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An Analytic Toeplitz Operator that Commutes with a Compact Operator and a Related Class of Toeplitz Operators*

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For a bounded analytic function, f, on the unit disk, D, let T_f and M_f denote the operators of multiplication by f on $H^2(\partial D)$ and $L^2(\partial D)$, respectively. In their 1973 paper, Deddens and Wong asked whether there is an analytic Toeplitz operator T_f that commutes with a nonzero compact operator, and whether every operator that commutes with an analytic Toeplitz operator has an extension that commutes with the corresponding multiplication operator on L^2. In the first part of this paper, we give an explicit example of an analytic Toeplitz operator T_ϕ that settles both of these questions. This operator commutes with a nonzero compact operator (a composition operator followed by an analytic Toeplitz operator). The only operators in the commutant of T_ϕ that extend to commute with M_ϕ are analytic Toeplitz operators. Although the commutant of T_ϕ contains more than just analytic Toeplitz operators, T_ϕ is irreducible. The remainder of the paper seeks to explain more fully the phenomena incorporated in this example by introducing a class of analytic functions, including the function ϕ, and giving additional conditions on functions g in the class to determine whether T_g commutes with nonzero compact operators, whether T_g is irreducible, and which operators in the commutant of T_g extend to the commutant of M_g. In particular, we find representations for operators in the commutant and second commutant of T_g.

The primary purpose of this paper is to give an example that answers several questions concerning commutants of analytic Toeplitz operators. After presenting the example, we investigate more carefully a class of analytic Toeplitz operators that this example suggests.

For ϕ in H^∞ of the unit disk D, the analytic Toeplitz operator, T_ϕ, is the operator on the Hardy space H^2 of multiplication by ϕ. We denote $\{A \mid AT_\phi = T_\phi A\}$, the commutant of T_ϕ, by $\{T_\phi\}'$.

In their 1973 paper, Deddens and Wong asked if there is a nonconstant

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function ϕ in H^∞ such that T_ϕ commutes with a nonzero compact operator [8, p. 272]. We will find such a function ϕ (Theorem 1.1). This question is related to the invariant subspace problem. Lomonosov [11] has shown that an operator has a nontrivial invariant subspace if it commutes with a nonscalar operator that commutes with a nonzero compact operator. Pearcy and Shields, in their survey of the Lomonosov technique [18, p. 225], point out that it is possible, though not likely, that this theorem solves the invariant subspace problem, that is, maybe every operator commutes with an operator that commutes with a compact. There is no operator for which Lomonosov's hypothesis is known to fail, however, the unilateral shift, T_z, was considered a good candidate for such an example. Since the commutant of the shift is the set of analytic Toeplitz operators, the Deddens–Wong question can be restated: "Does the unilateral shift, T_z, satisfy Lomonosov's hypothesis?" and we answer "yes." Such an example has been found, see [25].

It is well known that if ϕ is a nonconstant function in H^∞, the analytic Toeplitz operator T_ϕ is a pure subnormal operator and its minimal normal extension is the operator M_ϕ of multiplication by $\hat{\phi}$, the boundary function of ϕ, on L^2 of the unit circle [9, Lemma 3.8]. One might hope to find the commutant of a subnormal operator by studying the operators that commute with its minimal normal extension. This can be an effective approach if every operator in the commutant comes from an operator in the commutant of the normal extension. (If A commutes with the subnormal operator S, we call an operator $\tilde{A}$ a *lifting of* A if it is an extension of A that commutes with the minimal normal extension of A. If every operator in $\{S\}'$ lifts, we say the *commutant of* S *lifts*. Bram showed [5, p. 87] that if A lifts, then its lifting is unique.) The problem of lifting commutants of general subnormal operators has been studied by several authors and examples of subnormals whose commutants do not lift are known (see, for example, [2, 5, 13, 17]). Deddens and Wong [8, p. 272] asked if the commutant of every analytic Toeplitz operator lifts. In Theorem 1.2, we give an example whose commutant does not lift.

Several authors have studied questions concerning reducing subspaces of analytic Toeplitz operators and determining which analytic Toeplitz operators are irreducible (i.e., have only the trivial reducing subspaces) (see, for example, [1, 4, 8, 14, 22, 23]). Since the only analytic Toeplitz operators that are projections are 0 and I, the shift T_z is irreducible; indeed if $\{T_\phi\}' = \{T_z\}'$, then T_ϕ is also irreducible. Several results on reducing subspaces suggested that the converse of this statement might also be true, and this question was explicitly raised in a letter to the author by Abrahamse. Abrahamse also asks [2, problem 1] if the commutant of an irreducible subnormal operator lifts. Our example answers both of these questions negatively (Theorem 1.3).

To recapitulate, in Section 1, we exhibit a function ϕ in H^∞ such that T_ϕ commutes with a nonzero compact operator; $\{T_\phi\}'$ does not lift; and T_ϕ is irreducible.

It seems appropriate to review some of the earlier work on commutants of analytic Toeplitz operators. If ϕ is a nonconstant inner function (i.e., $|\hat{\phi}| = 1$ almost everywhere on the unit circle), T_ϕ is unitarily equivalent to a direct sum of copies of the unilateral shift T_z. Using this equivalence, it is easy to get a matrix characterization of operators in $\{T_\phi\}'$. From the matrix characterization, it follows that T_ϕ does not commute with any nonzero compact operators; that $\{T_\phi\}'$ lifts; and that T_ϕ is irreducible if and only if $\{T_\phi\}' = \{T_z\}'$. In [8], Deddens and Wong give some sufficient conditions on f for $\{T_f\}'$ to equal $\{T_\phi\}'$ for some inner function ϕ, and, although not explicitly stated, the questions they raise seem in spirit to be "For f in H^∞, is there an inner function ϕ so that $\{T_f\}' = \{T_\phi\}'$?"

