An Empirical Likelihood Approach Of Testing of High Dimensional Symmetries

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Outline

Jackknife Empirical likelihood for Multivariate U-statistics

Let $(\mathscr{X}, \mathscr{S})$ be a measurable space and P be a probability measure on this space. Let $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$ be independent copies of a $\mathscr{X}$-valued random variable $\mathbf{Z}$ with cumulative distribution function F under P. Let $\mathbf{h} : \mathbb{R}^m \mapsto \mathbb{R}^d$ be a known function that is permutation symmetric in its m arguments. A multivariate or vector U-statistic with kernel $\mathbf{h}$ of order m is defined as

$$\mathbf{U}_n \equiv \mathbf{U}_{nm}(h) = \binom{n}{m}^{-1} \sum_{1 \leq i_1 < \ldots < i_m \leq n} \mathbf{h}(\mathbf{Z}_{i_1}, \ldots, \mathbf{Z}_{i_m}), \quad n \geq 2.$$

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- $\mathbf{h} \in \mathbf{L}_2(F^m)$, where $L_2(F^m) = \left\{\mathbf{f} : \int \|\mathbf{f}\|^2 \, dF^m < \infty\right\}$

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Lemma

(The Central Limit Theorem for Multivariate U-statistics)
Suppose the kernel $\mathbf{h}$ *is square-integrable and the dispersion matrix* $\boldsymbol{\Sigma} = \mathrm{Var}(\mathbf{U})$ *is positive definite. Then* $\sqrt{n}(\mathbf{U}_n - \boldsymbol{\theta})$ *and* $\sqrt{n}(\mathbf{U}_n^{\#} - \boldsymbol{\theta})$ *are asymptotically equivalent, hence* $\sqrt{n}(\mathbf{U}_n - \boldsymbol{\theta})$ *is asymptotically normal with mean zero and covariance matrix* $m^2\boldsymbol{\Sigma}$*, that is,*

$$\sqrt{n}(\mathbf{U}_n - \boldsymbol{\theta}) \implies \mathcal{N}(0, m^2\boldsymbol{\Sigma}).$$

Jackknife pseudo values of U-statistics

- Let $\mathbf{U}_{n-1}^{(-j)}$ denote the U-statistic based on the $n-1$ observations $\mathbf{Z}_1, \ldots, \mathbf{Z}_{j-1}, \mathbf{Z}_{j+1}, \ldots, \mathbf{Z}_n$.

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- One has

$$\tilde{\mathbf{V}}_{nj} = m\tilde{\mathbf{h}}_1(\mathbf{Z}_j) + O_p(n^{-1/2}), \quad j = 1, \ldots, n.$$

where

$$h_c(\mathbf{z}_1, \ldots, \mathbf{z}_c) = E(\mathbf{h}(\mathbf{Z}_1, \ldots, \mathbf{Z}_m)|\mathbf{Z}_1 = \mathbf{z}_1, \ldots, \mathbf{Z}_c = \mathbf{z}_c),$$
$$c = 1, \ldots, m-1.$$

- It shows that each Jackknife value $\mathbf{V}_{nj}$ depends asymptotically on $\mathbf{Z}_j$, so that $\mathbf{V}_{nj}, j = 1, \ldots, n$ are asymptotically *independent*.

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- If π_j is a probability mass placed at $\mathbf{Z}_j$, then approximately the same probability mass π_j is placed at the Jackknife value $\mathbf{V}_{nj}$ for $j = 1, \ldots, n$.
- The joint likelihood is approximately the product of these π_j's.

Jackknife Empirical likelihood with side information

$$\mathscr{R}_n(\mathbf{h},\mathbf{g}) = \sup\left\{ \prod_{j=1}^{n} n\pi_j : \pi \in \mathscr{P}_n, \sum_{j=1}^{n} \pi_j \tilde{\mathbf{V}}_{nj} = 0, \quad \sum_{j=1}^{n} \pi_j \mathbf{g}(\mathbf{Z}_j) = 0 \right\},$$

- $\mathbf{g}$ is a measurable functions from $\mathscr{Z}$ to $\mathbb{R}^r$ such that $\int \mathbf{g}\, dF = 0$ and $\int \|\mathbf{g}\|^2\, dF$ is finite.
- r is the number of equalities that express the side information, and we shall call them *constraints*.
- We allow r to depend on the sample size n, $r = r_n$, and to grow to infinity slowly with n and study the asymptotic behaviors of the empirical likelihood.

