Reliability and maintenance policies for a two-stage shock model with self-healing mechanism

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Topics

Overview

- A two-stage shock model with self-healing mechanism is proposed : Change point is defined as the moment when the cumulative number of **valid** shocks reaches d.
- Before the change point, the system can heal when δ-**invalid** shocks reaches k in the trailing run of invalid shocks. Equivalently, the damage caused by previous i valid shocks can be healed when the number of delta-invalid shocks falls in $[ik, (i+1)k)$ in the run of invalid shocks.

- The system loses self-healing ability upon reaching change point, and then fails when the cumulative number of valid shocks reaches a prefixed value $n(n > d)$.
- Finite Markov chain imbedding approach is employed to obtain the pmf, cdf and the mean of shock length.
- Three preventive maintenance policies are proposed for the system under different monitoring condition.

Introduction (Contd.)

- Valid shock: a shock that results in a certain degree of system damage.
- δ- invalid shock: an invalid shock whose time lag with the preceding shock exceeds a given threshold δ.

Model Assumptions

- The system is operating in random environment, which is subject to a sequence of randomly occurring shocks over time
- Each shock has two possible types: Valid shock with the probability of p, and invalid shock with the probability of q (i.e., $p + q = 1$).
- X_i is time lag between the$(i - 1)$-th and i-th shock, $i \geq 1$.
  X_i are i.i.d $\sim Exp(\lambda)$. They are independent of type of shock.

Introduction: Motivating example

Consider the example of aviation industry. An aircraft can be considered as a functioning system exposed to shocks.

- Valid Shocks : Turbulences that can cause damage to the airframe.
- δ-invalid shocks : External stimuli like heat, light, electrical fields or moisture.
- These stimuli can trigger self healing mechanism that heals the system from damages.
- When the cumulative damage reaches a certain amount, the aircraft will lose the self-healing ability and then the damage caused by valid shocks starts to accumulate.
- The reliability of air craft can be increased and its service life can be extended effectively by applying the self-healing materials.

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Let us consider $n = 5, d = 3, k = 2, \delta = 1$. Consider $1', 0', 0_I, (X_1, \ldots, X_{11})$ indicate valid shocks, invalid shocks, the initial state of the sequence of shocks and the time lags, respectively

Illustration

For $n = 6, d = 3, k = 2, \delta = 1$

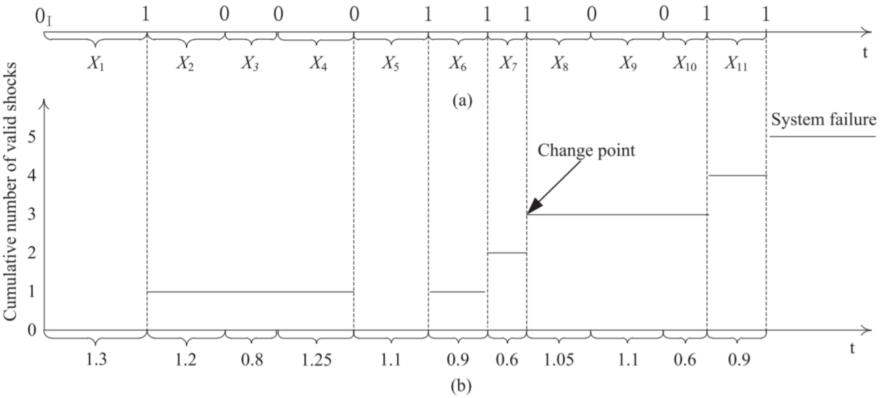

Fig. 2. Failure process of the system.

