb{C} \setminus \text{supp}(\mathcal{E})$ such that $z_0 \notin \text{supp}(\sigma)$ and set $M_{z_0}(z) := 1/(z - z_0)$. Then, all the sets $M_{z_0}(E_n)$ are contained in a common compact set and their counting measures converge weak* to σ' such that $\sigma'(B) := \sigma(M_{z_0}^{-1}(B))$ for any Borel set $B \subset \mathbb{C}$. What we require is then the finiteness of the logarithmic energy of σ' .

To describe asymptotic behavior of multipoint Padé approximants, we need two more concepts. Let h be a Dini-continuous function on $[a, b]$. Then the *geometric mean* of h , given by

$$G_h := \exp \left\{ \int \log h(t) d\omega_{[a,b]}(t) \right\},$$

is independent of the actual choice of the branch of the logarithm [13, Sec. 3.3]. Moreover, the *Szegő function* of h , defined as

$$S_h(z) := \exp \left\{ \frac{w(z)}{2} \int \frac{\log h(t)}{z-t} d\omega_{[a,b]}(t) - \frac{1}{2} \int \log h(t) d\omega_{[a,b]}(t) \right\}, \quad z \in D,$$

does not depend on the choice of the branch either (as long as the same branch is taken in both integrals) and is the unique non-vanishing holomorphic function in D that has continuous boundary values from each side of $[a, b]$ and satisfies $h = G_h S_b^+ S_b^-$ and $S_b(\infty) = 1$. The following theorem was proved in [13, Thm. 4] when $r = 0$ and in [31] for the general case.

Theorem 3. *Let \mathfrak{f} be as in Theorem 1, \mathcal{E} an admissible interpolation scheme, and $\{\Pi_n\}$ the sequence of diagonal Padé approximants to \mathfrak{f} associated with \mathcal{E} . Then*

$$(5.3) \quad (\mathfrak{f} - \Pi_n)\omega = [2G_\mu + o(1)](S_\mu R_n/R)^2$$

locally uniformly in $D_\mathfrak{f}$, where R_n is as in (5.2) and

$$R(z) := \prod (\phi(z) - \phi(e)) / (1 - \phi(z)\phi(e)),$$

the product defining R being taken over the poles of r according to their multiplicity.

Let now $\{q_n\}$ be a sequence of irreducible critical points for $\Psi_{\mathfrak{f},n}$. Put $q_n(z) = \prod_{1 \leq j \leq n} (z - \xi_{j,n})$. It follows from Proposition 2 that L_{q_n}/q_n interpolates \mathfrak{f} at every $1/\xi_{j,n}$ with order 2, hence L_{q_n}/q_n is the n -th diagonal Padé approximants associated with $\mathcal{E}_{\{q_n\}} := \{1/\xi_{j,n}\}_{j=1}^n$. This interpolation scheme of course depends on q_n , which accounts for the nonlinear character of the L^2 -best rational approximation problem. The next theorem contains in its statement the Green equilibrium distribution of $\text{supp}(\mu) = [a, b]$, for the definition of which we refer the reader to [28].

Theorem 4. *Let \mathfrak{f} be as in Theorem 1 and $\{q_n\}$ be a sequence of irreducible critical points for \mathfrak{f} . Then $\mathcal{E}_{\{q_n\}}$ is an admissible interpolation scheme, and moreover*

$$(5.4) \quad \sum_{j=1}^n |\text{Im}(\xi_{j,n})| \leq \text{const.}$$

where *const.* is independent of n . Also, the probability counting measures of the zeros of q_n converges to the Green equilibrium distribution on $\text{supp}(\mu)$. In addition, it holds that

$$(5.5) \quad (\mathfrak{f} - L_{q_n}/q_n)\omega = [2G_\mu + o(1)](S_\mu R_n/R)^2$$

locally uniformly in $D_\mathfrak{f}$, where R_n is as in (5.2) and R is as in Theorem 3.

A few comments on Theorem 4 are in order. First, the weak* convergence of the counting measures of the q_n was obtained in [15, Thm. 2.1]. It entails that the probability counting measures of the sets $E_{\{q_n\}}$ converge weak* to the reflection of the Green equilibrium measure across \mathbb{T} , which has finite energy. The admissibility of $\mathcal{E}_{\{q_n\}}$ follows easily from this and from the bound (5.4) which was proven in [4], see [31, Lem. 8]. Then relation (5.5) is a consequence of (5.3).

