You must provide necessary arguments to support your claims. You may use any theorems and basic examples from the textbooks but not from homework. Solutions must be shortened for simplicity.

1. (20) Let $A_6$ be the alternating group on six letters.
   (a) Does $A_6$ have a subgroup of order 72? Justify your answer.
   (b) Does $A_6$ have a subgroup of order 45? Justify your answer.

2. (20) If $G$ is a finite group with order $pqr$, where $p < q < r$ are prime numbers. Let $P$, $Q$ and $R$ be Sylow $p$, $q$ and $r$-subgroup, respectively.
   (a) Show that either $P$, $Q$ or $R$ is a normal subgroup.
   (b) If $P$ (or $Q$) is normal in $G$, show that $G/P$ (resp. $G/Q$) has a normal subgroup of order $r$.
   (c) Show that $R$ is a normal subgroup in $G$.

3. (20) Let $\mathbb{Z}[i]$ be the ring of Gaussian integers.
   (a) Show that $\mathbb{F} = \mathbb{Z}[i]/(3 + 8i)$ is a field, where $(3 + 8i)$ is the ideal generated by $3 + 8i$.
   (b) Determine the characteristic $p$ and the number of elements in $\mathbb{F}$ from part (a).
   (c) Find the irreducible polynomial $f \in \mathbb{Z}_p[x]$ so that the field $\mathbb{F}$ is a splitting field of $f$ over $\mathbb{Z}_p$.
   (d) What is the largest degree among all irreducible factors of $x^{p^2} - x$ in $\mathbb{Z}_p[x]$? Why?

4. (20) Let $R$ be a UFD. Let $a$ and $b$ be non-zero-non-unit elements in $R$ such that $a$ is a factor of $b$. Show that there is a natural surjective homomorphism $\pi : R/(b) \longrightarrow R/(a)$ which induces a surjection $\pi : (R/(b))^\times \longrightarrow (R/(a))^\times$. Here $R^\times$ is the set of units of $R$. (Hint: Chinese Remainder Theorem may be helpful).

5. (20) Let $K = \mathbb{Q}(\sqrt{2}, i)$ and $F = \mathbb{Q}(\sqrt{2})$.
   (a) Show that the extension $\mathbb{Q}(i, \sqrt{2})/\mathbb{Q}(\sqrt{2})$ is a Galois extension.
   (b) Find the order of the Galois group $\text{Aut}_{\mathbb{Q}(\sqrt{2})}(\mathbb{Q}(i, \sqrt{2})$.
   (c) Determine the structure of the Galois group $\text{Aut}_{\mathbb{Q}(\sqrt{2})}(\mathbb{Q}(i, \sqrt{2})$.}