Professor Carl Cowen **Math 44400** **Fall 14**

'A' LIST PROBLEMS

Solutions to problems from this list may be handed in at any time before 5:00pm on December 17. The problems will be read and either accepted as correct or returned for rewriting and resubmission. Only one of these problems will be counted for credit but this problem will be worth the same number of points as two regular homework assignments. In order to receive an 'A' or 'A+' for the course, you must have one of the *'A' List* problems accepted as correct.

1. (Construction of $\mathbb{Q}$)

Peano (1889) and Dedekind (1888) gave a careful construction of the integers from the axioms for set theory. We will take the integers, $\mathbb{Z}$, the subset $\mathbb{N}$ of natural numbers (that is, the positive integers), and their properties as given. The goal of this problem is, using the integers and their properties, to construct the set $\mathbb{Q}$, define the operations of 'addition' and 'multiplication' for elements of $\mathbb{Q}$, define the set $\mathcal{P}$ of 'positive' elements of $\mathbb{Q}$, and prove that $\mathbb{Q}$ with these operations and the distinguished subset $\mathcal{P}$ is an ordered field which we can recognize as being the (usual) rational numbers.

An *equivalence relation* on a set $\mathcal{X}$ is a binary relation $\sim$ that satisfies (i) for every x in $\mathcal{X}$, $x \sim x$ (reflexivity), (ii) for x and y in $\mathcal{X}$, $x \sim y$ implies $y \sim x$ (symmetry), and (iii) for x, y, and z in $\mathcal{X}$, $x \sim y$ and $y \sim z$ implies $x \sim z$ (transitivity). A equivalence relation on a set can be used to define *equivalence classes*: For x in $\mathcal{X}$, the equivalence class of x, denoted $[x]$, is the subset of $\mathcal{X}$

$$[x] = \{y \in \mathcal{X} : y \sim x\}$$

From the properties of an equivalence relation, we see that $[x] = [y]$ if and only if $x \sim y$. An equivalence relation therefore allows us to break up a set into disjoint pieces, the equivalence classes. An easy example is modular arithmetic: we say integers m and n are equivalent modulo 2 if $m - n$ is divisible by 2. This equivalence relation breaks the integers into two disjoint subsets usually called the 'even integers' (the equivalence class of 2) and the 'odd integers' (the equivalence class of 1).

The Construction:

Let $\mathcal{X}$ be the set of ordered pairs of integers with second entry non-zero, that is,

$$\mathcal{X} = \{(m, n) : m \in \mathbb{Z} \text{ and } n \in \mathbb{Z} \text{ but } n \neq 0\}$$

We define a binary relation $\sim$ on $\mathcal{X}$ by the following: for integers m_1, m_2, n_1, and n_2, we write $(m_1, n_1) \sim (m_2, n_2)$ when $m_1 n_2 = m_2 n_1$.

(a) Show that $\sim$, defined above, is an equivalence relation on $\mathcal{X}$.

(b) Show that if n_1 and n_2 non-zero integers, then $(0, n_1) \sim (0, n_2)$ and show that if m and n are integers with $n \neq 0$ for which $(0, 1) \sim (m, n)$, then $m = 0$. This shows that $[(0, 1)] = \{(0, n) : n \in \mathbb{Z} \text{ but } n \neq 0\}$. We let $\underline{0} = [(0, 1)]$.

(c) Describe $[(1, 1)]$ as a subset of $\mathcal{X}$. We let $\underline{1} = [(1, 1)]$.

1. (Continued)

(d) Suppose (m_1, n_1) and (m_2, n_2) are in $\mathcal{X}$ with $(m_1, n_1) \sim (m_2, n_2)$. Show that $(-m_1, n_1) \sim (-m_2, n_2)$. This means that letting $(-[(m,n)])$ be $[(-m,n)]$ cannot cause confusion; we say "$(-[(m,n)]) = [(-m,n)]$ is well-defined."

(e) For m and n non-zero integers, let $(1/[(m,n)]) = [(n,m)])$. Show that $(1/[(m,n)])$ is well-defined.

(f) Suppose (m_1, n_1), (m_2, n_2), (m_1', n_1'), and (m_2', n_2') are in $\mathcal{X}$ and $(m_1, n_1) \sim (m_2, n_2)$ and $(m_1', n_1') \sim (m_2', n_2')$. Show that

$$(m_1 n_1' + m_1' n_1, n_1 n_1') \sim (m_2 n_2' + m_2' n_2, n_2 n_2')$$

Conclude that, for (m,n) and (m',n') in $\mathcal{X}$, letting

$$[(m,n)] \oplus [(m',n')] = [(mn' + m'n, nn')]$$

is well-defined.