6. PROOF OF THEOREM 1

To prove Theorem 1, we follow the line of argument developed in [10, Thm. 1.3]. The main difference is that in the present case the critical points are no longer a priori irreducible and their poles no longer belong to the convex hull of the support of the measure. As we shall see, these difficulties can be resolved with the help of Theorem 4.

Proof of Theorem 1. We claim there exists $N = N(f) \in \mathbb{N}$ such that all the critical points of $\Psi_n = \Psi_{f,n}$ in $\overline{\mathcal{M}}_n$ are irreducible for $n > N$. Indeed, assume to the contrary that there exists an infinite subsequence of reducible critical point, say $\{q_{n_j}\}$. It follows from Propositions 2 and 4 that each q_{n_j} has a factor $q_{n_j}^*$ such that $q_{n_j}^* \in \mathcal{M}_{n_j - k_{n_j}}$ is an irreducible critical point of $\Psi_{n_j - k_{n_j}}$, and the difference $f - L_{q_{n_j}^*}/q_{n_j}^*$ vanishes at the zeros of $\widetilde{q_{n_j}^*}^2 d_{n_j}$ where d_{n_j} is a non-constant polynomial of degree at least $\lfloor (k_j + 1)/2 \rfloor \geq 1$ having all its zeros in $\{|z| \geq 1\}$. Suppose first that $(n_j - k_{n_j}) \rightarrow \infty$ as $j \rightarrow \infty$. Then, the asymptotic behavior of $f - L_{q_{n_j}^*}/q_{n_j}^*$ is governed by (5.5), in particular it can only vanish at the zeros of $\widetilde{q_{n_j}^*}^2$ for all large n which contradicts the assumption that d_{n_j} is non-constant. Second, suppose that $n_j - k_{n_j}$ remains bounded. Up to a subsequence, we may suppose that $n_j - k_{n_j} = l$ for some integer l . As $\overline{\mathcal{M}}_l$ is compact, we may assume that $q_{n_j}^*$ converges to some $q \in \overline{\mathcal{M}}_l$. Since L_v is a smooth function of v in some neighborhood of $\overline{\mathcal{M}}_l$ (see Subsection 3.3), the polynomials $L_{q_{n_j}^*}$ converge to L_q hence $q_{n_j}^* f - L_{q_{n_j}^*}$ converges to $qf - L_q$ locally uniformly in D_f . In particular, if we pick $0 < \rho < 1$ such that \mathbb{D}_ρ contains the zeros of q , we get that $f - L_{q_{n_j}^*}/q_{n_j}^*$ is a normal family of functions converging to $f - L_q/q$ in $|z| > \rho$. But since the number of zeros it has in $\overline{\mathbb{C}} \setminus \mathbb{D}$ increases indefinitely (because it vanishes at the zeros of d_{n_j} which are at least $\lfloor (n - l + 1)/2 \rfloor$ in number), we conclude that $f = L_q/q$, which is impossible since f is not rational. This contradiction proves the claim.

As we just showed, each critical point q of Ψ_n in $\overline{\mathcal{M}}_n$, is irreducible for all n large enough, in particular it belongs to \mathcal{M}_n and moreover L_q/q does not interpolate f on \mathbb{T} . Assume further that, for all such n , there exists a rational function $\Pi_q \in \mathcal{R}_{n-1}$ such that (4.15) holds with $f = f$. Then q is a local minimum by Theorem 2, and therefore it is the unique critical point of order n in view of the Index Theorem. Thus, to finish the proof, we need only construct some appropriate function Π_q for each critical point q of Ψ_n , provided that n is large enough.