- Let $\mathbf{h}^{(i)}$ be a measurable functions from $\mathscr{X}^{m_i}$ to $\mathbb{R}^{d_i}$ which is argument-symmetric and square-integrable for $i = 1, \ldots, r$.
- Let $\tilde{\mathbf{V}}_{nj}(\mathbf{h}^{(i)})$ be the centered jackknife pseduo value of the U-statistic $\mathbf{U}_{nm_k}(\mathbf{h}^{(k)})$ of order m_i.

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- Let $\tilde{\mathbf{V}}_{nj}(\mathbf{h}^{(i)})$ be the centered jackknife pseduo value of the U-statistic $\mathbf{U}_{nm_k}(\mathbf{h}^{(k)})$ of order m_i.
- With the U-statistics as side information, we associate the empirical likelihood

$$\mathscr{R}_n(\mathbf{h}^{(1)}, \ldots, \mathbf{h}^{(r)}) = \sup\Big\{ \prod_{j=1}^{n} n\pi_j : \pi \in \mathscr{P}_n, \sum_{j=1}^{n} \pi_j \tilde{\mathbf{V}}_{nj}(\mathbf{h}^{(k)}) = 0, \quad k = 1, \ldots, r \Big\}.$$

Theorem

THEOREM 1 Let $r_n = r$ *for all n. Suppose* $\mathbf{h}^{(1)}, \ldots, \mathbf{h}^{(r)}$ *are argument-symmetric and square-integrable kernels. Assume* $\mathbf{W}(\mathbf{h}^{(1)}, \ldots, \mathbf{h}^{(r)})$ *is positive definite. Then*

$$-2 \log \mathscr{R}_n(\mathbf{h}^{(1)}, \ldots, \mathbf{h}^{(r)}) \implies \chi^2_{d_1+\cdots+d_r}.$$

Testing Uniformity

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- Let $2 \leq m \leq n$.
- $\bar{\mathbf{X}}_m = m^{-1} \sum_{i=1}^{m} \mathbf{X}_i$ (the sample mean vector);
  $\bar{\mathbf{X}}_m^o = \bar{\mathbf{X}}_m / \|\bar{\mathbf{X}}_m\|$ (the direction of sample mean);
  $R_m = \|\bar{\mathbf{X}}_m\|$ (length of the sample resultant).

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  $R_m = \|\bar{\mathbf{X}}_m\|$ (length of the sample resultant).
- Kent, Mardia and Rao (1979) proved that Uniformity $\Leftrightarrow \bar{\mathbf{X}}_m^o \perp\!\!\!\perp R_m$, which implies

$$E\big(a_k(R_m)\bar{\mathbf{X}}_m^o\big) = 0, \quad a_k \in L_{2,0}(F), k = 1, 2, \ldots,$$

where F is the distribution function of R_m, and $\{a_k\}$ is a basis of $L_{2,0}(F)$.

- For convenience, let $m = 2$ and $d = 3$.

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- $\mathbf{X} \sim \mathscr{U}(\mathcal{S}^2)$, $\mathbf{X} = (x, y, z)^\top$, and the density is $f(x, y, z) = \frac{1}{4\pi}, (x, y, z) \in \mathcal{S}^2$.
- Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be i.i.d copies of $\mathbf{X}$.

$$R(\mathbf{X}_i, \mathbf{X}_j) = \|\mathbf{X}_i + \mathbf{X}_j\|, \quad S(\mathbf{X}_i, \mathbf{X}_j) = \frac{\mathbf{X}_i + \mathbf{X}_j}{\|\mathbf{X}_i + \mathbf{X}_j\|}, \quad i, j = 1, \ldots, n.$$

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- It follows that $E\Big(a_k(R(\mathbf{X}_i, \mathbf{X}_j))S(\mathbf{X}_i, \mathbf{X}_j)\Big) = 0, \quad a_k \in L_{2,0}(F), k = 1, 2, \ldots,$

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- Assume that F is continuous. Then a basis of $L_{2,0}(F)$ is $\{\varphi_k \circ F\}$, where $\{\varphi_k\} = \{\varphi_k : k = 1, 2, \ldots\}$ is a basis of $L_{2,0}(\mathscr{U})$ with $\mathscr{U}$ the uniform distribution over $[0, 1]$.