(g) Show that, for (m,n) and (m',n') in $\mathcal{X}$, letting

$$[(m,n)] \odot [(m',n')] = [(mm', nn')]$$

is well-defined.

The Definition:

Define $\mathbb{Q}$ be the set of equivalence classes of ordered pairs in $\mathcal{X}$, that is,

$$\mathbb{Q} = \{[(m,n)] : (m,n) \in \mathcal{X}\}$$

We usually denote elements of $\mathbb{Q}$ with single characters such as p or q, but it is important to remember that each such element of $\mathbb{Q}$ is $p = [(m,n)]$ for some (many!) pair (m,n) in $\mathcal{X}$. Thus, proofs about $\mathbb{Q}$ very often rely on choosing appropriate pairs (m,n) to use in the calculations involving the elements of $\mathbb{Q}$.

(h) Show that $\mathbb{Q}$ with operations $\oplus$ and $\odot$, the elements $\underline{0}$ and $\underline{1}$, and the definitions of $(-p)$ and $(1/p)$ given in parts (d) and (e) above makes $\mathbb{Q}$ into a field, that is, show that the axioms (A1), (A2), (A3), (A4), (M1), (M2), (M3), (M4), and (D) on page 23 of the text are satisfied.

The Positives:

Let $\mathcal{P}$ be defined by

$$\mathcal{P} = \{[(m,n)] : \text{ for } m \in \mathbb{N} \text{ and } n \in \mathbb{N}\}$$

(i) Show that $\mathcal{P}$ as a subset of $\mathbb{Q}$ satisfies the order axioms (O1), (O2), and (O3) on page 25 of the text.

We conclude that the set $\mathbb{Q}$ as defined above is an ordered field. We call this ordered field the *rational numbers* and we write '0' for the element called $\underline{0}$ above, '1' for $\underline{1}$, '+' for $\oplus$, '1/2' for $[(1,2)] = [(12,24)] = [(39,78)]$, etc.

In a similar process, starting with the rational numbers (as defined above, for example) it is possible to define the set $\mathbb{R}$, the real numbers, so that all of the axioms/properties described in the text for the real numbers (in Sections 2.1 and 2.3) are true. In fact, to do so will be an 'A' List problem for Math 44500.

Definition (for Problem 2.) A set E of real numbers is said to be *dense* if for each real number r and each $\epsilon > 0$, the ϵ-neighborhood of r includes a point of E.

In particular, a set E is dense if and only if every real number is a cluster point of E.

2. (Open Dense Sets)
The set $U = \{x \in \mathbb{R} : x \neq \sqrt{2}\} = (-\infty, \sqrt{2}) \cup (\sqrt{2}, \infty)$ is an open and dense set in $\mathbb{R}$.

(a) Prove: If V_1, V_2, $\cdots$, V_n are each open and dense sets in $\mathbb{R}$, then $\cap_{j=1}^{n} V_j$ is also an open and dense set.

(b) Find a sequence U_1, U_2, U_3, $\cdots$ of open and dense subsets of $\mathbb{R}$ such that

$$\bigcap_{j=1}^{\infty} U_j = \{x \in \mathbb{R} : x \text{ is irrational } \}$$

3. Sometimes it is annoying that a sequence does not converge and you wish you could 'make it converge'. This is a useful idea and, often, it can be done. We will call this extension of the idea of limit a *weighted limit.* This exercise will show that if a sequence has a limit, then the weighted limit is the same number, but that weighted limits allow *more* sequences to 'converge'. Let (a_n) be a sequence. For each positive integer n, let b_n be defined by

$$b_n = \frac{2}{3n}\left(a_1 + 2a_2 + a_3 + 2a_4 + \cdots + \rho_n a_n\right)$$

where $\rho_n = 2$ if n is even and $\rho_n = 1$ if n is odd. This is sets up a weighted limit with weights 1 and 2.

(a) Show that if $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

(b) Find an example of a sequence (a_n) that does not converge, but for which the weighted limit with weights 1 and 2, that is, the sequence (b_n), *does* converge.

The more general idea is as follows: For ℓ a positive integer, let w_1, w_2, $\cdots$, w_ℓ be positive numbers and let $W = w_1 + w_2 + \cdots + w_\ell$. Given a sequence (a_n), for each positive integer n, let c_n be defined by

$$c_n = \frac{\ell}{nW}\left(w_1 a_1 + w_2 a_2 + w_3 a_3 + \cdots + \rho_n a_n\right)$$

where $\rho_n = w_j$ if $n \equiv j \bmod \ell$.