Let $\mathcal{E}_{\{q_n\}}$ be the interpolation scheme induced by $\{q_n\}$ and \mathcal{E}_v some admissible interpolation scheme, with $\text{supp}(\mathcal{E}_v) \subset \{|z| > 1\}$. Set $\{\Pi_n\}$ to be the sequence of diagonal Padé approximants to f associated with \mathcal{E}_v . Then Theorems 3 and 4 imply that when $n \rightarrow \infty$

$$(6.1) \quad \frac{(f - \Pi_{n-1})(z)}{(f - L_{q_n}/q_n)(z)} = [1 + o(1)] \left(\frac{R_{n-1}(\mathcal{E}_v; z)}{R_n(\mathcal{E}_{\{q_n\}}; z)} \right)^2 \quad \text{uniformly on } \mathbb{T}.$$

Moreover, for all n large enough, $(f - \Pi_{n-1})$ is holomorphic outside of \mathbb{D} by (5.3) and it has $2n - 1$ zeros there, namely those of $R_{n-1}(\mathcal{E}_v; \cdot)$, counting multiplicities, plus one at infinity. Consequently $w_{\mathbb{T}}(f - \Pi_{n-1}) = 1 - 2n$ for all such n , and so (4.15) will follow from (6.1) upon constructing \mathcal{E}_v

such that, for n large enough,

$$(6.2) \quad \left| 1 - \left(\frac{R_{n-1}(\mathcal{E}_\nu; z)}{R_n(\mathcal{E}_{\{q_n\}}; z)} \right)^2 \right| > 2 \quad \text{on } \mathbb{T}.$$

For convenience, let us put $I := [a, b] = \text{supp}(\mu)$ and $I^{-1} := \{x : 1/x \in I\}$, together with $\Omega := \overline{\mathbb{C}} \setminus (I \cup I^{-1})$. Set $\varrho : \Omega \rightarrow \{1 < |z| < A\}$ to be the conformal map such that $\varrho(I) = \mathbb{T}$, $\varrho(I^{-1}) = \mathbb{T}_A$, $\lim_{z \rightarrow b^+} \varrho(z) = 1$; as is well known, the number A is here uniquely determined by the so-called condenser capacity of the pair (I, I^{-1}) [28]. Note also that, by construction, ϱ is conjugate-symmetric. Define

$$h_\nu(z) = \frac{1 - (\varrho(z)A)^2}{\varrho^2(z) - A^2}, \quad z \in \Omega,$$

which is a well-defined holomorphic function in $\Omega := \overline{\mathbb{C}} \setminus (I \cup I^{-1})$. It is not difficult to show (cf. the proof of [10, Thm 1.3] after eq. (6.24)) that $|1 - h_\nu| > 2$ on \mathbb{T} . Thus, to prove our theorem, it is sufficient to find \mathcal{E}_ν such that

$$(6.3) \quad \left(\frac{R_{n-1}(\mathcal{E}_\nu; z)}{R_n(\mathcal{E}_{\{q_n\}}; z)} \right)^2 = [1 + o(1)] h_\nu(z) \quad \text{uniformly on } \mathbb{T}.$$

For this, we shall make use of the fact, also proven in the course of [10, Thm 1.3], that h_ν can be represented as

$$h_\nu(z) := \exp \left\{ \int \log \frac{1 - \phi(z)\phi(x)}{\phi(z) - \phi(x)} d\nu(x) \right\},$$

where ν is a signed measure of mass 2 supported on I^{-1} .

Denote by $\{\xi_{j,n}\}_{j=1}^n$ the zeros of q_n and by $\{x_{j,n}\}_{j=1}^n$ their real parts. Observe from (5.5) that any neighborhood of the poles of r which is disjoint from I , contains exactly m zeros of q_n for all n large enough. We enumerate these as $\xi_{n-m+1,n}, \dots, \xi_{n,n}$. The rest of the zeros of q_n we order in such a manner that

$$a < x_{1,n} < x_{2,n} < \dots < x_{d_n,n} < b,$$

while those $j \in \{d_n + 1, \dots, n - m\}$ for which $x_{j,n}$ either lies outside of (a, b) or else coincides with $x_{k,n}$ for some $k \in \{1, \dots, d_n\}$, are numbered arbitrarily. Again from (5.5), any open neighborhood of I contains $\{\xi_{j,n}\}_{j=1}^{n-m}$ for all n large enough, and therefore

$$(6.4) \quad \delta_n^{im} := \max_{j \in \{1, \dots, d_n\}} |\text{Im}(\xi_{j,n})| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

