Honors Project 17b: Measure Theory

Objective
To introduce the concept of measure and relate it to integration.

Background Required
Integration and the Fundamental Theorem of Calculus.

Narrative
Wherever quantitative analysis is involved, so are measurements. And this includes a great deal of territory: it includes many (if not all) areas of Science and Engineering, as well as subjects such as Economics. This being the case, it makes sense that we take a closer look at the concept of measure. It makes sense particularly since a closer look at one of our earliest experiences with measure reveals certain serious limitations of it.

The experience we're referring to is the use of measurement — or scale — in reading geographical maps. The scale provided with most maps is useful in providing a rough idea of distance. However, the use of a map's scale has limitations. One arises from the fact that the earth is curved, and a flat map cannot be made of a curved surface without introducing some distortion. This problem is reduced if we focus attention on a small part of the earth’s surface; but another serious problem remains. The problem is that we cannot make an accurate map of a hilly or mountainous region: in such a region, the distance between two points on a map might be small even though the actual distance between the places in the real world to which they correspond is quite large because of differences in elevation.

Thus we cannot simply use a ruler to accurately measure distance on a map. In fact, not only are there problems with measuring distances on a map, there are also problems measuring areas. We discuss these issues — the measurement of distances and areas — further in MATH 261. In this project we restrict our attention to establishing a framework for discussing accurate measurement of length.

Before discussing what others have done, it might be a good idea to ask yourself a few questions:

1. What do I (and others) mean by “measure”?
2. What properties should the measurement of distances, for example, have?
3. What kind of quantitative framework might embody the answer to (1) and satisfy the properties of (2)?

Well, the way mathematicians look at it, a measure on the real number line $\mathbb{R}$ is a function $m$ which associates to each closed interval $[a, b] \subset \mathbb{R}$ a non-negative real number $m([a, b])$ for which:

1. $m([a, a]) = 0$ for each $a$.
2. If $c \in (a, b)$ then $m([a, b]) = m([a, c]) + m([c, b])$.
3. For any two non-overlapping intervals $I_1$ and $I_2$, $m(I_1 \cup I_2) = m(I_1) + m(I_2)$.
4. For any two intervals $I_1$ and $I_2$, $m(I_1 \cup I_2) = m(I_1) + m(I_2) - m(I_1 \cap I_2)$.

For example, if $g : \mathbb{R} \to \mathbb{R}$ is a piecewise continuous function for which $g(x) > 0$ for all $x \in \mathbb{R}$ then

\[
m([a, b]) = \int_{x=a}^{b} g(x) \, dx
\]

defines a measure on $\mathbb{R}$.
Exercise 1: If \( g(x) = 1 \), what is \( m([a,b]) \)?

Exercise 2: If \( g(x) = 2 \), what is \( m([a,b]) \)?

Exercise 3: If \( g(x) = x^2 \), what is \( m([a,b]) \)?

Exercise 4: Explain why \( g(x) > 0 \) for all \( x \in \mathbb{R} \) implies that \( m([a,b]) > 0 \).

Exercise 5: Why don’t we use \( g(x) \geq 0 \) (instead of \( g(x) > 0 \)) in the definition of measure?

Exercise 6: Explain why, for a measure defined by (*),

1. \( m([a,a]) = 0 \) for each \( a \).
2. If \( c \in (a, b) \) then \( m([a,b]) = m([a,c]) + m([c,b]) \).
3. For any two non-overlapping intervals \( I_1 \) and \( I_2 \), \( m(I_1 \cup I_2) = m(I_1) + m(I_2) \).
4. For any two intervals \( I_1 \) and \( I_2 \), \( m(I_1 \cup I_2) = m(I_1) + m(I_2) - m(I_1 \cap I_2) \).

To obtain further insight into this framework for discussing measure, observe that if \( G \) is any antiderivative (such as \( G(x) = \int_{t=a}^{x} g(t) \, dt \)) of \( g \) then the Fundamental Theorem of Calculus implies that

\[
m([a,b]) = \int_{x=a}^{b} g(x) \, dx = G(b) - G(a).
\]

Thus, for example, if \( g(x) = 0.5 \) then \( G(x) = \int_{t=0}^{x} 0.5 \, dt = 0.5x \) and

\[
m([a,b]) = \int_{x=a}^{b} 0.5 \, dx = 0.5(b-a).
\]

The insight this last computation provides is illustrated in Fig. 1: Since (see Fig. 1(b))

\[
m([a,b]) = G(b) - G(a) = dG = g(x) \, dx = g(x) (b - a),
\]

we can convert \( x \)-distances to \( y \)-distances by projecting an \( x \)-segment up vertically to the graph of \( G \), and then projecting horizontally back to the \( y \)-axis.