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- We usually use the trignometric basis $\varphi_k(t) = \sqrt{2}\cos k\pi t$.

- In this case, an estimator of the above expected value is the vector U-statistics

$$\mathbf{U}_n = \binom{n}{m}^{-1} \sum_{1 \le i < j \le n} a_k(R(\mathbf{X}_i, \mathbf{X}_j)) S(\mathbf{X}_i, \mathbf{X}_j)$$

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  This suggests the jackknife empirical likelihood

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$$-2\log\mathscr{R}_n \implies \chi^2_{r\times d}$$
- In this case, we accept H_0 if $-2\log\mathscr{R}_n \leq \chi^2_{3\times r}(1-\alpha)$, where α is the level of significance. $\chi^2_r(1-\alpha)$ is the $(1-\alpha)\times 100\%$ percentile of χ^2 distribution with degrees of freedom r.

Testing Symmetries

- Spherical Symmetry
- Rotational Symmetry

Testing Spherical Symmetry

- Suppose a random vector $\mathbf{X} \in \mathcal{R}^d$ has a distribution spherically symmetric about $\boldsymbol{\theta}$, i.e.,

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- $V = \|\mathbf{X} - \boldsymbol{\theta}\|$, $\mathbf{U} = (\mathbf{X} - \boldsymbol{\theta})/\|\mathbf{X} - \boldsymbol{\theta}\|$. We have $\mathbf{U} \sim \mathscr{U}(\mathcal{S}^{d-1})$.

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- Spherical symmetry
  $\Leftrightarrow V \perp\!\!\!\perp \mathbf{U} \Leftrightarrow E[a(V)b(\mathbf{U})] = 0, \quad a \in L_{2,0}(F_V),\ b \in L_{2,0}(F_{\mathbf{U}}).$
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- Using the first few basis functions, we can construct empirical likelihood ratio:

$$\mathscr{R}_n^{ss} = \sup\left\{ \prod_{i=1}^{n} n\pi_i : \boldsymbol{\pi} \in \mathscr{P}_n, \sum_{i=1}^{n} \pi_i a_j(V_i) b_k(\mathbf{U}_i) = 0, \quad j = 1, \ldots, J, \quad k = 1, \ldots, K \right\}$$

where $(V_i, \mathbf{U}_i), i = 1, \ldots, n$ is a random sample of $(V, \mathbf{U})$.

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  where $(V_i, \mathbf{U}_i), i = 1, \dots, n$ is a random sample of $(V, \mathbf{U})$.
- By Owen's theorem, $-2\log \mathscr{R}_n^{ss} \Rightarrow \chi^2_{JK}$.

- Consider a vector function $\mathbf{U} \to \mathbf{f}(\mathbf{U})$ for some known function $\mathbf{f} : \mathcal{R}^d \to \mathcal{R}^e$. (For example, $\mathbf{f}(\mathbf{U}) = \mathbf{U}$.)

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- The empirical likelihood takes the form

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- Consider a vector function $\mathbf{U} \to \mathbf{f}(\mathbf{U})$ for some known function $\mathbf{f} : \mathcal{R}^d \to \mathcal{R}^e$. (For example, $\mathbf{f}(\mathbf{U}) = \mathbf{U}$.)
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- In this case, under certain conditions we have

$$-2\log \mathscr{R}_n^{ssh} \Rightarrow \chi^2_{Je}$$

Simulations with Jackknife pseudo values

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- Let $m = 2$, for any $p, q = 1, \ldots, n$, let $\mathbf{R} = \mathbf{U}_p + \mathbf{U}_q$, $\mathbf{R}^0 = \mathbf{R}/\|\mathbf{R}\|$.
- We have the fact that $\mathbf{U}_i \sim \mathscr{U}(\mathcal{S}^{d-1}) \Leftrightarrow \|\mathbf{R}\| \perp\!\!\!\perp \mathbf{R}^0$.

- Let $b_k = \varphi_k \circ G$, $\varphi_k(t) = \sqrt{2}\cos k\pi t$, $k = 1, \ldots, K$.
  $\mathbf{b}_K = (b_1, \ldots, b_K)^\top$.
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- Let the kernel function
  $\mathbf{h}(\mathbf{U}_p, \mathbf{U}_q) = \mathbf{b}_K(\|\mathbf{U}_p + \mathbf{U}_q\|) \otimes ((\mathbf{U}_p + \mathbf{U}_q)/\|\mathbf{U}_p + \mathbf{U}_q\|)$, which is argument-symmetric.