In addition, as the probability counting measures of the zeros of q_n converge to a measure supported on the whole interval I , namely the Green equilibrium distribution, we deduce that $d_n/n \rightarrow 1$ and that

$$(6.5) \quad \delta_n^{re} := \max \left\{ (x_{1,n} - a), (b - x_{d_n,n}), \max_{j \in \{2, \dots, d_n\}} (x_{j,n} - x_{j-1,n}) \right\} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Define $\check{\nu}$ to be the image of ν under the map $t \rightarrow 1/t$, so that $\check{\nu}$ is a signed measure on I of mass 2. Let further $\varphi(z) := \phi(1/z)$ be the conformal map of $\overline{\mathbb{C}} \setminus I^{-1}$ onto \mathbb{D} , normalized so that $\varphi(0) = 0$ and $\varphi'(0) > 0$. Finally, set

$$K(z, t) := \log \left| \frac{\varphi(z) - \varphi(t)}{1 - \varphi(z)\varphi(t)} \right|.$$

To define an appropriate interpolation scheme \mathcal{E}_n , we consider the coefficients:

$$\begin{aligned} c_{1,n} &:= \check{\nu} \left(\left[a, \frac{x_{1,n} + x_{2,n}}{2} \right] \right), \\ c_{j,n} &:= \check{\nu} \left(\left[\frac{x_{j-1,n} + x_{j,n}}{2}, \frac{x_{j,n} + x_{j+1,n}}{2} \right] \right), \quad j \in \{2, \dots, d_n - 1\}, \\ c_{d_n,n} &:= \check{\nu} \left(\left[\frac{x_{d_n-1,n} + x_{d_n,n}}{2}, b \right] \right). \end{aligned}$$

Subsequently, we define two other sets of coefficients

$$\begin{cases} b_{j,n} := \sum_{k=1}^j c_{k,n} = \check{\nu} \left(\left[a, \frac{x_{j,n} + x_{j+1,n}}{2} \right] \right) \\ a_{j,n} := 2 - b_{j,n} = \check{\nu} \left(\left[\frac{x_{j,n} + x_{j+1,n}}{2}, b \right] \right) \end{cases} \quad j \in \{1, \dots, d_n - 1\},$$

and $b_{0,n} = a_{d_n,n} := 0$. It follows in a straightforward manner from the definitions that

$$2 - c_{j,n} = b_{j-1,n} + a_{j,n}, \quad j \in \{1, \dots, d_n\},$$

and therefore

$$(6.6) \quad 2 \sum_{j=1}^{d_n} K(z, \xi_{j,n}) - \sum_{j=1}^{d_n} c_{j,n} K(z, \xi_{j,n}) = \sum_{j=1}^{d_n-1} b_{j,n} K(z, \xi_{j+1,n}) + \sum_{j=1}^{d_n-1} a_{j,n} K(z, \xi_{j,n}).$$

Next, we introduce auxiliary points $y_{j,n}$ by setting

$$y_{j,n} := \frac{a_{j,n} \xi_{j,n} + b_{j,n} \xi_{j+1,n}}{2}, \quad j \in \{1, \dots, d_n - 1\}.$$

Observe that

$$(6.7) \quad |y_{j,n} - \xi_{j,n}| = \left| \frac{b_{j,n}}{2} (\xi_{j+1,n} - \xi_{j,n}) \right| \leq \frac{\|\check{\nu}\|}{2} |\xi_{j+1,n} - \xi_{j,n}|,$$

where $\|\check{\nu}\|$ is the total variation of $\check{\nu}$.

Let \mathcal{K} be compact in Ω and $\mathcal{U} \subset \mathbb{D}$ be a neighborhood of I whose closure is disjoint from \mathcal{K} .

By (6.4), (6.5), and (6.7) we see that both $\{\xi_{j,n}\}_{j=1}^{d_n} \subset \mathcal{U}$ and $\{y_{j,n}\}_{j=1}^{d_n-1} \subset \mathcal{U}$ for all n large enough.

Thus, for such n and $z \in \mathcal{K}$, we can write the first-order Taylor expansions:

$$(6.8) \quad K(z, \xi_{j,n}) - K(z, y_{j,n}) = \frac{\partial}{\partial t} K(z, y_{j,n}) (\xi_{j,n} - y_{j,n}) + O((\xi_{j,n} - y_{j,n})^2),$$

$$(6.9) \quad K(z, \xi_{j+1,n}) - K(z, y_{j,n}) = \frac{\partial}{\partial t} K(z, y_{j,n}) (\xi_{j+1,n} - y_{j,n}) + O((\xi_{j+1,n} - y_{j,n})^2),$$

