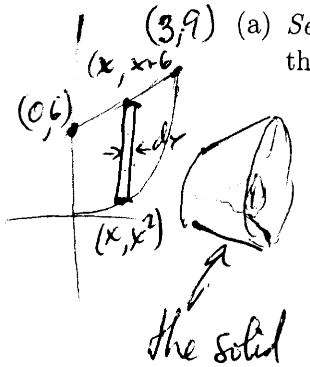
NAME: KEY

Math S165 Test 2 **24 November 2008**

There are 6 pages, 6 questions, and 100 points on this test. **The test finishes at 10:10am!** Follow the instructions for each question and show enough of your work that I can understand what you are doing.

(15 points) **1.** Let R be the region in the first quadrant bounded on the top by the line $y - x = 6$, on the right by the parabola $y = x^2$, and on the left by the y-axis.

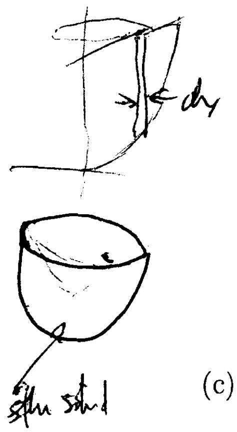
(a) *Set up* an integral to find the volume of the solid obtained by revolving the region R around the x-axis.



Use washers

$$\int_{x=0}^{3} \pi (x+6)^2 - \pi (x^2)^2 \, dx$$

(b) *Set up* an integral to find the volume of the solid obtained by revolving the region R around the y-axis.

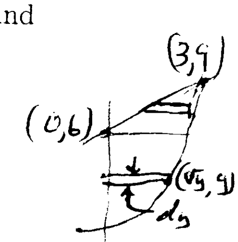


Use Shells

$$\int_{x=0}^{3} 2\pi x \left((x+6) - x^2\right) dx$$

or Use disks on bottom part + washers on top

$$\int_{y=0}^{6} \pi (\sqrt{y})^2 \, dy + \int_{y=6}^{9} \pi (\sqrt{y})^2 - \pi (y-6)^2 \, dy$$



(c) Evaluate *one* of the integrals found in (a) or (b) to find the volume (as a number) of that solid.

(a) $$\int_0^3 \pi (x+6)^2 - \pi (x^2)^2 \, dx = \int_0^3 \pi (x^2 + 12x + 36) - \pi x^4 \, dx$$

$$= \pi\left(\frac{3^3}{3} + \frac{12(3^2)}{2} + \frac{36(3)}{1}\right) - \pi\left(\frac{0^3}{3} + \frac{12(0^2)}{2} + 6\left(\frac{0}{1}\right)\right) - \pi \frac{3^5}{5} + \pi \frac{0^5}{5}$$

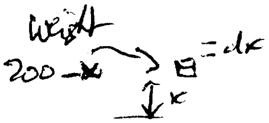
$$= \pi\left[9 + 54 + 18 - \frac{243}{5}\right] = \cancel{\frac{\pi 162}{5}} \; \pi\left(171 - \frac{243}{5}\right) = \frac{\pi 612}{5}$$

(b) $$\int_0^3 2\pi x (x + 6 - x^2) \, dx = 2\pi \int_0^3 x^2 + 6x - x^3 \, dx = 2\pi\left[\frac{3^3}{3} + \frac{6(3^2)}{2} - \frac{3^4}{4} - 0\right]$$

$$= 2\pi\left[9 + 27 - \frac{81}{4}\right] = \frac{126\pi}{4} = \frac{63\pi}{2}$$

(15 points) **2.** A barrel full of water is being lifted by a crane to the top of a 100 foot building. When the barrel is full, it weighs 200 pounds. Unfortunately, the barrel is leaking: in fact during the time it is being lifted to the top of the building, it loses 100 pounds of water! Assuming the water is being lost at a constant rate, that the crane lifts the barrel at a constant rate, and that the weight of the cable holding the barrel is negligible (that is, the work done in lifting the cable can be ignored), *set up* an integral to find the work done in lifting the barrel to the top of the building. (The assumptions imply that the weight of the barrel and water at the beginning is 200 pounds, when the height is 20 feet, the weight is 180 pounds, when the height is 40 feet, the weight is 160 pounds, etc.)

$$\int_{x=0}^{100} (200-x)\,dx \quad \text{ft}\cdot\text{lbs}$$



Work in raising barrel from x feet high to $x+dx$ feet high is $(200-x)\,dx$

Note: as a check notice that the work done must be less than raising 200 lbs 100 feet $= 200\times 100$ ft·lbs

and more than raising 100 lbs 100 feet $= 100\times 100$ ft-lbs

$$W=\int_0^{100} 200-x\,dx = \left(200\cdot\frac{100}{1}-\frac{100^2}{2}\right)-\left(200\frac{0}{1}-\frac{0^2}{2}\right)$$

$$= 200\cdot 100 - \frac{10000}{2} = 15000 \text{ ft-lbs}$$

(20 points) **3.**

(a) Let a be a real number. Use the definition of derivative as a limit of difference quotients (that is, not the rules for finding derivatives that we have proved) to find $f'(a)$ when f is the function

$$f(x) = x^2 + 3x + 5$$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{[(a+h)^2 + 3(a+h) + 5] - [a^2+3a+5]}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{a^2} + 2ah + h^2 + \cancel{3a} + 3h + \cancel{5} - \cancel{a^2} - \cancel{3a} - \cancel{5}}{h}$$

$$= \lim_{h\to 0} \frac{2ah + h^2 + 3h}{h} = \lim_{h\to 0} \frac{h(2a+h+3)}{h}$$

$$= \lim_{h\to 0} 2a + h + 3 = 2a + 3$$

(b) Use the definition of derivative as a limit of difference quotients to explain why $g'(0)$ is not defined when g is the absolute value function,

$$g(x) = |x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

$$g'(0) = \lim_{h\to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0} \frac{|h| - 0}{h}$$