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- The U-statistics with the kernel **h** is given by

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$$\mathbf{U}_n(\mathbf{b}_K) = \binom{n}{2}^{-1} \sum_{1 \le p < q \le n} \mathbf{h}(\mathbf{U}_p, \mathbf{U}_q)$$

- The Jackknife pseudo values of the U-statistics is given by

$$\mathbf{V}_{ni} = n\mathbf{U}_n(\mathbf{b}_K) - (n-1)\mathbf{U}_{n-1}(\mathbf{b}_K), \quad i = 1, \ldots, n.$$

- Let $\mathbf{a}_J = (a_1, \ldots, a_J)^\top$.

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- Combine two parts together, we get the Jackknife empirical likelihood with side information as follows,

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- By Theorem 1, under certain conditions we have

$$-2\log \mathscr{R}_n(\mathbf{h}, \mathbf{g}) \to \chi^2_{(Je+Kd)}$$

Simulation results of samples from normal distribution

- We still calculate powers of this test with different settings.
- For convenience, we set $d = e = dim$, and $J = K = r$.
- r is basically the number of basis functions.
- The null hypothesis $H_0 : \boldsymbol{\theta} = (0, 0, 0)^\top$.
  $rep(x, dim)$ denotes the alternative hypothesis
  $H_1 : \boldsymbol{\theta} = (x, x, x)^\top$.

Multivariate Normal Distribution						
Comparison of Power for Different H1 with H0:theta=0						
n=100 m=2000						
d=e=dim J=K=r						
		r=1	r=2	r=3	r=4	r=5
dim=2	rep(0, dim)	0.044	0.031	0.04	0.056	0.0785
	rep(0.1, dim)	0.1215	0.084	0.081	0.1015	0.135
	rep(0.2, dim)	0.4185	0.329	0.2825	0.2935	0.3595
	rep(0.3, dim)	0.793	0.7365	0.663	0.6715	0.707
	rep(0.4, dim)	0.9225	0.971	0.928	0.9215	0.9265
	rep(0.5, dim)	0.901	0.9985	0.9955	0.9945	0.9935
dim=3	rep(0, dim)	0.0325	0.0345	0.0715	0.1355	0.2825
	rep(0.1, dim)	0.0775	0.097	0.131	0.201	0.363
	rep(0.2, dim)	0.293	0.335	0.381	0.471	0.6185
	rep(0.3, dim)	0.531	0.7425	0.7355	0.7995	0.8665
	rep(0.4, dim)	0.6055	0.954	0.9365	0.956	0.9665
	rep(0.5, dim)	0.566	0.9465	0.9395	0.9695	0.9795
dim=4	rep(0, dim)	0.0235	0.054	0.145	0.349	0.639
	rep(0.1, dim)	0.068	0.109	0.214	0.443	0.717
	rep(0.2, dim)	0.209	0.351	0.477	0.6965	0.8825
	rep(0.3, dim)	0.3425	0.7225	0.772	0.9135	0.9595
	rep(0.4, dim)	0.374	0.836	0.8735	0.9515	0.981
	rep(0.5, dim)	0.3155	0.757	0.842	0.933	0.9745

Simulation results of samples from t distribution

- We still calculate powers of this test with different settings.
- For convenience, we set $d = e = dim$, and $J = K = r$, df denotes the degrees of freedom of t distribution.
- r is basically the number of basis functions.
- The null hypothesis $H_0 : \boldsymbol{\theta} = (0, 0, 0)^\top$.
  $rep(x, dim)$ denotes the alternative hypothesis
  $H_1 : \boldsymbol{\theta} = (x, x, x)^\top$.