In [1] Abrahamse gave an example of a function π, a covering map of the disk onto an annulus considering the disk as the universal covering space of the annulus, for which $\{T_\pi\}' \neq \{T_\phi\}'$ for any inner function ϕ. Although π cannot be written $\pi = h \circ \phi$ in a nontrivial way for ϕ an inner function, T_π has many reducing subspaces [1, Theorem 2]. In [6, 7], the author extended the example of Abrahamse, showing that, usually, the commutant of a Toeplitz operator whose symbol is a covering map of the disk onto some plane domain is different from commutants of operators arising from inner functions [6, p. 22]. On the other hand, it was shown that even if ϕ is a covering map, T_ϕ does not commute with any nonzero compact operators [6, p. 27]; $\{T_\phi\}'$ lifts [7, Theorem 5]; and T_ϕ is irreducible if and only if $\{T_\phi\}' = \{T_z\}'$.

There are several theorems which give necessary conditions for an analytic Toeplitz operator to commute with a compact operator. To paraphrase these results, we have that if T_ϕ commutes with a nonzero compact operator, $\hat{\phi}$ must map (most of) the boundary of the disk into the image of the open disk [6, Theorem 10]; ϕ must be an infinite-to-one map of the disk [6, p. 19]; and the compact operator must be quasinilpotent [6, p. 26]. In addition, ϕ must satisfy a technical condition which eliminates functions automorphic under a group but allows functions automorphic under a semigroup [6, p. 27]. The questions on lifting and compacts are related: if $\{T_\phi\}'$ lifts, then T_ϕ cannot commute with any nonzero compact operators [7, Theorem 1]. However, as will be clear from the proofs that follow, $\{T_\phi\}'$ may fail to lift even when T_ϕ does not commute with compacts.

The example of Section 1 is based on a function, related to a covering map, which is automorphic under a semigroup of maps from the disk into the disk. In Section 2, we introduce a class of such functions, which we call evenly semiautomorphic functions and find a general form for operators in $\{T_\phi\}'$ and $\{T_\phi\}''$ when ϕ is in this class.

In Section 3, we give a geometric condition on the maps in the semigroup which enables one to construct compact operators in $\{T_\phi\}'$.

Section 4 considers the lifting question for this class of analytic Toeplitz operators. We find that if the semigroup is not a group, then the commutant does not lift. However, the operators in the commutant associated only with invertible elements of the semigroup do have extensions, with the same norm, that commute with M_ϕ.

In Section 5, we discuss reducibility. If there is a nontrivial invertible element of the semigroup, then T_ϕ has reducing subspaces. As a partial converse, we give an unfortunately complicated algebraic condition on the semigroup which enables us to show that no nontrivial projections commute with T_ϕ, hence that every operator similar to T_ϕ is irreducible.

As will become clear, this work makes extensive use of composition operators. My thoughts on composition operators were inspired by the excellent lectures of Eric Nordgren at the conference on Concrete Representations of Operators on Hilbert Space held June 1977 at Long Beach [16]. I would also like to thank Abrahamse for suggesting the proof of Theorem 1.3 as given (my original proof is the proof of Theorem 5.1) and Davidson for suggesting simplifications to the proof of Theorem 1.1.

1. The Example

To obtain the example, we need to define several functions and operators. As a help to our intuition, we define the domain $\Omega = \{z \mid \operatorname{Im}(z^2) < -1 \text{ and } \operatorname{Re}(z) < 0\}$, in the second quadrant bounded by a branch of the hyperbola $2xy = -1$. We let $\sigma(z) = (-1 + i)(z + 1)^{-1/2}$ (where $x^{1/2} > 0$ for $x > 0$) so that σ is a Riemann map of D onto Ω, $\sigma(-1) = \infty$. Define ϕ on D by $\phi(z) = \exp(\sigma(z)) - \exp(\sigma(0))$. We see that ϕ maps the disk onto a translate of the unit disk punctured at 0, and $\hat{\phi}$ maps the boundary of the disk, one-to-one, onto a spiral that approaches the boundary of the image asymptotically.

Define the map J of the disk into itself by $J(z) = \sigma^{-1}(\sigma(z) + 2\pi i)$. In fact, J extends to a continuous map of the closed disk, and

$$J(\bar{D}) \cap \partial D = \{-1\} = \{J(-1)\}.$$

Since the exponential map is periodic, we have $\phi \circ J = \phi$. Let C be the operator on H^2 of composition with J, that is, $Cf = f \circ J$ for f in H^2. This operator is bounded [15, 19], but not compact. The operators C and T_ϕ commute since, for f in H^2, $CT_\phi f = (\phi \circ J)(f \circ J) = \phi(f \circ J) = T_\phi Cf$.

Finally, we let $\psi(z) = \frac{1}{2}(z + 1)$, so ψ is continuous on the closed disk, $\|\psi\|_\infty = 1$, and $\psi(-1) = 0$.

Theorem 1.1. *The operator CT_ψ is compact and commutes with T_ϕ.*

Proof. Clearly CT_ψ commutes with T_ϕ since both C and T_ψ do.

To see that CT_ψ is compact, it is sufficient to show that $\lim_{n\to\infty} \|CT_\psi|_{z^n H^2}\| = 0$. Given $\epsilon > 0$, let $K_\epsilon = \{w \mid w \in J(\bar{D}) \text{ and } |\psi(w)| = |\frac{1}{2}(w + 1)| \geq \epsilon\}$. Since

$J(\bar{D}) \subset D \cup \{-1\}$, we see that K_ϵ is a compact subset of D. Choose N so that for $n > N$ and w in K_ϵ, we have $|w|^n \leqslant \epsilon$. For z in D and $n > N$ either $J(z)$ is in K_ϵ, in which case $|\psi(J(z))\, J(z)^n| \leqslant |\psi(J(z))|\, \epsilon \leqslant \epsilon$, or $J(z)$ is not in K_ϵ, in which case $|\psi(J(z))\, J(z)^n| \leqslant \epsilon\, |J(z)|^n \leqslant \epsilon$. Thus, we have $\|(\psi \circ J)(J^n)\|_\infty \leqslant \epsilon$ for $n > N$.

Now, for f in H^2 and $n > N$,

$$\| CT_\psi(z^n f)\| = \|(\psi \circ J)(J^n) f \circ J\| \leqslant \|(\psi \circ J)\, J^n\|_\infty \|f \circ J\|$$
$$\leqslant \epsilon \| C \| \|f\| = \epsilon \| C \| \| z^n f \|.$$

Therefore, $\| CT_\psi |_{z^n H^2} \| \leqslant \epsilon \| C \|$ for $n > N$, and we see that CT_ψ is compact. ∎

Although the nonlifting of $\{T_\phi\}'$ follows from the above result and the theorem "Lifting implies no commuting compacts" [7, Theorem 1], a direct proof gives a better understanding of how lifting fails. (In this proof, the interplay between arguments on the boundary and inside the disk is important. For f in H^2, $\hat{f}$ will denote the boundary function of f in L^2.)