![Figure 1](image-url)

**Figure 1**: The case in which \( g(x) = 0.5 \) and \( G(x) = 0.5x \).

A more interesting example is illustrated in Fig. 2. In this example, the graph of \( G \) is not linear, so the scaling is not uniform. To compute the actual measure \( m([a,b]) \) of any \( x \)-interval \([a,b]\) in this case we:

1. subdivide \([a,b]\) into a large number of subintervals, each of length \( dx = \Delta x \),
2. find the length \( dG = g(x_i) \, dx \) of each corresponding \( y \)-interval, and
3. sum up the results: 
\[ m([a, b]) = \lim_{n \to \infty} g(x_i) \Delta x = \int_{x=a}^{b} g(x) \, dx. \]

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**Exercise 7:** If \( g(x) = 2 \),

1. find \( G \),
2. find \( m([1, 2]) \), and
3. draw a figure similar to Fig. 1(b) illustrating the \( x \)-interval \([1, 2]\), the corresponding \( y \)-interval, and the mapping of one interval to the other.

**Exercise 8:** Repeat Exercise 7 with \( g(x) = x^2 \).

This approach to measure arises is in Physics in measuring mass. If we have a thin rod whose density is \( \rho = \rho(x) \) at each point \( x \) in the interval, then the total mass of the rod

\[ m = \int_{x=a}^{b} \rho(x) \, dx. \]

To find \( m \) we:

1. subdivide \([a, b]\) into a large number of subintervals, each of length \( dx = \Delta x \),
2. find the mass \( \rho(x) \, dx \) of each small piece of the rod, and
3. sum up the results, obtaining \( \int_{x=a}^{b} \rho(x) \, dx \).

One variation on this application involves finding the center of mass

\[ \bar{m} = \frac{1}{m} \int_{x=a}^{b} x \rho(x) \, dx \]

of the rod. To find the center of mass we weight the mass \( \rho(x) \, dx \) of each small piece of the rod by \( x \), and sum up the results. Another variation involves finding the higher order moments

\[ \int_{x=a}^{b} x^n \rho(x) \, dx, \quad n > 1 \]
of the rod. And yet another variation involves finding the mass
\[ \int_{x=a}^{b} f(x)\rho(x) \, dx \]
and moments
\[ \int_{x=a}^{b} x^n f(x)\rho(x) \, dx, \quad n > 1 \]
of a lamina (such as illustrated in Fig. 3) whose density \( \rho = \rho(x) \) is constant along each vertical line, but which varies as \( x \).

![Figure 3: A lamina of nonuniform density.](image)

This approach to measure also arises in probability theory where a function of the form
\[ P(a \leq X \leq b) = \int_{X=a}^{b} p(X) \, dX \]
is often referred to as a probability measure. (It should be noted that for a probability measure it is required that the integral of \( p \) over its entire domain is 1.) Indeed, in this context the concept of center of mass from Physics corresponds to the mean
\[ \bar{m} = \int_{X=a}^{b} X p(X) \, dX \]
and the higher order central moments
\[ \int_{x=a}^{b} (x - \bar{m})^n \rho(x) \, dx, \quad n > 1 \]
represent some other special numbers: the second central moment is the variance, the square root of the second central moment is the standard deviation, the third central moment is the coefficient of skewness, and the square of the fourth central moment is the kurtosis. (What distinguishes a central moment from a moment is the term \((x - \bar{m})^n \) as opposed to the term \( x^n \).)

**Exercise 9**: The density of a thin metal rod is \( \rho(x) = 0.5x + 1 \) grams per cm, where \( x \) is the distance / length in cm from one end of the rod. If the rod is 8 cm long, find:

1. the total mass of the rod,
2. the center of mass of the rod, and
3. the second moment of the rod.
Exercise 10: If the probability density function

\[ p(X) = \begin{cases} 
    X + 1 & \text{if } -1 \leq X \leq 0 \\
    -X + 1 & \text{if } 0 \leq X \leq 1 
\end{cases} \]

find:
1. the mean,
2. the first order central moment, and
3. the second order central moment.

We will discuss many of the above applications in greater detail in MATH 164 and 261. At this time our point is simply that measure theory is an important topic in subjects such as Probability and Physics.

We close this project with one final comment. Not only do the concepts of Physics arise in Probability, but Probability arises in Physics (and Chemistry) in the study of the atom: as indicated in the text ([1], p. 609, Exercise 13), “The hydrogen atom is composed of one proton and one electron, which moves about the nucleus. In the quantum theory of atomic structure, it is assumed that the electron does not move in a well defined orbit. Instead, it occupies a state known as an orbital, which may be though of as a “cloud” of negative charge surrounding the nucleus. At the state of lowest energy, called the ground state or 1s-orbital, the shape of this cloud is assumed to be a sphere centered at the nucleus. This sphere is described in terms of the probability density function . . . .” (See also [2].)

References