Break into cases:
$h > 0$
$h < 0$

$$\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1$$

But

$$\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} -1 = -1$$

So $\lim_{h\to 0} \frac{|h|}{h}$ does not exist: $\lim_{h\to 0^+} \frac{|h|}{h} \neq \lim_{h\to 0^-} \frac{|h|}{h}$ so $g'(0)$ does not exist.

(20 points) **4.** Let $h(x) = \dfrac{4x}{x^2 - 3}$

(a) For which values of x, if any, is h not continuous?

A rational function is continuous for all x for which the denominator is non-zero; $x^2-3=0 \Leftrightarrow x=\sqrt{3}$ or $x=-\sqrt{3}$ Since $4\sqrt{3} \neq 0$ and $-4\sqrt{3} \neq 0$ the numerator of h is not zero at $x = \pm\sqrt{3}$ so h is not continuous for $x = \pm\sqrt{3}$

(b) Find $\lim_{x\to 1} h(x)$

$$\lim_{x\to 1} h(x) = \frac{4(1)}{1^2-3} = \frac{4}{-2} = -2$$

(c) Find $h'(x)$

$$h'(x) = \frac{(x^2-3)(4) - (4x)(2x)}{(x^2-3)^2} = \frac{4x^2-12-8x^2}{(x^2-3)^2} = -\frac{4x^2+12}{(x^2-3)^2}$$

(d) Find an equation of the line tangent to the graph of h at $(2, 8)$

Slope of tangent line is $h'(2) = -\dfrac{4(2)^2+12}{(2^2-3)^2} = -\dfrac{16+12}{1} = -28$

so $\dfrac{y-8}{x-2} = -28$ or $y = -28x + 64$ are equations of tangent line

(e) Find $h''(x)$

$$h''(x) = \frac{d}{dx} h'(x) = -\frac{(x^2-3)^2(8x) - (4x^2+12)\left(2(x^2-3)(2x)\right)}{((x^2-3)^2)^2}$$

$$= -\frac{4x(x^2-3)\left[(x^2-3)(2) - (4x^2+12)\right]}{(x^2-3)^4} = -\frac{4x(2x^2-6-4x^2-12)}{(x^2-3)^3}$$

$$= \frac{4x(2x^2+18)}{(x^2-3)^3} = \frac{8x^3+72x}{(x^2-3)^3}$$

(f) Find $h^{(3)}(x) = h'''(x)$

$$h^{(3)}(x) = \frac{d}{dx} h''(x) = \frac{(x^2-3)^3(24x^2+72) - (8x^3+72x)\left(3(x^2-3)^2(2x)\right)}{((x^2-3)^3)^2}$$

$$= \frac{(x^2-3)(24x^2+72) - (8x^3+72x)(6x)}{(x^2-3)^4}$$

(15 points) **5.** Assume that the equation

$$x^3 + 2x \sin y - (\cos y)^2 = 7$$

defines y as a function (or functions) of x.

(a) Find y' as a function of x and y.

Take $\frac{d}{dx}$ of equation above:

$$3x^2 + 2\sin y + 2x(\cos y)\, y' - 2\cos y(-\sin y)\, y' = 0$$

$$(2x\cos y)\, y' + (2\cos y \sin y)\, y' = -3x^2 - 2\sin y$$

$$y'(2x\cos y + 2\cos y \sin y) = -(3x^2 + 2\sin y)$$

$$y' = -\frac{3x^2 + 2\sin y}{2x\cos y + 2\cos y \sin y}$$

(b) Find an equation for the line tangent to the curve at the point $(2, 0)$.

at $(2,0)$ i.e. $x=2$, $y=0$

which is on the curve $x^3 + 2x\sin y - (\cos y)^2 = 7$

we see the slope of the tangent line is $y' = -\dfrac{3(2)^2 + 2\sin(0)}{2(2)\cos(0) + 2(\cos 0)\sin}$

So $y' = -\dfrac{12}{4} = -3$

and an equation for the tangent line is $\dfrac{y-0}{x-2} = -3$

or $y = -3x + 6$

(15 points) **6.** Use the epsilon(ϵ) – delta(δ) definition of limit to prove that

$$\lim_{x\to 2} 3x+1=7$$

Proof: Suppose $\varepsilon > 0$ is given.

Let $\delta = \varepsilon/3$. If $|x-2|<\delta$

then $|x-2| < \frac{\varepsilon}{3}$

So $3|x-2|<\varepsilon$

or $|3x-6|<\varepsilon$

and $|(3x+1)-7|<\varepsilon$.

Thus, for any $\varepsilon>0$, we can choose $\delta>0$ so that $|x-2|<\delta$ implies $|(3x+1)-7|<\varepsilon$.

This means $\lim_{x\to 2} 3x+1=7$

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We want $|(3x+1)-7|<\varepsilon$

or $|3x-6|<\varepsilon$

or $|3(x-2)|<\varepsilon$

or $|x-2|<\frac{\varepsilon}{3}$

that is $\delta=\varepsilon/3$ works.