THEOREM 1.2. *C has no (bounded) extension to L^2 that commutes with M_ϕ.*

Proof. Suppose $\tilde{C}$ is an extension of C defined on L^2 such that $\tilde{C}M_\phi = M_\phi \tilde{C}$. We will show that $\| \tilde{C}(\bar{z}^n)\| \to \infty$ as $n \to \infty$, even though $\| \bar{z}^n \| = 1$ for all n.

For $n = 1, 2, 3,\ldots$ we have $\hat{\phi}^n \tilde{C}(\bar{z}^n) = M_\phi{}^n \tilde{C}(\bar{z}^n) = \tilde{C}(\hat{\phi}^n \bar{z}^n)$. Now $\phi(0) = 0$, so z divides ϕ and $\hat{\phi}^n \bar{z}^n = (z^{-n}\phi^n)^\wedge$ where $z^{-n}\phi^n$ is in H^2. Thus $\tilde{C}(\hat{\phi}^n \bar{z}^n) = [C(z^{-n}\phi^n)]^\wedge = [J^{-n}(\phi \circ J)^n]^\wedge = [J^{-n}\phi^n]^\wedge = (\hat{J})^{-n}\, \hat{\phi}^n$. Now $\hat{\phi} \neq 0$ almost everywhere, so $\hat{\phi}^n \tilde{C}(\bar{z}^n) = (\hat{J})^{-n}\, \hat{\phi}^n$ implies $\tilde{C}(\bar{z}^n) = (\hat{J})^{-n}$. Since $\rho = \max_{-\pi/2<\theta<\pi/2} |\hat{J}(e^{i\theta})|$ is less than 1, we see that

$$\|\hat{J}^{-n}\|_2 = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |\hat{J}|^{-2n}\, d\theta\right)^{1/2} \geqslant \left(\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \rho^{-2n}\, d\theta\right)^{1/2} = 2^{-1/2}\rho^{-n},$$

which is unbounded as $n \to \infty$.

Thus $\tilde{C}$ cannot be a bounded operator on L^2. ∎

The crucial fact in the proof of Theorem 1.2 is that $\hat{J}$ maps *some* of the boundary inside the disk (in this case $\{e^{i\theta} \mid -\pi/2 < \theta < \pi/2\}$ was used), whereas for Theorem 1.1, we needed the fact that $\hat{J}$ maps *nearly all* of the boundary inside the disk. Thus, if we modify the above construction by taking $\Omega_0 = \{z \mid \operatorname{Im} z > 0 \text{ and } \operatorname{Re} z < 0\}$; σ_0 to be a Riemann map of D onto Ω_0; $\phi_0(z) = \exp(\sigma_0(z)) - \exp(\sigma_0(0))$; etc.; then the operator T_{ϕ_0} does not commute with any compact operators [6, Theorem 10], but C_0 does not have a bounded extension to L^2 that commutes with M_{ϕ_0}.

The following theorem is a special case of a theorem of Nordgren [14, Theorem 1].

THEOREM 1.3. *T_ϕ is irreducible.*

Proof. If K is a reducing subspace for T_ϕ, the orthogonal projection of H^2 onto K commutes with T_ϕ. We will show that if P is a self-adjoint projection in $\{T_\phi\}'$, then P is either zero or the identity.

Since P is self-adjoint, P has an extension $\tilde{P}$ that commutes with M_ϕ [5, p. 88], and $\tilde{P}^2 = \tilde{P}$ by the uniqueness of lifting [5, p. 87]. However, since $\hat{\phi}$ is continuous and univalent on $\partial D \backslash \{-1\}$, the only operators that commute with M_ϕ are other multiplication operators. Thus, $\tilde{P} = M_h$ for some h in L^∞. But $h = M_h 1 = \tilde{P}1 = P1$ so h is actually in H^∞ and $P = T_h$. Since P is a projection, $h = h^2$ and h is either identically zero or identically one, that is, P is either the zero operator or the identity operator. ∎

More is true than this theorem states. The above argument actually shows that the only operators in $\{T_\phi\}'$ that lift are the analytic Toeplitz operators. Later, in a more general setting, we show that T_ϕ does not commute with any projections, so that every operator similar to T_ϕ is irreducible (Theorem 5.1).

2. SEMIAUTOMORPHIC FUNCTIONS

In this section we will define a class of functions which includes the function ϕ of Section 1, and compute the commutant and double commutant of analytic Toeplitz operators whose symbol is in this class. Later sections will deal with compact commuting operators, lifting, and reducing subspaces for these operators. Background material on covering maps may be found in standard references, for example, Chap. 3 of Veech's work [24].

If ϕ is a bounded analytic function on the unit disk, we say ϕ is *evenly semiautomorphic with respect to the semigroup Γ* if there is

(1) a plane domain Σ, whose universal covering space is a half plane or disk U, and (2) a univalent analytic map σ from the disk D *into* U such that $\phi = \pi \circ \sigma$, where π: $U \rightarrow \Sigma$ is an analytic covering map. Let G be the group of deck transformations of U under which π is automorphic, let $\Omega = \sigma(D)$, and let Γ be the semigroup of maps of D into D

$$\{J = \sigma^{-1} \circ \tau \circ \sigma \mid \tau \in G \text{ and } \tau(\Omega) \subset \Omega\}.$$

Throughout the rest of the paper, ϕ, Σ, U, π, σ, G, Ω, and Γ will be as in the above definition. For brevity, we may say "ϕ is semiautomorphic" when this structure is meant. Clearly, if J is in Γ, we have $\phi \circ J = \phi$. We note that if J in Γ maps D *onto* D, then J is a Möbius transformation and J^{-1} is also in Γ.

It is apparent that the function ϕ of the preceding example is evenly semiautomorphic with respect to the semigroup $\Gamma = \{J \circ J \circ \cdots \circ J$ (n times) $|$

$n = 0, 1, 2, 3, \ldots\}$. It is also clear that this class includes the class of covering maps (in which case $\Omega = \Sigma$), but for our present purposes, we are less interested in these special cases.