The Picturization of the two-stage model

Therefore essentially, the two-stage process looks like this:

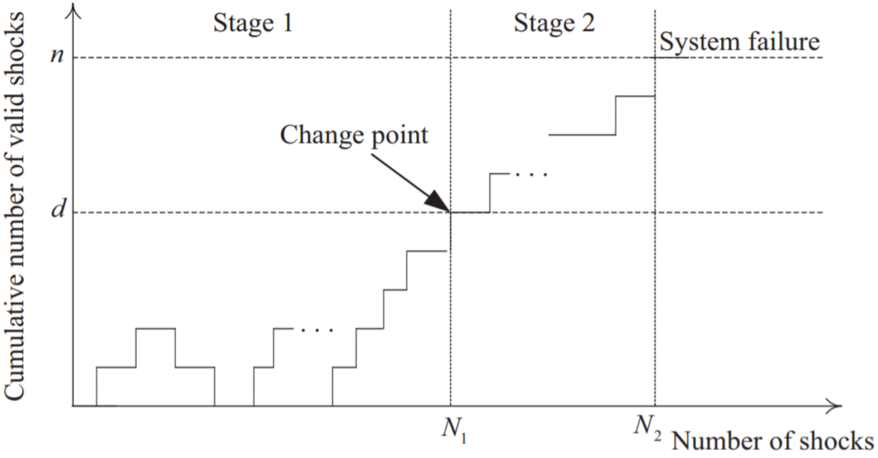

Fig. 1. Two-stage failure process of the system.

Reliability Analysis of the shock model

- S_m denotes the cumulative number of valid shocks.
- T_m represents the number of δ-invalid shocks in the trailing run of invalid shocks.
- R_m denotes the number of δ-invalid shocks which can trigger a next self-healing behavior if the system is in **stage 1** where $R_m = T_m - [\frac{T_m}{k}]k$, otherwise, $R_m = \Theta$ where Θ means the system is in **stage 2**.
- Markov chain $Y_m, m \geq 0$ associated with S_m and R_m is defined as

$$Y_m = (S_m, R_m), m \geq 0$$

The state-space

$$\Omega_m = \{(s,r) : 0 \le s \le d{-}1, 0 \le r \le k{-}1\} U \{(s,\Theta) : d \le s \le n{-}1\} U \{E_a\}$$

Initial State : $Y_0 = (0,0)$
E_a: absorbing state, cumulative number of valid shocks reaches n and the system fails.

Forming the transition matrix

The probability q of invalid shock can be further broken down:

- q_1 = probability that m-th shock is δ-invalid.
- q_2 = probability that m-th shock is an invalid shock but not δ-invalid. $(q_1 + q_2 = q)$

Example: $n = 4, d = 2, k = 2$

$$\Omega_m = \{(0,0),(0,1),(1,0),(1,1),(2,\Theta),(3,\Theta)\} \bigcup \{E_a\}$$

The state transition diagram

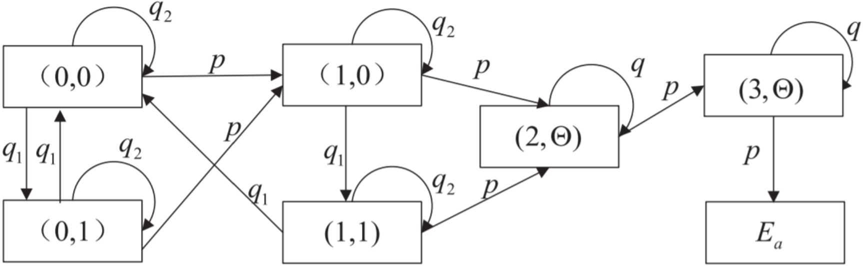

Fig. 3. State transition diagram for the Markov chain.

One Step transition probability Matrix

The one step transition probability matrix for the state-space is as follows:

$$\Lambda = \begin{array}{c} (0,0) \\ (0,1) \\ (1,0) \\ (1,1) \\ (2,\Theta) \\ (3,\Theta) \\ E_a \end{array} \left[\begin{array}{cccccc|c} q_2 & q_1 & p & 0 & 0 & 0 & 0 \\ q_1 & q_2 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & q_2 & q_1 & p & 0 & 0 \\ q_1 & 0 & 0 & q_2 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & q & p & 0 \\ 0 & 0 & 0 & 0 & 0 & q & p \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]_{7\times 7}$$

