Math 554 Qualifying Exam

August, 2014

You may use any theorems from the book. Other results you use must be proved. Make sure to double check your calculations and support your arguments.

1. (20) Let

\[ A = \begin{pmatrix} -7 & -1 & -1 \\ -21 & -3 & -3 \\ 70 & 10 & 10 \end{pmatrix}. \]

Find the Jordan canonical form \( J \) of \( A \) and an invertible matrix \( P \) such that \( P^{-1}AP = J \).
2. (20) Let $T$ be a linear operator on an $n$-dimensional vector space $V$ ($n > 1$) and $W$ is a $k$-dimensional ($0 < k < n$) $T$-invariant subspace. Show that if $T$ has $n$ distinct eigenvalues, then for any $T$-invariant direct sum decomposition of $V = W_1 \oplus W_2 \oplus \cdots \oplus W_s$, $W = (W_1 \cap W) \oplus (W_2 \cap W) \oplus \cdots \oplus (W_s \cap W)$.
3. (20) Let $T$ be a linear transformation on a finite dimensional vector space over the field $\mathbb{F}$. Let $p_T$ and $f_T$ be the minimal and respectively characteristic polynomial of $T$. If $p_T = f_T = q^k$ for some irreducible polynomial $q \in \mathbb{F}[x]$ and $k > 1$, show that no nonzero proper $T$-invariant subspace can have a $T$-invariant complement.
4. (20) Let $A$ be the $2n \times 2n$ complex matrix

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & a_{2n} \\
0 & 0 & \cdots & a_{2n-1} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & a_2 & \cdots & 0 & \cdots \\
a_1 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

Find a necessary and sufficient condition that the matrix $A$ is diagonalizable. Justify your answer.
5. (20) Recall that an operator $N$ on a finite dimensional inner product space is normal if $N^*N = NN^*$. Show that the product $ST$ of two normal operators $S$ and $T$ on a finite dimensional inner product space $V$ is normal if $ST = TS$. 