It is somewhat less obvious that this class includes some familiar functions. For instance, $\phi(z) = (z+1)^3$ is evenly semiautomorphic. Let U be the half plane $\{z \mid \operatorname{Re} z < \ln 8\}$; let $\pi(z) = e^z$; and let $\sigma(z) = 3\ln(z+1)$ (usual branch of the logarithm). Then $\phi(z) = \exp(3\ln(z+1)) = \pi \circ \sigma(z)$. The group G, of automorphisms of U, is $\{\tau(z) = z + 2\pi n i \mid n = 0, \pm 1, \pm 2, \ldots\}$. Since $|\operatorname{Im}(\sigma(z))| < \frac{3}{2}\pi$, the only τ in G that maps $\sigma(D)$ into $\sigma(D)$ is the identity map, so $\Gamma = \{\text{identity}\}$. We will see below that this implies $\{T_{(z+1)^3}\}' = \{T_z\}'$. (This corollary is hardly new; it is a special case of Theorem 2 of [8].)

For α in D, let K_α denote the kernel function for evaluation of H^2 functions at α, that is, $K_\alpha(z) = (1 - \bar{\alpha} z)^{-1}$.

LEMMA A. *Let ϕ be an evenly semiautomorphic function. For each α in D, every function f in $[(\phi - \phi(\alpha))H^2]^\perp$ has a unique representation $f = \sum_{\phi(\beta)=\phi(\alpha)} d_\beta K_\beta$ where the series converges in norm.*

Proof. We will show first that the inner factor of $\phi - \phi(\alpha)$ is a Blaschke product and that $\{\beta \in D \mid \phi(\beta) = \phi(\alpha)\}$ is an interpolating sequence.

Without loss of generality, we may assume that the covering space U associated with ϕ is the unit disk. Let $\alpha_0 = \alpha, \alpha_1, \alpha_2, \alpha_3, \ldots$ be an enumeration of the set $\{\beta \in D \mid \phi(\beta) = \phi(\alpha)\}$. For $n = 0, 1, 2, \ldots$, let $\zeta_n = \sigma(\alpha_n)$ and let $\zeta_{-1}, \zeta_{-2}, \zeta_{-3}, \ldots$ be an enumeration of the set $\{\xi \in U \mid \pi(\xi) = \pi(\zeta_0),\ \xi \notin \Omega\}$. From a result of Stout [21, Theorem 6.4] (or see [6, Theorem 6]), we find that $\{\zeta_n\}_{n=-\infty}^{\infty}$ is an interpolating sequence, and that the inner–outer factorization of π is

$$\pi(z) = \left(\prod_{-\infty}^{\infty} \frac{\bar{\zeta}_n}{|\zeta_n|} \frac{\zeta_n - z}{1 - \bar{\zeta}_n z}\right) g(z)$$

where g is invertible.

Since F in H^∞ is outer if and only if $\log|F(0)| = (1/2\pi)\int_{-\pi}^{\pi} \log|F(e^{i\theta})|\, d\theta$, [10, p. 62], we see from the Lebesque dominated convergence theorem that if $F = \prod_{n=1}^{\infty} f_n$ is nonzero, $\|f_n\|_\infty \leqslant 1$ for all n, and each f_n is outer, then F is also outer.

Now

$$\pi \circ \sigma(z) = \left(\prod_{n=0}^{\infty} \frac{\bar{\alpha}_n}{|\alpha_n|} \frac{\alpha_n - z}{1 - \bar{\alpha}_n z}\right) \cdot \left(\prod_{n=0}^{\infty} \frac{|\alpha_n|}{\bar{\alpha}_n} \frac{\bar{\zeta}_n}{|\zeta_n|} \frac{1 - \bar{\alpha}_n z}{\alpha_n - z} \frac{\zeta_n - \sigma(z)}{1 - \bar{\zeta}_n \sigma(z)}\right)$$

$$\cdot \left(\prod_{n=-\infty}^{-1} \frac{\bar{\zeta}_n}{|\zeta_n|} \frac{\zeta_n - \sigma(z)}{1 - \bar{\zeta}_n \sigma(z)}\right) \cdot g(\sigma(z)).$$

Since each factor

$$\frac{|\alpha_n|}{\bar{\alpha}_n} \frac{\bar{\zeta}_n}{|\zeta_n|} \frac{1-\bar{\alpha}_n z}{\alpha_n - z} \frac{\zeta_n - \sigma(z)}{1-\bar{\zeta}_n\sigma(z)}$$

is less than or equal to one in modulus and since σ is univalent, each is outer, so the second factor in the expression for $\pi \circ \sigma$ is outer. Similarly, since $\zeta_n \notin \Omega$ for $n < 0$ and σ is univalent, each factor

$$\frac{\bar{\zeta}_n}{|\zeta_n|} \frac{\zeta_n - \sigma(z)}{1-\bar{\zeta}_n\sigma(z)}$$

is outer and bounded by 1 in modulus, so the third factor in the above expression is outer. Since g is invertible in H^∞, so is $g \circ \sigma$ and $g \circ \sigma$ is outer. Thus we have found that the inner factor of $\pi \circ \sigma$ is just the Blaschke product with zeros $\alpha_0, \alpha_1, \alpha_2, \ldots$.

We claim that $\{\alpha_n\}_{n=0}^\infty$ is an interpolating sequence in D. Indeed, suppose $\{a_n\}_{n=0}^\infty$ is a bounded sequence. We enlarge this sequence by setting $a_n = 0$ for $n = -1, -2, -3, \ldots$. Since $\{\zeta_n\}_{-\infty}^\infty$ is an interpolating sequence in U and $\{a_n\}_{-\infty}^\infty$ is a bounded sequence, there is a function h in H^∞ so that $h(\zeta_n) = a_n$ for all n. Therefore, $h \circ \sigma$ is in H^∞, $h(\sigma(\alpha_n)) = h(\zeta_n) = a_n$ for $n = 0, 1, 2, \ldots$, and $h \circ \sigma$ interpolates the given sequence.