Multivariate t Distribution						
Comparison of Power for Different H1 with H0:theta=0						
n=100 m=2000 df=1						
d=e=dim J=K=r						
dim=2	rep(0, dim)	0.044	0.0295	0.0335	0.0505	0.0835
	rep(0.1, dim)	0.0745	0.053	0.0595	0.079	0.1145
	rep(0.2, dim)	0.2145	0.1635	0.159	0.1755	0.225
	rep(0.3, dim)	0.484	0.3955	0.367	0.3635	0.4045
	rep(0.4, dim)	0.733	0.6925	0.6085	0.5995	0.643
	rep(0.5, dim)	0.8725	0.889	0.8275	0.8045	0.8195
dim=3	rep(0, dim)	0.036	0.037	0.07	0.1245	0.274
	rep(0.1, dim)	0.06	0.0625	0.1025	0.19	0.3195
	rep(0.2, dim)	0.198	0.2255	0.255	0.35	0.5115
	rep(0.3, dim)	0.4035	0.485	0.4915	0.599	0.7065
	rep(0.4, dim)	0.619	0.762	0.7265	0.817	0.8765
	rep(0.5, dim)	0.734	0.918	0.89	0.929	0.9585
dim=4	rep(0, dim)	0.0295	0.0495	0.1365	0.326	0.6315
	rep(0.1, dim)	0.062	0.097	0.1985	0.418	0.713
	rep(0.2, dim)	0.205	0.284	0.4025	0.6295	0.8315
	rep(0.3, dim)	0.4385	0.574	0.6575	0.8205	0.929
	rep(0.4, dim)	0.6315	0.8235	0.8495	0.9405	0.9735
	rep(0.5, dim)	0.7275	0.912	0.9215	0.977	0.9935

Multivariate t Distribution						
Comparison of Power for Different H1 with H0:theta=0						
n=100 m=2000 df=2						
d=e=dim J=K=r						
		r=1	r=2	r=3	r=4	r=5
dim=2	rep(0, dim)	0.042	0.033	0.0375	0.056	0.092
	rep(0.1, dim)	0.0805	0.055	0.0615	0.0795	0.1265
	rep(0.2, dim)	0.2415	0.1925	0.15	0.176	0.241
	rep(0.3, dim)	0.514	0.462	0.38	0.3995	0.452
	rep(0.4, dim)	0.7585	0.7725	0.664	0.6695	0.708
	rep(0.5, dim)	0.7985	0.9355	0.8875	0.8755	0.8755
dim=3	rep(0, dim)	0.034	0.041	0.065	0.129	0.2615
	rep(0.1, dim)	0.0605	0.068	0.0945	0.1775	0.332
	rep(0.2, dim)	0.14	0.222	0.236	0.3485	0.5085
	rep(0.3, dim)	0.2845	0.4875	0.5135	0.6155	0.7195
	rep(0.4, dim)	0.3755	0.769	0.749	0.834	0.8875
	rep(0.5, dim)	0.3895	0.883	0.8785	0.934	0.964
dim=4	rep(0, dim)	0.027	0.05	0.1375	0.3335	0.636
	rep(0.1, dim)	0.049	0.101	0.2045	0.4035	0.691
	rep(0.2, dim)	0.107	0.2535	0.3665	0.5895	0.824
	rep(0.3, dim)	0.191	0.5205	0.624	0.8155	0.925
	rep(0.4, dim)	0.2765	0.735	0.8075	0.9265	0.9725
	rep(0.5, dim)	0.306	0.8055	0.8535	0.947	0.985

Multivariate t Distribution						
Comparison of Power for Different H1 with H0:theta=0						
n=100 m=2000 df=3						
d=e=dim J=K=r						
		r=1	r=2	r=3	r=4	r=5
dim=2	rep(0, dim)	0.0425	0.0365	0.0385	0.054	0.092
	rep(0.1, dim)	0.102	0.0565	0.0655	0.083	0.118
	rep(0.2, dim)	0.2615	0.199	0.1665	0.197	0.24
	rep(0.3, dim)	0.5625	0.5165	0.435	0.4375	0.476
	rep(0.4, dim)	0.7745	0.821	0.7565	0.7355	0.738
	rep(0.5, dim)	0.8225	0.957	0.9245	0.912	0.8975
dim=3	rep(0, dim)	0.029	0.039	0.075	0.135	0.2665
	rep(0.1, dim)	0.0575	0.0705	0.108	0.186	0.3405
	rep(0.2, dim)	0.1405	0.198	0.251	0.3485	0.5215
	rep(0.3, dim)	0.2545	0.4995	0.5325	0.6355	0.7465
	rep(0.4, dim)	0.3165	0.804	0.7775	0.84	0.902
	rep(0.5, dim)	0.305	0.874	0.8835	0.932	0.9585
dim=4	rep(0, dim)	0.031	0.0505	0.1425	0.332	0.642
	rep(0.1, dim)	0.047	0.093	0.191	0.4155	0.705
	rep(0.2, dim)	0.0915	0.231	0.3715	0.6105	0.817
	rep(0.3, dim)	0.147	0.5055	0.619	0.7995	0.935
	rep(0.4, dim)	0.1705	0.6985	0.7985	0.9105	0.9775
	rep(0.5, dim)	0.1745	0.7145	0.7995	0.9435	0.981