and adding up (6.8) multiplied by $a_{j,n}$ to (6.9) multiplied by $b_{j,n}$ we obtain

$$(6.10) \quad b_{j,n} K(z, \xi_{j+1,n}) + a_{j,n} K(z, \xi_{j,n}) - 2K(z, y_{j,n}) = O((\xi_{j+1,n} - \xi_{j,n})^2),$$

where we took (6.7) into account and, of course, the three symbols big “Oh” used above indicate different functions. By the smoothness of K on $\overline{\mathbb{C}} \setminus I^{-1} \times \overline{\mathbb{C}} \setminus I^{-1}$ and the compactness of $\mathcal{K} \times \overline{\mathcal{U}}$,

these big “Oh” can be made uniform with respect to $z \in \mathcal{X}$, being majorized by

$$\zeta \mapsto 2\|\check{\nu}\| \sup_{(z,t) \in \mathcal{X} \times \overline{\mathcal{U}}} \left| \frac{\partial^2 K}{\partial t^2}(z, t) \right| |\zeta|^2.$$

In another connection, it is an immediate consequence of (6.4), (6.5), and (5.4) that

$$(6.11) \quad \begin{aligned} \sum_{j=1}^{d_n-1} |\xi_{j+1,n} - \xi_{j,n}|^2 &\leq \sum_{j=1}^{d_n-1} |x_{j+1,n} - x_{j,n}|^2 + 2 \sum_{j=1}^{d_n} |\operatorname{Im}(\xi_{j,n})|^2 \\ &\leq (b-a)\delta_n^{re} + \operatorname{const}.\delta_n^{im} = o(1). \end{aligned}$$

Therefore, we derive from (6.11) upon adding equations (6.10) for $j \in \{1, \dots, d_n - 1\}$ that

$$(6.12) \quad \left| \sum_{j=1}^{d_n-1} b_{j,n} K(z, \xi_{j+1,n}) + \sum_{j=1}^{d_n-1} a_{j,n} K(z, \xi_{j,n}) - 2 \sum_{j=1}^{d_n-1} K(z, y_{j,n}) \right| = o(1),$$

where $o(1)$ is uniform with respect to $z \in \mathcal{X}$. In view of (6.6), equation (6.12) can be rewritten as

$$(6.13) \quad \left| 2 \sum_{j=1}^{d_n} K(z, \xi_{j,n}) - 2 \sum_{j=1}^{d_n-1} K(z, y_{j,n}) - \sum_{j=1}^{d_n} c_{j,n} K(z, \xi_{j,n}) \right| = o(1).$$

Now, it follows from (6.5) and the definitions of $c_{j,n}$ and h_ν that

$$(6.14) \quad \sum_{j=1}^{d_n} c_{j,n} K(z, x_{j,n}) \rightarrow \int K(z, t) d\check{\nu}(t) = -\log |h_\nu(1/z)| \quad \text{as } n \rightarrow \infty,$$

uniformly with respect to $z \in \mathcal{X}$. Moreover, we deduce from (6.4) and (5.4) that

$$(6.15) \quad \begin{aligned} \sum_{j=1}^{d_n} |c_{j,n} (K(z, \xi_{j,n}) - K(z, x_{j,n}))| &\leq C \sum_{j=1}^{d_n} |c_{j,n}| |\operatorname{Im}(\xi_{j,n})| \\ &\leq C \|\check{\nu}\| \delta_n^{im} \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$, where $C = \sup_{(z,t) \in \mathcal{X} \times \overline{\mathcal{U}}} |\partial K / \partial t(z, t)|$. Hence, combining (6.14) and (6.15) with (6.13), we get

$$(6.16) \quad 2 \sum_{j=1}^{d_n-1} K(z, y_{j,n}) - 2 \sum_{j=1}^{d_n} K(z, \xi_{j,n}) \rightarrow \log |h_\nu(1/z)| \quad \text{as } n \rightarrow \infty,$$

uniformly on \mathcal{X} . Define

$$g_n(z) := \left(\prod_{j=1}^{d_n-1} \frac{\varphi(z) - \varphi(y_{j,n})}{1 - \varphi(z)\varphi(y_{j,n})} / \prod_{j=1}^{d_n} \frac{\varphi(z) - \varphi(\xi_{j,n})}{1 - \varphi(z)\varphi(\xi_{j,n})} \right)^2,$$