Let $\{e_n\}_{n=0}^\infty$ be the usual basis for l^2. Shapiro and Shields showed [20, Theorem 2] that since $\{\alpha_n\}_{n=0}^\infty$ is an interpolating sequence, the map Φ: $[(\phi - \phi(\alpha)) H^2]^\perp \to l^2$, given by $\Phi f = \sum_{n=0}^\infty (1 - |\alpha_n|^2)^{1/2} f(\alpha_n) e_n$, is an isomorphism. This means that Φ^* is also an isomorphism and a simple calculation shows that $\Phi^* e_n = (1 - |\alpha_n|^2)^{1/2} K_{\alpha_n}$. It follows that if d_n, $n = 0, 1, 2, \ldots$ are the unique complex numbers such that $\Phi^{*-1} f = \sum_{n=0}^\infty d_n (1 - |\alpha_n|^2)^{-1/2} e_n$, then $f = \Phi^*(\sum_{n=0}^\infty d_n (1 - |\alpha_n|^2)^{-1/2} e_n) = \sum_{n=0}^\infty d_n K_{\alpha_n}$, where the series converges in norm. ▮

We are now ready to compute the commutant of the Toeplitz operator T_ϕ where ϕ is semiautomorphic.

THEOREM 2.1. *Suppose ϕ is evenly semiautomorphic with respect to the semigroup Γ of maps of D into D.*

If S is in $\{T_\phi\}'$, then for each J in Γ, there is a unique analytic function a_J defined on D so that for each α in D and each g in H^2

$$(Sg)(\alpha) = \sum_{J \in \Gamma} a_J(\alpha)\, g(J(\alpha)), \qquad (*)$$

where the series converge absolutely for each α and uniformly on compact subsets of D.

Conversely, if S is a bounded operator on H^2 that has a representation (*), then S commutes with T_ϕ.

If J is an analytic map of D into D, we may define the composition operator C_J on H^2 by $C_J g = g \circ J$. (C_J is a bounded operator [15, 19].) We saw in the first section that C_J commutes with an analytic Toeplitz operator if and only if its symbol is automorphic with respect to J. A more revealing, but less precise, way of stating the theorem is that an operator S on H^2 commutes with T_ϕ if and only if S can be represented by a series $S \sim \sum_{J \in \Gamma} a_J C_J$, where the coefficients a_J are analytic functions defined in D. (The a_J need not be bounded functions [7].) The crucial point here is that the composition operators that appear in the series are associated with the maps J such that $\phi \circ J = \phi$ and $J(D) \subset D$. The work of the proof is in showing that the maps $\tilde{J}$ such that $\phi \circ \tilde{J} = \phi$ but $\tilde{J}(D) \not\subset D$, which may exist because ϕ comes from a covering map, make no contribution to the commutant.

Proof of Theorem 2.1. Suppose ϕ is semiautomorphic with respect to Γ (as above) and S commutes with T_ϕ. For α in D, by the Fundamental Lemma of [6, p. 3], $S^* K_\alpha$ is in $[(\phi - \phi(\alpha)) H^2]^\perp$, so by Lemma A there are complex numbers d_τ^α, τ in G, such that $S^* K_\alpha = \Sigma d_\tau^\alpha K_\beta$ where the sum is taken over those τ in G for which $\tau(\sigma(\alpha)) \in \Omega$, writing β for $\sigma^{-1}(\tau(\sigma(\alpha)))$.

For α in D and τ_0 in G, the function $h_\alpha = (\phi - \phi(\alpha))(\sigma - \tau_0(\sigma(\alpha)))^{-1}$ is bounded on D since ϕ is in H^∞ and when $\sigma(z) - \tau_0(\sigma(\alpha))$ is near zero, then $h_\alpha(z) = (\pi(\sigma(z)) - \pi(\tau_0(\sigma(\alpha))))(\sigma(z) - \tau_0(\sigma(\alpha)))^{-1}$ is near $\pi'(\tau_0(\alpha))$ which is finite. Thus the map $\alpha \to (Sh_\alpha)(\alpha) = \langle h_\alpha, S^* K_\alpha \rangle$ is an analytic function defined on the disk. Now if $\tau_0(\Omega) \not\subset \Omega$ and α is a point of D so that $\tau_0(\sigma(\alpha)) \notin \Omega$, then h_α vanishes at all points β in D such that $\phi(\alpha) = \phi(\beta)$ which means that $(Sh_\alpha)(\alpha) = 0$. Since Ω is simply connected and τ_0 is one-to-one, if $\tau_0(\Omega) \not\subset \Omega$, there are uncountably many α in D so that $\tau_0(\sigma(\alpha)) \notin \Omega$ and we conclude that for this τ_0 the map $\alpha \to (Sh_\alpha)(\alpha)$ is identically zero.

On the other hand, if α is a point of D so that $\tau_0(\sigma(\alpha)) \in \Omega$, then $(Sh_\alpha)(\alpha) = \langle h_\alpha, S^* K_\alpha \rangle = \langle h_\alpha, \Sigma d_\tau^\alpha K_\beta \rangle = \Sigma \overline{d_\tau^\alpha} \pi'(\tau_0(\sigma(\alpha)))$. Therefore, for α in the disk the map $\alpha \to d_{\tau_0}^\alpha$ is the complex conjugate of the analytic function $\alpha \to (Sh_\alpha)(\alpha)(\pi'(\tau_0(\sigma(\alpha))))^{-1}$. It follows that if $\tau_0(\Omega) \not\subset \Omega$, then $d_{\tau_0}^\alpha$ is zero for all α in D.

For J in Γ, let a_J be the analytic function $a_J(\alpha) = \overline{d_\tau^\alpha}$ where $J = \sigma^{-1} \circ \tau \circ \sigma$. We have shown that $S^* K_\alpha = \sum_{J \in \Gamma} \overline{a_J(\alpha)}\, K_{J(\alpha)}$, where (from Lemma A) the series converges in norm. Since $(Sg)(\alpha) = \langle g, S^* K_\alpha \rangle$, the representation (*) follows immediately.

The proof that this series converges uniformly on compact subsets of D is a modification of the proof of the lemma to Theorem 5 of [7] and will be omitted.

