Honors Project 7a: The Iteration Method

Objective

In addition to the Bisection Method and Newton’s Method, an approximate solution to the equation \( f(x) = 0 \) can be found with the Iteration Method. The Iteration Method often arises in applications such as economics in the form of “cobweb cycles”. In this project we discuss the Iteration Method.

Background Required

Differentiation and Maple programming.

Narrative

To find an approximate solution to the equation \( f(x) = 0 \) by the Iteration Method, we look for a value of \( x \) for which \( g(x) = x \) for an appropriate function \( g(x) \): an \( x \) for which \( g(x) = x \) is called a fixed point of \( g \). For example, to find an approximate solution to the equation \( x^2 - 2 = 0 \) we might choose an arbitrary value for \( x_0 \in [1, 1.99] \) and repeatedly compute \( x_n \) using the formula

\[
x_n = g(x_{n-1}), \quad n = 1, 2, 3, \ldots
\]

where \( g(x) = -\frac{1}{2}(x^2 - 2) + x \). The reason the iteration method works is that

\[
g(x) = -\frac{1}{2}(x^2 - 2) + x = x \quad \text{if and only if} \quad x^2 - 2 = 0.
\]

The reason we didn’t simply let \( g(x) = x^2 - 2 + x \) is that with this choice of \( g \), the Iteration Method does not converge. We illustrate these statements below.

One strength of the Iteration Method is that after the appropriate function \( g \) and initial value \( x_0 \) are identified, it is fairly easy to implement. One weakness of it is that given \( f \) you have to come up with \( g \), and that’s not always easy.

Task

1. a) Type the commands below into Maple in the order in which they are listed. These commands illustrate that the first method described in the Narrative converges. (Note how little it takes to implement this method!)

\[
\text{> # Honors Project 1a: The Iteration Method}
\text{> restart:}
\text{> # Task 1, Part a}
\text{> g := x -> -(x^2-2)/2+x;}
\text{> g(1.5);}
\text{> for n from 1 to 20 do g(%)end do;}
\]

b) Continue by typing the following commands into Maple. These command lines illustrate that the second method described in the Narrative diverges.

\[
\text{> # Part b}
\text{> g := x -> x^2-2+x;}
\text{> g(1.5);}
\text{> for n from 1 to 20 do g(%) end do;}
\]
At this time make a hard-copy of your typed input and Maple’s responses. Then, ...

The condition that determines whether the iteration method converges is $|g'(x)| < 1$ for all $x$ in some closed interval $I$ containing the root of the equation $g(x) = 0$.

c) Show by hand that if $g(x) = -\frac{1}{2}(x^2 - 2) + x$ then $|g'(x)| = | - x + 1 | < 1$ for all $x \in [1, 1.99]$.

d) Show by hand that if $g(x) = (x^2 - 2) + x$ then $|g'(x)| = |2x + 1| \neq 1$ for any $x \in [1, 1.99]$.

2. Use the iteration method to find a solution to the equation $\sin(\cos x) = -x$ accurate to 8 decimal places.

3. Use the iteration method to find a solution to the equation $x^3 - x - 1 = 0$ accurate to 8 decimal places.

4. The ancient Babylonians had an “averaging iteration” method

$$x_{n+1} = \frac{1}{2} \left( x_n + \frac{A}{x_n} \right)$$

for yielding a sequence $x_0, x_1, x_2, \ldots$ of successive approximations to $\sqrt{A}$.

(a) Show that if this method converges then it converges to $\sqrt{A}$.

(b) Implement this method.

(c) For what initial values $x_0$ does this method converge? Justify your answer.

Comments

1. As indicated above, one of the strengths of the iteration method is that after setting up a problem, it is very easy to program. On the other hand, since setting up a problem includes finding a function $g$ satisfying the convergence condition, and since it is not always “obvious” how to do this, getting through the “set-up” of a problem can be a weakness of the iteration method.

2. The iteration method can be extended to multiple variables.

3. The use of the iteration method in applications is one of the factors that motivates the study of “fixed point theory” in mathematics.