4. (Sequel to Exercise 11, page 75 of Bartle and Sherbert, 3^{rd} Edition.)
Later this semester, we will define the natural logarithm function to be

$$\ln(x) = \int_1^x \frac{1}{t}\, dt$$

From this, we will prove the usual properties of the logarithm and the exponential functions. You may use the properties of these functions as well as the usual properties of the Riemann integral to do this problem, even if we have not yet covered the material this semester. This exercise recapitulates work of Euler on estimating the size of $\sum_{k=1}^{n} 1/k$.

(a) Use an easy estimate connected to Riemann sums to show that

$$\frac{1}{n} \leq \ln(n) - \ln(n-1) \leq \frac{1}{n-1}$$

(b) Use part (a) to show that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \leq \ln(n) \leq 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

Define the sequence (E_n) by

$$E_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln(n)$$

(c) Use parts (a) and (b) to show that the sequence (E_n) is a non-negative, bounded sequence.

(d) Prove that the sequence (E_n) is decreasing.

(e) Notice that the definition of E_n shows that

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} = \ln(n) + E_n$$

is an *exact* equality! Use this to compute

$$\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} \right)$$

History: (Much of this note is from the *Wikipedia* article on the topic.) The limit of the sequence (E_n) is denoted γ and is called *Euler's constant* or perhaps more properly the *Euler–Mascheroni constant.*

$$\gamma \approx 0.5772156649015328606065120900824024310421593359399\,2$$

Surprisingly, it is not known whether γ is rational or irrational, but it is widely believed to be irrational. Its decimal expansion has been computed to about 30 billion places, and it has been proved that if it is rational, its denominator must be more than 10^{242080}.

5. Let $\mathcal{F}$ be the family of real valued functions

$$\mathcal{F} = \{f : f \text{ is continuous on } [0,1] \text{ and } f(0) = f(1) = 0\}$$

If f is a function in $\mathcal{F}$ and ℓ is a number, ℓ is called a *secant of* f if there is a number a such that $0 \le a \le a + \ell \le 1$ such that $f(a) = f(a + \ell)$. We call ℓ a *ubiquitous secant* if ℓ is a secant of every f in $\mathcal{F}$. Clearly $\ell = 0$ and $\ell = 1$ are ubiquitous secants. Find all ubiquitous secants (and prove your answer).

6. (Open Dense Sets, II)
Can you find a sequence V_1, V_2, V_3, $\cdots$ of open and dense subsets of $\mathbb{R}$ such that

$$\bigcap_{j=1}^{\infty} V_j = \{x \in \mathbb{R} : x \text{ is rational }\}$$

7. (Open Dense Sets, III)
Let V_1, V_2, V_3, $\cdots$ be a sequence of open and dense subsets of $\mathbb{R}$. Is the set

$$\bigcap_{j=1}^{\infty} V_j$$

always a dense subset of $\mathbb{R}$?

Definition (for Problem 8.) Let $\mathcal{Z}$ be a set of real numbers. A point p in $\mathcal{Z}$ is said to be *an isolated point of* $\mathcal{Z}$ if there is a neighborhood U of p such that for all z in $\mathcal{Z}$, with $z \neq p$, then z is *NOT* in U. (Recall that U is a neighborhood of p if there is a number $\delta > 0$ such that $U = \{x \in \mathbb{R} : |x - p| < \delta\}$.)

Definition: Let $\mathcal{Z}$ be a set of real numbers. The set $\mathcal{Z}$ is said to be *a discrete set* if *every point* of $\mathcal{Z}$ is an isolated point of $\mathcal{Z}$.

8. (a) Give an example of a set that is countably infinite and discrete.

(b) Give an example of a set that is countably infinite and *NOT* discrete.

(c) Show that every discrete set is countable.

9. (a) Let χ (for characteristic) be the function defined by

$$\chi(t) = \begin{cases} 1 & t \text{ is rational} \\ 0 & t \text{ is irrational} \end{cases}$$

For which real numbers t, if any, is χ continuous? (Prove your answer.)

(b) Let σ (for stair) be the function defined by

$$\sigma(t) = \begin{cases} \frac{1}{q} & t = \frac{p}{q} \text{ for } p, q \in \mathbb{Z} \text{ with } q > 0 \text{ where } p \text{ and } q \text{ have no common factors} \\ 0 & t \text{ is irrational} \end{cases}$$

For which real numbers t, if any, is σ continuous? (Prove your answer.)

(c) For which real numbers t, if any, is σ differentiable? (Prove your answer.)