Honors Project 12: More on Summations

**Objective**

Summation analysis is an important technical tool which is introduced in first semester calculus and which is eventually used in numerous other places in science and engineering. In this project we pursue summation analysis a little further than it is pursued in the text.

**Background Required**

Familiarity with summation notation.

**Narrative**

Recall that
\[ \sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n, \]
and remember that this is just notation: there is generally more than one way to write the same summation. For example,
\[ \sum_{i=1}^{n} a_i = \sum_{i=0}^{n-1} a_{i+1} = \sum_{i=2}^{n+1} a_{i-1} \]

Also the variable (or index) \( i \) is a dummy variable: we could just as easily use any other available letter of the alphabet:
\[ \sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j = \sum_{k=1}^{n} a_k = \sum_{l=1}^{n} a_l \]

The following identities follow from the basic arithmetic properties of addition and multiplication:
\[ \sum_{i=1}^{n} (ka_i + lb_i) = k \sum_{i=1}^{n} a_i + l \sum_{i=1}^{n} b_i \]
\[ \sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i \quad \text{or} \quad \sum_{i=1}^{n} b_i = \sum_{i=1}^{m} a_i - \sum_{i=m+1}^{n} a_i \]

Further, we have the following identities which are useful when evaluating Riemann sums:
\[ \sum_{i=1}^{n} c = nc, \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \left( \frac{n(n+1)}{2} \right)^2 \]

We will illustrate how these (and further related identities) are derived below. First, however, we observe that these identities allow us to easily compute
\[ \sum_{i=1}^{n} (a_0 + a_1 i + a_2 i^2 + \ldots + a_k i^k) \]

**Example:**
\[ \sum_{i=1}^{50} (2 + 3i - 4i^2 + i^3) = \sum_{i=1}^{50} 2 + 3 \sum_{i=1}^{50} i - 4 \sum_{i=1}^{50} i^2 + \sum_{i=1}^{50} i^3 \]
Exercise 1: Find \( \sum_{i=1}^{100} (3 + 2i - i^2) \).

A useful technique to use in evaluating summations is “change of index of summation”.\(^1\) We illustrate this technique by rewriting \( \sum_{i=k}^{n} a_i \) in another form: First observe that

\[
\sum_{i=k}^{n} a_i = \sum_{i=k}^{n-k} a_i = \sum_{i=0}^{n-k} a_{i+k}.
\]

Now if we let \( j = i - k \), so \( i = j + k \), then

\[
\sum_{i=k}^{n} a_i = \sum_{j=0}^{n-k} a_{j+k} = \sum_{i=0}^{n} a_{i+k},
\]

the later inequality resulting from the replacement of the dummy variable \( j \) by \( i \).

Let us now illustrate how we can derive the formula for \( \sum_{i=1}^{n} i^k \) from \( \sum_{i=1}^{n} i^k \) where \( t < k \). We proceed by example, specifically addressing the case in which \( k = 2 \). We begin by looking at the sum \( \sum_{i=1}^{n+1} i^3 \). (Note that the exponent 3 of \( i \) in this sum is 1 greater than the exponent of \( i \) in the sum \( \sum_{i=1}^{n} i^2 \).)

\[
\sum_{i=1}^{n+1} i^3 = \sum_{i=0}^{n} (i+1)^3 = \sum_{i=0}^{n} (i^3 + 3i^2 + 3i + 1) = \sum_{i=0}^{n} i^3 + 3 \sum_{i=0}^{n} i^2 + 3 \sum_{i=0}^{n} i + n + 1
\]

Since the first \( n \) terms in the summation on the left-hand side of this equality are the same as the first \( n \) terms in the first sum on the right-hand side of this equality, we have

\[
(n+1)^3 = 3 \sum_{i=1}^{n} i^2 + 3 \frac{n(n+1)}{2} + n + 1
\]

Exercise 2: Solve this equation for \( \sum_{i=1}^{n} i^2 \) and put it in the form presented earlier, completing the derivation.

Exercise 3: Derive the formula presented earlier for \( \sum_{i=1}^{n} i^3 \).

**Geometric Series**

If \( r \in (-1,1) \) then we have the following identities:

\[
\sum_{i=0}^{n} r^i = 1 + r + r^2 + \ldots + r^n = 1 - r^{n+1} \frac{1}{1-r}, \quad \sum_{i=1}^{n} r^i = r + r^2 + \ldots + r^n = r \frac{1 - r^{n+1}}{1-r},
\]

\[
\sum_{i=0}^{n} ar^i = a \sum_{i=0}^{n} r^i = a(1+r+r^2+\ldots+r^n) = a \frac{1 - r^{n+1}}{1-r}, \quad \sum_{i=1}^{n} ar^i = a \sum_{i=1}^{n} r^i = a(r+r^2+\ldots+r^n) = ar \frac{1 - r^{n+1}}{1-r}
\]

\(^1\)This is a precursor to \( u \)-substitution in integration theory.
All of these identities follow easily from the first, and the first can be derived as follows: If

\[ S = \sum_{i=0}^{n} r^i = 1 + r + r^2 + \ldots + r^n \]

then

\[ rS = r + r^2 + \ldots + r^{n+1} \]

so (subtracting the later from the former) we find that

\[ S - rS = (1 + r + r^2 + \ldots + r^n) - (r + r^2 + \ldots + r^{n+1}) = 1 - r^{n+1}. \]

Since

\[ (1 - r)S = 1 - r^{n+1}, \]

it follows that

\[ S = \sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}. \]

Exercise 4: Find \( \sum_{i=0}^{50} (2 + 3r). \)
Exercise 5: Derive a formula for \( \sum_{i=0}^{n} r^{2i}. \)
Exercise 6: Derive a formula for \( \sum_{i=0}^{n} r^{i+m}. \)
Exercise 7: Derive a formula for \( \sum_{i=m}^{n} r^i. \)

The Binomial Series

The combinatorial coefficient \( \binom{n}{i} \) is defined by

\[ \binom{n}{i} = \frac{n!}{k!(n-i)!}. \]

Exercise 8: Find \( \binom{6}{2}, \binom{6}{0}, \binom{6}{1}, \) and \( \binom{6}{6}. \)

Combinatorial coefficients satisfy the identities

\[ \binom{n}{i} = \binom{n}{n-i} \quad \text{and} \quad \binom{n-1}{i-1} + \binom{n-1}{i} = \binom{n}{i}. \]

Exercise 9: Verify these two identities.

The later of these two identities allows us to construct Pascal’s Triangle:
You might recall this triangle from algebra: Observe that the upper left and upper right edges are all 1’s, and that each other entry can, starting from the top, be obtained by adding together the two neighboring integers in the row immediately above that of the entry in which we are interested.

Pascal’s Triangle is useful in expanding polynomials such as \((x + y)^n\) for integers \(n\). For example, observe that the coefficients of the third row are precisely the coefficients of

\[(x + y)^3 = x^3y^0 + 3x^2y^1 + 3x^1y^2 + x^0y^3 = x^3 + 3x^2y + 3xy^2 + y^3.\]

Exercise 10: Find \((x + y)^6\), \((x + 3)^5\), and \((x - 2)^7\).

Indeed, the Binomial Theorem states that:

\[(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}.\]

Hence, for example,

\[(x + 1)^n = \sum_{i=0}^{n} \binom{n}{i} x^i.\]

Exercise 11: Observe that (as far as we have listed Pascal’s Triangle) the sum of the entries in each row of Pascal’s Triangle is a power of 2. Prove that this is always true! That is, prove that \(\sum_{i=0}^{n} \binom{n}{i} = 2^n\).

References