Now if the bounded operator S has a representation (*), then for f in H^2,

$$\langle (\phi - \phi(\alpha)) f, S^* K_\alpha \rangle = S((\phi - \phi(\alpha)) f)(\alpha)$$

$$= \sum_{J \in \Gamma} a_J(\alpha)(\phi(J(\alpha)) - \phi(\alpha)) f(J(\alpha)) = 0.$$

Therefore, S^*K_α is in $[(\phi - \phi(\alpha))\, H^2]^\perp$ for all α in D and by the Fundamental Lemma [6, p. 3], S commutes with T_ϕ. ∎

Knowing the commutant of T_ϕ for a semiautomorphic function ϕ enables us to find the double commutant easily.

THEOREM 2.2. *If ϕ is evenly semiautomorphic with respect to the semigroup Γ of maps of D into D, then*

$$\{T_\phi\}'' = \{T_h \mid h \text{ is in } H^\infty \text{ and } h \circ J = h \text{ for all } J \text{ in } \Gamma\}.$$

Proof. Since T_ϕ commutes with T_z, $\{T_\phi\}'' \subset \{T_z\}' = \{T_h \mid h \in H^\infty\}$. Since T_ϕ commutes with C_J for J in Γ, T_h does also when T_h is in $\{T_\phi\}''$, and $h \circ J = h$ for J in Γ.

On the other hand, if h is in H^∞ and $h \circ J = h$ for all J in Γ, then T_h commutes with S when T_ϕ does. Indeed, by the representation (*) for S, we have

$$(ST_h g)(\alpha) = \sum_{J \in \Gamma} a_J(\alpha)\, h(J(\alpha))\, g(J(\alpha)) = h(\alpha) \sum_{J \in \Gamma} a_J(\alpha)\, g(J(\alpha))$$

$$= (T_h S_g)(\alpha) \qquad \text{for all } \alpha \text{ in } D \text{ and all } g \text{ in } H^2. \quad ∎$$

Theorem 2.2 characterizes $\{T_\phi\}''$ in terms of the semigroup Γ, but we would prefer a characterization in terms of ϕ, in particular we want to know when all the operators in $\{T_\phi\}''$ are functions of T_ϕ. For the example of Section 1, it is fairly easy to prove that $\{T_\phi\}'' = \{T_h \mid h = g \circ \phi$ where $g \in H^\infty(\phi(D))\}$. On the other hand, $\{T_{(z+1)^3}\}'' = \{T_h \mid h$ is in $H^\infty\}$ but not every h in H^∞ is a function of $(z+1)^3$. In general, it seems to be difficult to decide when $\{T_\phi\}''$ consists of functions of T_ϕ.

In [6, p. 4] a function, f, in H^∞ was called ancestral if, for each α in D, the subspace spanned by $\{S^*K_\alpha \mid S \in \{T_f\}'\}$ is exactly $[(f - f(\alpha))\, H^2]^\perp$. Also in [6], several properties of $\{T_f\}'$ for an ancestral function f were found and the question was raised "For h in H^∞ can we always find an ancestral function f so that $\{T_h\}' = \{T_f\}'$?" We see from Theorem 2.1 that the example of Section 1 answers this question negatively. If $\{T_f\}' = \{T_\phi\}'$ then, by the above, $f = g \circ \phi$ so that $[(f - f(\alpha))\, H^2]^\perp \supset [(\phi - \phi(\alpha))\, H^2]^\perp$. For α in the disk, the subspace spanned by $\{S^*K_{J(\alpha)} \mid S \in \{T_\phi\}'\}$ does not include K_α, so is not all of $[(\phi - \phi(J(\alpha))\, H^2]^\perp$.

3. COMPACTS IN THE COMMUTANT

We have seen in Section 1 that, if ϕ is semiautomorphic, it is possible, but not necessary, for T_ϕ to commute with nonzero compact operators. We will give a sufficient condition that $\{T_\phi\}'$ contain a compact. Theorem 10 of [6] showed that if ϕ maps a large subset of the circle onto the boundary of $\phi(D)$, then T_ϕ does not commute with any compacts. If ϕ does map a large part of the circle to the boundary of $\phi(D)$, then since ϕ is automorphic with respect to Γ each J in Γ must map a large part of the circle into the circle. We assume in Theorem 3.1 that this is not the case.

THEOREM 3.1. *Let J be an analytic map of D into D and let $F = \overline{J(D)} \cap \partial D$. If ψ is analytic in D, continuous on $\bar{D}$, and vanishes on F, then the operator $C_J T_\psi$ is compact.*

Proof. The proof of this theorem is the same as the proof of Theorem 1.1, where in Theorem 1.1, $F = \{-1\}$. ▌

COROLLARY. *If ϕ is semiautomorphic with respect to the semigroup Γ and there is J in Γ such that $F = \overline{J(D)} \cap \partial D$ has measure zero, then T_ϕ commutes with a nonzero compact operator.*

Proof. Since F is a closed set of measure zero, we can find a nonzero function ψ in the disk algebra that vanishes on F [10, p. 80]. $C_J T_\psi$ is compact and commutes with T_ϕ. ▌

Since there are compact composition operators, the reader may wonder if there is an example of a compact composition operator that commutes with an analytic Toeplitz operator.

THEOREM 3.2. *If h is a nonconstant function in H^∞ and J is an analytic map of D into D such that C_J commutes with T_h, then C_J is not compact.*

Proof. We have seen that C_J and T_h commute if and only if $h \circ J = h$, that is, if h is an eigenvector for C_J with eigenvalue 1. But if $h \circ J = h$, we also have $h^n \circ J = h^n$ for $n = 2, 3, 4, 5,\ldots$. Therefore, if T_h and C_J commute, $1, h, h^2, h^3,\ldots$ are all eigenvectors for C_J with eigenvalue 1. Since h is nonconstant, these are linearly independent and C_J is not compact. ▌

4. LIFTING COMMUTANTS

In Section 1 we noted that whether a composition operator commuting with an analytic Toeplitz operator has an extension to L^2 commuting with the multiplication operator depends on the boundary values of the map inducing the composition operator. (As before, $\hat{f}$ will denote the boundary function of f.) For J in Γ, $\hat{J}(e^{i\theta})$ is on the unit circle for almost all θ if and only if J is a Möbius transformation if and only if J is invertible in Γ. In this section we state two results that relate invertibility of elements of Γ to lifting the commutant of T_ϕ.