which is holomorphic in Ω . By (6.16), it holds that $\log |g_n(z)| \rightarrow \log |h_\nu(1/z)|$ as $n \rightarrow \infty$ uniformly on \mathcal{X} , and since the latter was arbitrary in Ω this convergence is in fact locally uniform there. Thus, $\{g_n\}$ is a normal family in Ω , and any limit point of this family is a unimodular multiple of $h_\nu(1/\cdot)$. However, $\lim_{z \rightarrow b^+} h_\nu(z) = 1$ while it follows immediately from the properties of φ that each g_n has a well-defined limit at $1/b$ which is also 1. So, $\{g_n\}$ is, in fact, a locally uniformly convergent sequence in Ω and its limit is $h_\nu(1/\cdot)$.

Finally, set $\mathcal{E}_v := \{E_{v,n}\}$, where $E_{v,n} = \{\zeta_{j,n}\}$, $\zeta_{j,n} := 1/y_{j,n+1}$ when $j \in \{1, \dots, d_{n+1} - 1\}$, and $\zeta_{j,n} := 1/\bar{\xi}_{j+1,n+1}$ when $j \in \{d_{n+1}, \dots, n\}$. Then

$$\left(R_{n-1}(\mathcal{E}_v; z) / R_n(\mathcal{E}_{\{q_n\}}; z) \right)^2 = g_n(1/z)$$

and (6.3) follows from the limit just proved that $\{g_n\} \rightarrow h_v(1/\cdot)$. Thus, it only remains to prove that \mathcal{E}_v is admissible. To show the first admissibility condition, put

$$X_n := \sum_{j=1}^{n-1} |\phi(\zeta_{j,n-1}) - \phi(\bar{\zeta}_{j,n-1})|.$$

Then, since

$$|\operatorname{Im}(y_{j,n})| \leq \frac{\|\bar{y}\|}{2} (|\operatorname{Im}(\xi_{j,n})| + |\operatorname{Im}(\xi_{j+1,n})|), \quad 1 \leq j \leq d_n - 1,$$

by the very definition of $y_{j,n}$, we get

$$\begin{aligned} X_n &= \sum_{j=1}^{d_n-1} |\varphi(y_{j,n}) - \varphi(\bar{y}_{j,n})| + \sum_{j=d_n}^{n-1} |\varphi(\xi_{j+1,n}) - \varphi(\bar{\xi}_{j+1,n})| \\ &\leq 2 \sup_{\bar{\mathcal{U}}} |\varphi'| \left(\sum_{j=1}^{d_n-1} |\operatorname{Im}(y_{j,n})| + \sum_{j=d_n}^{n-1} |\operatorname{Im}(\xi_{j+1,n})| \right) \\ &< 2 \sup_{\bar{\mathcal{U}}} |\varphi'| \left(2\|\bar{y}\| \sum_{j=1}^{d_n} |\operatorname{Im}(\xi_{j,n})| + \sum_{j=d_n+1}^n |\operatorname{Im}(\xi_{j,n})| \right) \end{aligned}$$

which is uniformly bounded by (5.4). Further, since each $E_{v,n}$ is contained in \mathcal{U}^{-1} , we have that $\operatorname{supp}(\mathcal{E}_v) \subset \{|z| > 1\}$. So, it only remains to show that the probability counting measures of $E_{v,n}$ converges weak* to some Borel measure with finite logarithmic energy. Now, since $d_n/n \rightarrow 1$ as $n \rightarrow \infty$, and by the remark made in footnote 3, it is enough to prove that this property holds for the probability counting measures of the points $\{y_{j,n}\}$. But from (6.7) and (6.11), the latter have the same asymptotic distribution as the points $\{\xi_{j,n}\}$, namely the Green equilibrium distribution on I by Theorem 4. This finishes the proof of Theorem 1. \square

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INRIA, PROJECT APICS, 2004 ROUTE DES LUCIOLES — BP 93, 06902 SOPHIA-ANTIPOLIS, FRANCE
E-mail address: laurent.baratchart@sophia.inria.fr

INRIA, PROJECT APICS, 2004 ROUTE DES LUCIOLES — BP 93, 06902 SOPHIA-ANTIPOLIS, FRANCE
E-mail address: myattsel@sophia.inria.fr