Testing Rotational Symmetry

- Suppose a random vector $\mathbf{X} \in \mathcal{S}^{d-1}$ is rotationally symmetric about direction $\boldsymbol{\theta}$, that is,

$$\mathbf{X} - \boldsymbol{\theta} \stackrel{d}{=} O(\mathbf{X} - \boldsymbol{\theta}),$$

for every $d \times d$ rotation matrix O about a fixed direction $\boldsymbol{\theta}$ in $\mathcal{R}^d$.

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  for every $d \times d$ rotation matrix O about a fixed direction $\boldsymbol{\theta}$ in $\mathcal{R}^d$.
- Let $T = \boldsymbol{\theta}^\top \mathbf{X}$ be the projection of $\mathbf{X}$ onto the direction $\boldsymbol{\theta}$.
- Let $\boldsymbol{\xi}$ be the unit tangent at $\boldsymbol{\theta}$ to $\mathcal{S}^{d-1}$.
  $\boldsymbol{\xi} \sim \mathscr{U}(\mathcal{S}^{d-2}(\boldsymbol{\theta}))$, where
  $\mathcal{S}^{d-2}(\boldsymbol{\theta}) = \left\{\mathbf{x} \in \mathcal{R}^d : |\mathbf{x}| = 1, \mathbf{x}^\top \boldsymbol{\theta} = 0\right\}$.
- Rotational Symmetry $\Rightarrow T = \boldsymbol{\theta}^\top \mathbf{X} \perp\!\!\!\perp \boldsymbol{\xi} = \frac{\mathbf{X} - T\boldsymbol{\theta}}{\|\mathbf{X} - T\boldsymbol{\theta}\|}$.

- Independence implies
  $\mathbf{E}(a(T)b(\boldsymbol{\xi})) = 0, \quad a \in L_{2,0}(F_T),\ b \in L_{2,0}(G_{\boldsymbol{\xi}}).$
- Similar to the spherical symmetry case, take $a_j = \varphi_j \circ F_T$, $j = 1, \ldots, J$, and a vector function $\boldsymbol{\xi} \to \mathbf{f}(\boldsymbol{\xi})$ for some known function $\mathbf{f} : \mathcal{R}^d \to \mathcal{R}^e$.

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- We have $E(a_j(T)\boldsymbol{\xi}) = 0$. Let $\mathbf{a_J} = (a_1, \ldots, a_J)^\top$. The empirical likelihood ratio takes the form

$$\mathscr{R}_n^{rsh} = \sup\left\{ \prod_{i=1}^{n} n\pi_i : \boldsymbol{\pi} \in \mathscr{P}_n, \sum_{i=1}^{n} \pi_i \mathbf{a_J}(T) \otimes \boldsymbol{\xi} = 0 \right\}$$

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- Under certain conditions we have

$$-2\log \mathscr{R}_n^{rsh} \Rightarrow \chi^2_{Je}$$

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- In our simulation, we generated the data distributed from Von Mises-Fisher distribution.
- We are testing $H_0 : \boldsymbol{\theta} = (0, 0, 1)^\top \quad V.S. \quad H_1 : \boldsymbol{\theta} = (0.14, 0.14, 0.98)^\top$.

- Calculate the powers of this test with different settings.
- For convenience, take $d = e = 3$, $J = K = r$.
- r is basically the number of basis functions.
- The results of simulations are showed below:

The level of significance of the testing of rotational symmetry			
H0: theta=(0, 0, 1)			
d=e=3 J=K=r m=2000			
	r=1	r=3	r=5
n=50	0.0505	0.099	0.2275
n=100	0.053	0.0685	0.0865

The power of the testing of rotational symmetry			
H0: theta=(0, 0, 1) V.S. H1: theta=(0.14, 0.14, 0.98)			
d=e=3 J=K=r m=2000			
	r=1	r=3	r=5
n=50	0.949	0.9995	0.9955
n=100	0.9995	1	1

Reference

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Thank you very much!