Problem 1. Let \( p \) be a prime number and \( G = \mathbb{Z}_p \times \mathbb{Z}_{p^2} \) the product of cyclic groups of orders \( p \) and \( p^2 \).

a) Count the number of subgroups of \( G \) of order \( p \).

b) Count the number of subgroups \( K \subset G \) such that \( G/K \) is isomorphic to \( \mathbb{Z}_{p^2} \).

c) Count the number of subgroups \( K \subset G \) such that \( G/K \) is isomorphic to \( \mathbb{Z}_p \times \mathbb{Z}_p \).

Problem 2. Let \( A_n \subset S_n \) be the alternating group. For which \( n \) is \( A_n \) a semidirect product of two non-trivial subgroups? Explain.

Problem 3. Let \( R \) be a commutative ring. Give the definition of a prime ideal \( I \subset R \). Suppose \( I, J, I \cap J \) are all prime ideals of \( R \). Show that \( I \subset J \) or \( J \subset I \).

Problem 4. Let \( F_5 = \mathbb{Z}/5\mathbb{Z} \) be the field with 5 elements. Show that the quotients \( F_5[x]/(x^2 + 2) \) and \( F_5[x]/(x^2 + x + 1) \) are fields.

Construct an explicit isomorphism of fields: \( F_5[x]/(x^2 + 2) \rightarrow F_5[x]/(x^2 + x + 1) \).

Problem 5. Give a definition of a Galois extension. Let \( K \subseteq M \subseteq L \) be fields. True or false? Explain!

a) If \( L : K \) is a Galois extension, then \( M : K \) is a Galois extension.

b) If \( L : K \) is a Galois extension, then \( L : M \) is a Galois extension.

c) If \( M : K \) and \( L : M \) are Galois extensions, then \( L : K \) is a Galois extension.

Problem 6. Give a polynomial \( p(x) \in \mathbb{Z}[x] \) with Galois group

a) \( \mathbb{Z}_6 \).

b) \( \mathbb{Z}_3 \).