Therefore the matrix Λ

$$\Lambda = \begin{array}{c} (0,0) \\ (0,1) \\ (1,0) \\ (1,1) \\ (2,\Theta) \\ (3,\Theta) \\ E_a \end{array} \left[\begin{array}{cccccc:c} q_2 & q_1 & p & 0 & 0 & 0 & 0 \\ q_1 & q_2 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & q_2 & q_1 & p & 0 & 0 \\ q_1 & 0 & 0 & q_2 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & q & p & 0 \\ 0 & 0 & 0 & 0 & 0 & q & p \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]_{7\times 7}$$

can be written in general in the following format:

$$\Lambda = \begin{bmatrix} \mathbf{A}_m & \mathbf{B}_m & & & & & \\ \mathbf{D}_m & \mathbf{C}_m & \ddots & & & & \\ & \ddots & \ddots & \mathbf{B}_m & & & \\ & & \mathbf{D}_m & \mathbf{C}_m & \mathbf{E}_m & & \\ & & & & \mathbf{F}_m & \mathbf{G}_m & \\ & & & & & 1 & \end{bmatrix}_{\gamma\times\gamma}$$

Where $\gamma = [dk + (n - d + 1)]$ is the cardinality of the state-space Ω.

Λ can be transitioned as follows: $\Lambda = \begin{bmatrix} \mathbf{P}_{(\gamma-1)X(\gamma-1)} & \mathbf{Q}_{(\gamma-1)X1} \\ \mathbf{0}_{1X(\gamma-1)} & \mathbf{E}_{1X1} \end{bmatrix}$

One step transition probabilities:

- $\mathbf{P}_{(\gamma-1)X(\gamma-1)}$ among transient states
- $\mathbf{Q}_{(\gamma-1)X1}$ transient states to absorbing states
- $\mathbf{0}_{1X(\gamma-1)}$ absorbing state to transient state
- $\mathbf{E}_{1X1}$ identity matrix

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Combining transient states in **stage 2** and absorbing states into new absorbing state E_f (system moves from**stage 1**). $\Phi = \begin{bmatrix} \mathbf{U}_{(\gamma_1)X(\gamma_1)} & \mathbf{V}_{(\gamma_1)X1} \\ \mathbf{0}_{1X(\gamma_1)} & \mathbf{E}_{1X1} \end{bmatrix}$ where $\gamma_1 = dXk$

The general format of matrices **P**, **Q**,**U** and **V**

$$\mathbf{P} = \begin{bmatrix} \mathbf{A}_m & \mathbf{B}_m & & & \\ \mathbf{D}_m & \mathbf{C}_m & \ddots & & \\ & \ddots & \ddots & \mathbf{B}_m & \\ & & \mathbf{D}_m & \mathbf{C}_m & \mathbf{E}_m \\ & & & & \mathbf{F}_m \end{bmatrix}_{(\gamma-1)\times(\gamma-1)}, \mathbf{Q} = \begin{bmatrix} \mathbf{0}_{(d\times k)\times 1} \\ \mathbf{G}_m \end{bmatrix}_{(\gamma-1)\times 1},$$

$$\mathbf{U} = \begin{bmatrix} \mathbf{A}_m & \mathbf{B}_m & & \\ \mathbf{D}_m & \mathbf{C}_m & \ddots & \\ & \ddots & \ddots & \mathbf{B}_m \\ & & \mathbf{D}_m & \mathbf{C}_m \end{bmatrix}_{\gamma_1\times\gamma_1}, \text{ and } \mathbf{V} = \begin{bmatrix} \mathbf{0}_{(d-1)\times k} \\ \mathbf{e}'_k \end{bmatrix}_{\gamma_1\times 1}.$$

The distribution of N_1

N_1 is the total number of shocks until change point appears.
Let $\boldsymbol{\alpha} = (1, 0, \ldots, 0)_{1X\gamma_1}$, $\mathbf{I}_c = \mathbf{U}^0$ and $\mathbf{e}_{\gamma_1} = (1, \ldots, 1)_{1X\gamma_1}$
The pmf of the shock length N_1 is

$$P(N_1 = k) = \boldsymbol{\alpha}\mathbf{U}^{k-1}\mathbf{V}$$

The cdf of the shock length N_1 is

$$P(N_1 \leq k) = \sum_{i=1}^{k} \boldsymbol{\alpha}\mathbf{U}^{i-1}\mathbf{V}$$

The expected shocklength is

$$E(N_1) = \boldsymbol{\alpha}(\mathbf{I}_c - \mathbf{U})^{-1}\mathbf{e}_{\gamma_1}^{'}$$