THEOREM 4.1. *If ϕ is evenly semiautomorphic with respect to the semigroup Γ, and J is a noninvertible map in Γ, then the composition operator C_J does not have a bounded extension to L^2 that commutes with M_ϕ.*

Proof. Since J is not invertible in Γ, there is a set E with positive measure and a constant ρ, $0 < \rho < 1$ so that for $e^{i\theta}$ in E, $|\hat{J}(e^{i\theta})| < \rho$. The proof of Theorem 1.2 applies here with the interval $[-\pi/2, \pi/2]$ replaced by the set E. ∎

THEOREM 4.2. *Suppose ϕ is evenly semiautomorphic with respect to the semigroup Γ. Let H be the subgroup of Γ, $H = \{J \in \Gamma \mid J^{-1} \in \Gamma\}$, and let $\mathfrak{A}$ be the subalgebra of $\{T_\phi\}'$ of operators*

$$\mathfrak{A} = \{S \in \{T_\phi\}' \mid S \sim \Sigma a_J C_J \text{ where } a_J = 0 \text{ for } J \notin H\}.$$

The algebra $\mathfrak{A}$ lifts isometrically to the commutant of M_ϕ.

Proof. The proof of this result is a minor modification of the proof of Theorem 5 of [7]. The modifications are possible because each J in H is a Möbius transformation so that if B is an inner function, so is $B \circ J$. ∎

In the example of Section 1, the subgroup H is just the identity so this theorem gives the obvious conclusion that analytic Toeplitz operators have isometric liftings. The proof of Theorem 1.3 shows that the converse is true in this case.

The above theorems, of course, do not cover all cases. It seems likely that the converse of Theorem 4.2 is true in general, that is, that S in $\{T_\phi\}'$ has a lifting if and only if S is in $\mathfrak{A}$, but I know no proof.

5. PROJECTIONS IN THE COMMUTANT

For a semiautomorphic function ϕ, we would like to find the reducing subspaces for T_ϕ, or at least to determine when T_ϕ is irreducible. In some cases, we can find a few reducing subspaces easily. If J in Γ is invertible, it is a Möbius transformation and the operator $U_J = T_{(J')^{1/2}} C_J$ is unitary [12]. As Abrahamse and Ball noted, since it commutes with T_ϕ, the spectral subspaces of U_J are reducing subspaces for T_ϕ [3].

A subspace is a reducing subspace for T_ϕ if and only if the orthogonal projection onto the subspace commutes with T_ϕ. Rather than attacking the irreducibility question directly, we look at the more general question of determining when T_ϕ commutes with a nontrivial projection, orthogonal or not.

Before stating the theorem, we need the following definition.

A function ν, whose domain is the semigroup Γ and whose values are ordinal numbers, is called a *factorization function for Γ* if for g, g_1, g_2 in Γ

(1) $\nu(g) = 0$ if and only if g is the identity,

and

(2) if $g = g_1 g_2$, where g_1 and g_2 are not *both* the identity, then either $\nu(g) > \nu(g_1)$ or $\nu(g) > \nu(g_2)$.

If Γ has a factorization function ν, then the identity is the only invertible element in Γ, for if $g \neq id$ is invertible, $\nu(gg^{-1}) = \nu(id) = 0$ so either $\nu(g)$ or $\nu(g^{-1})$ is less than zero, which is impossible.

The existence of a factorization function should be regarded as a strengthening of the hypothesis on Γ that it have no invertible elements except the identity. It should be pointed out that any semigroup embedded in a finitely generated free group can be the semigroup of automorphisms for some evenly semiautomorphic function, so the existence of a factorization function is more an algebraic question than a geometric one. Although not every semigroup that is embedded in a free group and has no inverses has a factorization function, they do exist for some cases of interest. For example, if Γ is a free semigroup with generators $g_1, g_2, \ldots, g_n$, that is, Γ is the set of words in the generators with nonnegative exponents, then the sum of the exponents is a factorization function. Factorization functions need not be this simple however. Let Γ_1 be the semigroup in the free group with generators g and h with

$$\Gamma_1 = \{id\} \cup \left\{ g^{k_1}h^{l_1} \cdots g^{k_n}h^{l_n} \;\middle|\; \text{either } \sum_{j=1}^{n} k_j + l_j \geq 1 \text{ or } \sum_{j=1}^{n} k_j + l_j = 0 \text{ and } \sum_{j=1}^{n} k_j \geq 1 \right\}.$$

The function $\nu(w) = (\sum_{j=1}^{n} k_j + l_j)\omega + |\sum_{j=1}^{n} k_j|$, where ω is the first infinite ordinal, is a factorization function for Γ_1. (It can be shown that Γ_1 has no factorization function into the integers.)

The semigroup for the example of Section 1 is the free semigroup on one generator (i.e., the positive integers) so the following theorem applies. An examination of the proof in this case will show that we are generalizing the proof that a lower triangular Toeplitz matrix is not a nontrivial projection.

THEOREM 5.1. *If ϕ is evenly semiautomorphic with respect to the semigroup Γ and ν is a factorization function for Γ, then T_ϕ does not commute with any nontrivial projections.*

Proof. Suppose P is a projection that commutes with T_ϕ. We want to show that P is zero or the identity.

From the representation (*) of Theorem 2.1, we have $P^*K_\alpha = \sum_{J\in\Gamma} \overline{a_J(\alpha)}\, K_{J(\alpha)}$ for each α in D. Now

$$P^{*2}K_\alpha = P^*\left(\sum_{J\in\Gamma} \overline{a_J(\alpha)}\, K_{J(\alpha)}\right) = \sum_{I\in\Gamma}\sum_{J\in\Gamma} \overline{a_J(\alpha)}\, \overline{a_I(J(\alpha))}\, K_{I\circ J(\alpha)}$$

$$= \sum_{J\in\Gamma}\left(\sum_{J_1\circ J_2=J} \overline{a_{J_2}(\alpha)}\, \overline{a_{J_1}(J_2(\alpha))}\right) K_{J(\alpha)} .$$

Since P is a projection, we have $P^{*2} = P^*$, so we have $a_J(\alpha) = \sum_{J_1\circ J_2=J} a_{J_2}(\alpha)\, a_{J_1}(J_2(\alpha))$ for all J in Γ and α in D. (Lemma A justifies equating coefficients in this way.)

Since ν is a factorization function on Γ, the identity is the only invertible element of Γ, so the only factorization of the identity is $id = id \circ id$. Thus, we have $a_{id} = a_{id}^2$ and since a_{id} is analytic, it is the constant function 0 or 1. If P is a projection that commutes with T_ϕ, so is $I - P$ and if $a_{id} \equiv 1$, then the corresponding function for $I - P$ is zero. We therefore assume that $a_{id} \equiv 0$. We have shown that $a_J \equiv 0$ if $\nu(J) = 0$ and we proceed by induction.

Suppose η is an ordinal number and we have shown $a_J \equiv 0$ for $\nu(J) < \eta$. If $\nu(J) = \eta$, then whenever $J = J_1 \circ J_2$, either $\nu(J_1) < \eta$ or $\nu(J_2) < \eta$ so either $a_{J_1} \equiv 0$ or $a_{J_2} \equiv 0$, and we have

$$a_J(\alpha) = \sum_{J_1\circ J_2=J} a_{J_2}(\alpha)\, a_{J_1}(J_2(\alpha)) = 0.$$

We conclude that $a_J \equiv 0$ for all J in Γ so that either P or $I - P$ is zero. ∎

Although I do not know a proof, it seems likely that the conclusion remains true if we require only that the identity is the only invertible element of Γ.

6. Conclusions

In this paper we have examined a class of analytic Toeplitz operators whose commutants are quite different from the previously known examples. Some of these differences are apparently due to the fact that the boundary behavior of semiautomorphic functions is not as representative of the behavior inside the disk as for inner functions or covering maps. For example, the function in Section 1 is one–to–one on the boundary but infinite–to–one on the inside of the disk, whereas the boundary function of a covering map of the disk onto a plane region is in some sense a covering of the boundary of the region. These differences seem to be very important for lifting questions, but less important for the description of the commutant.

It is clear that the proofs herein depend heavily on the continuability of the automorphisms in Γ, which are just branches of the multiple valued analytic function $\phi^{-1} \circ \phi$. If f is a bounded analytic function for which f' vanishes at some points, or for which f' is never zero, but f is not semi-automorphic, the technical details needed to compute $\{T_f\}'$ will certainly be more difficult, and perhaps $\{T_f\}'$ will exhibit unexpected behavior.

It would be interesting to use the techniques of Section 3 to find compacts commuting with other, more general, operators. In cases where the operator to be studied is also multiplication by z and the Hilbert space is a Hilbert space of functions, the direction to proceed is clear. In other cases, a more general notion of composition operators and careful control of convergence of functions of the operator seem to be required.

References

1. M. B. Abrahamse, Analytic Toeplitz operators with automorphic symbol, *Proc. Amer. Math. Soc.* **52** (1975), 297–302.
2. M. B. Abrahamse, Some examples of lifting the commutant of a subnormal operator, preprint.
3. M. B. Abrahamse and J. A. Ball, Analytic Toeplitz operators with automorphic symbol, II, *Proc. Amer. Math. Soc.* **59** (1976), 323–328.
4. J. A. Ball, Hardy space expectation operators and reducing subspaces, *Proc. Amer. Math. Soc.* **47** (1975), 351–357.
5. J. Bram, Subnormal operators, *Duke Math. J.* **22** (1955), 75–94.
6. C. C. Cowen, The commutant of an analytic Toeplitz operator, *Trans. Amer. Math. Soc.* **239** (1978), 1–31.
7. C. C. Cowen, The commutant of an analytic Toeplitz operator, II, *Indiana J. Math*, **29** (1980), 1–12.
8. J. A. Deddens and T. K. Wong, The commutant of analytic Toeplitz operators. *Trans. Amer. Math. Soc.* **184** (1973), 261–273.
9. R. G. Douglas and C. Pearcy, Spectral theory of generalized Toeplitz operators, *Trans. Amer. Math. Soc.* **115** (1965), 433–444.
10. K. Hoffman, "Banach Spaces of Analytic Functions," Prentice-Hall, Englewood Cliffs, N.J., 1962.
11. V. Lomonosov, On invariant subspaces of families of operators commuting with a completely continuous operator, *Funkcional. Anal. i Priložen* **7** (1973), 55–56.
12. A. Lubin, Isometries induced by composition operators and invariant subspaces, *Illinois J. Math.* **19** (1975), 424–427.
13. A. Lubin, Weighted shifts and commuting normal extension, *J. Australian Math. Soc. (Ser. A)* **27** (1979), 17–26.
14. E. A. Nordgren, Reducing subspaces of analytic Toeplitz operators, *Duke J. Math.* **34** (1967), 175–181.
15. E. A. Nordgren, Composition operators, *Canad. J. Math.* **20** (1968), 442–449.
16. E. A. Nordgren, Composition operators on Hilbert spaces, *in* "Hilbert Space Operators". Lecture Notes in Mathematics, No. 693, Springer-Verlag, Berlin, 1978.

17. R. F. OLIN AND J. E. THOMSON, Lifting the commutant of a subnormal operator, preprint.
18. C. PEARCY AND A. L. SHIELDS, "A Survey of the Lomonosov Technique in Theory of Invariant Subspaces," Math. Surveys No. 13, pp. 219–229, Amer. Math. Soc., Providence, R.I., 1974.
19. J. V. RYFF, Subordinate H^p functions, *Duke Math. J.* **33** (1966), 347–354.
20. H. S. SHAPIRO AND A. L. SHIELDS, On some interpolation problems for analytic functions, *Amer. J. Math.* **83** (1961), 513–532.
21. E. L. STOUT, Bounded holomorphic functions on finite Riemann surfaces, *Trans. Amer. Math. Soc.* **120** (1965), 255–285.
22. J. E. THOMSON, The commutants of certain analytic Toeplitz operators, *Proc. Amer. Math. Soc.* **54** (1976), 165–169.
23. J. E. THOMSON, The commutant of a class of analytic Toeplitz operators, *Amer. J. Math.* **99** (1977), 522–529.
24. W. A. VEECH, "A Second Course in Complex Analysis," Benjamin, New York, 1967.
25. HADWIN, NORDGREN, RADJAVI, AND ROSENTHAL, An operator not satisfying Lomonosov's hypothesis, *J. Functional Analysis*, in press.