The distribution of N_2

N_2 is the total number of shocks until the system fails.
Let $\boldsymbol{\pi}_0 = (1, 0, \ldots, 0)_{1X(\gamma-1)}$, $\mathbf{I} = \mathbf{P}^0$ and $\mathbf{e}_{(\gamma-1)} = (1, \ldots, 1)_{1X(\gamma-1)}$
The pmf of the shock length N_2 is

$$P(N_2 = k) = \boldsymbol{\pi}_0 \mathbf{P}^{k-1} \mathbf{Q}$$

The cdf of the shock length N_2 is

$$P(N_2 \leq k) = \sum_{i=1}^{k} \boldsymbol{\pi}_0 \mathbf{P}^{i-1} \mathbf{Q}$$

The expected shocklength is

$$E(N_2) = \boldsymbol{\pi}_0 (\mathbf{I} - \mathbf{P})^{-1} \mathbf{e}'_{\gamma-1}$$

Illustrative Example

- Consider time lags between two successive shocks follow the exponential distribution with parameter λ.
- $n = 6, d = 3, k = 2$ the state space is $\Omega_m = \{(0,0),(0,1),(1,0),(1,1),(2,0),(2,1),(3,\Theta),(4,\Theta),(5,\Theta),\}\bigcup\{E_a\}$.
- We consider values of parameters are chosen to be $\lambda = 0.5, \delta = 1.5, p = 0.4$. Then, $q_1 = qP\{X_i > \delta\} = 0.2834$, $q_2 = qP\{X_i \leq \delta\} = 0.3166$. The transition probability matrices are shown in next slide.
- After calculation, we find $E(N_1) = 8.6464$, $E(N_2) = 16.1464$.

Illustrative Example (Contd.)

$$\Lambda = \begin{array}{c} (0,0) \\ (0,1) \\ (1,0) \\ (1,1) \\ (2,0) \\ (2,1) \\ (3,\Theta) \\ (4,\Theta) \\ (5,\Theta) \\ E_a \end{array} \left[\begin{array}{cccccc:ccc:c} q_2 & q_1 & p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ q_1 & q_2 & p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_2 & q_1 & p & 0 & 0 & 0 & 0 & 0 \\ q_1 & 0 & 0 & q_2 & p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_2 & q_1 & p & 0 & 0 & 0 \\ 0 & 0 & q_1 & 0 & 0 & q_2 & p & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & q & p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & p & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & p \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]_{10\times 10}$$

Illustrative Example (Contd.)

$$\mathbf{P} = \begin{bmatrix} q_2 & q_1 & p & 0 & 0 & 0 & 0 & 0 & 0 \\ q_1 & q_2 & p & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_2 & q_1 & p & 0 & 0 & 0 & 0 \\ q_1 & 0 & 0 & q_2 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_2 & q_1 & p & 0 & 0 \\ 0 & 0 & q_1 & 0 & 0 & q_2 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q & p & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & p \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q \end{bmatrix}_{9\times 9}, \quad \mathbf{Q} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ p \end{bmatrix}_{9\times 1}$$

$$\mathbf{U} = \begin{bmatrix} q_2 & q_1 & p & 0 & 0 & 0 \\ q_1 & q_2 & p & 0 & 0 & 0 \\ 0 & 0 & q_2 & q_1 & p & 0 \\ q_1 & 0 & 0 & q_2 & p & 0 \\ 0 & 0 & 0 & 0 & q_2 & q_1 \\ 0 & 0 & q_1 & 0 & 0 & q_2 \end{bmatrix}_{6\times 6}, \quad \mathbf{V} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ p \\ p \end{bmatrix}_{6\times 1}.$$

Table for some other choices of parameters

p	n	d	k	$E(N_1)$	$E(N_2)$
0.4	6	4	2	11.7959	16.7959
	6	2	2	5.5193	15.5193
	6	4	4	10.2333	15.2333
	10	4	2	11.7959	26.7959
0.7	6	4	2	5.8418	8.6989
	6	2	2	2.8988	8.6131
	6	4	4	5.7177	8.5749
	10	4	2	5.8418	14.4132

Thank you!